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HARMONIC MAPS INTO REAL HYPERBOLIC SPACE

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Introduction. In [2,4,5,6,7] Calabi, Barbosa and Chern showed that there is a one-to-one correspondence between arbitrary pairs of full isotropic (terminology as in [8]) harmonic maps $\pm\phi : M \to S^{2m}$ from a Riemann surface to Euclidean sphere and full totally isotropic holomorphic maps $f : M \to \mathbb{C}P^{2m}$ from the surface to complex projective space. In this paper we show, very explicity, how to construct a similar one-to-one correspondence when S^{2m} is replaced by real hyperbolic space H^{2m} with its standard metric. We get over a difficulty encountered by Barbosa of dealing with the zeros of certain wedge product by a technique adapted from [8]. (The case of indefinite complex hyperbolic and projective spaces will be considered in a separate paper).

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Let Q be a symmetric $\mathbb{C}$-bilinear form on $\mathbb{C}^{n+1}$ given by

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$$Q(x,y) = \sum_{k=1}^{n} x_k y_k - rt$$

where $x = (x_1,\dots,x_n,r)$, $y = (y_1,\dots,y_n,t) \in \mathbb{C}^{n+1}$. Set

$$\mathscr{H}^n = \{x = (x_1,\dots,x_n,r) \in R^{n+1} : Q(x,x) = -1\}.$$

$\mathscr{H}^n$ is called the hyperbolic space form or shortly hyperbolic space. We give $\mathscr{H}^n$ the metric Q by restricting Q to each tangent space at $x \in \mathscr{H}^n$ i.e. the linear subspace $\{y \in R^{n+1} : Q(y,x) = 0\}$. $\mathscr{H}^n$ with its standard metric Q is a real analytic complete Riemannian manifold of curvature -1. It has two disjoint connected componets namely

$$H_1^n = \{(x_1,\dots,x_n,r) \in \mathscr{H}^n : r>0\} \text{ and } H_2^n = \{(x_1,\dots,x_n,r) \in \mathscr{H}^n : r<0\}$$

We shall only deal with one component $H^n = H_1^n$.

Now let M be a Riemann surface (not necessarily compact) and $\phi : M \to H^n$ be a smooth map. Write $\Phi = j \circ \phi$ where $j : H^n \to R^{n+1}$ is the inclusion map and $N = n+1$.

Say that the map $\phi : M \to H^n$ is full if the image of Φ lies in no hyperplane of R^N. Let z be a local complex coordinate on M. Write $\partial = \partial/\partial z$ $\bar{\partial} = \partial/\partial \bar{z}$. Say that the smooth map $\phi : M \to H^n$ is Q-isotropic if $Q(\partial^k\Phi, \partial^r\Phi) = 0$ for every $r+k>0$. (This is equivalent to the condition that for the covariant derivative D in $\phi^{-1}T^C H^n$ $Q(D^k\phi, D^r\phi) = 0$). It is easy to see that if $\phi : M \to H^n$ is full, Q-isotropic harmonic map then n is even (cf [4]). So we write $n = 2m$ and hence $N = 2m+1$.

Definition: Say that a smooth map $f : M \to \mathbb{C}P^{2m}$ is totally Q-isotropic if for any local lift ξ of f, $Q(\partial^k\xi,\partial^k\xi) = 0$, $0 \le k \le (m-1)$. Equivalently $Q(\partial^k\xi,\partial^r\xi) = 0$, $0 \le r+k \le 2m-1$.

Note the different ranges of k,r make this a very different concept to Q-isotropy.

Let f_{m-1} be the $(m-1)^{th}$ associated curve of f (see for example [8]) given by span $\{\xi,\ldots\partial^{m-1}\xi\}$ at points where $\xi,\ldots\partial^{m-1}\xi$ are linearly independent. Here ξ is a local lift of f.

We now state our main result:

THEOREM : *There is a one-one correspondence between the set of all full* Q-*isotropic harmonic maps* $\phi : M \to H^{2m}$ *and the set of all full totally* Q-*isotropic holomorphic maps* $f : M \to \mathbb{C}P^{2m}$ *such that* Q *is positive definite on the* $f_{m-1}(p)$ *for every* $p \in M$. *The correspondence given by* $\phi \to f$ *such that*

$$f : M \xrightarrow{F} (\Lambda^{2m}\,\mathbb{C}^N)\setminus 0 \xrightarrow{*} \mathbb{C}^N\setminus 0 \xrightarrow{\Pi} \mathbb{C}P^{2m}$$

where $F = \Phi\Lambda\ldots\Lambda\partial^m\Phi\Lambda\bar\partial\Phi\Lambda\ldots\Lambda\bar\partial^{m-1}\Phi$ *and* $*$ *is the star operator. (The star operator* $* : \Lambda^{2m}\mathbb{C}^N \to \mathbb{C}^N$ *is given by*

$$\sum_{k=1}^{N} \lambda_k\Big(\bigwedge_{i\in S_k} e_i\Big) = \sum_{k=1}^{N} \bar\lambda_k\,\bar e_k$$

where $e_1,\ldots,e_N$ *are orthonormal basis for* $\mathbb{C}^N$ *with respect to the standard Hermitian form* $<\ >$ *on* $\mathbb{C}^N$ *and* $S_k \subset \{1,\ldots,N\}$ *contains all but* k.)

We prove this in two stages: we construct holomorphic map from *the harmonic one in section 1 and harmonic map from the holomorphic one in section 2.*

Section 1:

(1.1) Let $\phi : M \to H^{2m}$ be a full Q-isotropic harmonic map. Set $F = F_u = \Phi\Lambda\ldots\Lambda\partial^m\Phi\Lambda\ \bar\partial\Phi\Lambda\ldots\Lambda\bar\partial^{m-1}\Phi$ and $*F = * \circ F$ where $\partial = \partial/\partial z$, $\bar\partial = \partial/\partial\bar z$ and $U \subset M$ is a local coordinate set. By fullness $*F$ is non-zero on an open dense subset of M, see forexample [8].

Claim: $\bar\partial(*F) = \lambda(*F)$ at points where $*F \neq 0$ for some smooth function $\lambda : U \to \mathbb{C}$.

Proof. We have $\partial F = \Phi\Lambda\ldots\Lambda\partial^{m-1}\Phi\Lambda\partial^{m+1}\Phi\Lambda\bar\partial\Phi\Lambda\ldots\Lambda\bar\partial^{m-1}\Phi$ by the fact that ϕ is harmonic if and only if $\partial\bar\partial\Phi = \mu\Phi$ for some smooth function $\mu : U \to R$, see for example [1]. So $\partial F = \Phi\Lambda\ldots\Lambda\partial^{m-1}\Phi\Lambda$ $(\sum_{k=1}^{m} \lambda_k\ \partial^k\Phi)\Lambda\bar\partial\Phi\Lambda\ldots\ \Lambda\bar\partial^{m-1}\Phi = \lambda_m F$ at some points where $F \neq 0$ by the Q-isotropy of ϕ where $\lambda_k : U \to \mathbb{C}$ is smooth. Thus $\bar\partial(*F) = \bar\lambda_m(*F)$.

Now let $\Pi : \mathbb{C}^{2m+1}\setminus 0 \to \mathbb{C}P^{2m}$ be the natural projection. Set $f_u = \Pi \circ (*F)$ then f_u is defined on $U\setminus\Omega$ where $\Omega=\{z\in U: F_u(z) = 0\}$

LEMMA 1.2: $f = f_u$ *is holomorphic on* $U\setminus\Omega$.

Proof. Consider $\partial f/\partial\bar z = (\partial\Pi/\partial *F)(\partial *F/\partial\bar z) + (\partial\Pi/\partial *\bar F)(\partial *\bar F/\partial\bar z)$. We see that $\partial\Pi/\partial(*\bar F) = 0$ since Π is holomorphic and $(\partial\Pi/\partial *F)(\partial *F/\partial\bar z) = (\partial\Pi/\partial *F)(\lambda *F) = 0$ by the definition of Π. So $\partial f/\partial\bar z = 0$ i.e. f is holomorphic.

LEMMA 1.3: f *can be extended to holomorphic map on the whole of* U.

Proof: Observe that $F(p) = F_u(p) = 0$ if and only if $\sigma(p) = (\partial\Phi\Lambda\ldots\Lambda\partial^m\Phi)(p) = 0$ by the Q-isotropy of ϕ. On the other hand σ is a smooth section of the bundle $\beta = \Lambda^m\phi^{-1}T^{\mathbb{C}}H^{2m}$ over U. Now let ∇, $D, \mathscr{D}$ be the connections on $M\times\mathbb{C}^{2m+1}$, $\phi^{-1}T^{\mathbb{C}}H^{2m}$, β respectively with ∇ is a trivial connection, D induced by ∇ and $\mathscr{D}$ induced by D. Let ∇', ∇'' denote ∇_Z, $\nabla_{\bar{Z}}$ respectively where $Z=\partial/\partial z$, $\bar{Z}=\partial/\partial\bar{z}\in T^{\mathbb{C}}M$ (similarly for D', D'', $\mathscr{D}'$, $\mathscr{D}''$). Now consider $\mathscr{D}''\sigma=[(D''\partial\Phi)\Lambda\ldots\Lambda\partial^m\phi]+\ldots+[\partial\Phi\Lambda\ldots\Lambda(D''\partial^m\Phi)]=[pr_Q(\nabla''\partial\Phi)\Lambda\ldots\Lambda\partial^m\Phi]+\ldots+[\partial\Phi\Lambda\ldots\Lambda pr_Q(\nabla''\partial^m\Phi)]= 0$

since $pr_Q(\nabla''\partial^k\Phi) = pr_Q(\bar{\partial}\partial^k\Phi) = \sum_{r=1}^{k-1} a_r\partial^r\Phi$ where $a_r : U \to \mathbb{C}$, $k = 1,\ldots,m$, pr_Q = projection on to subbundle $\phi^{-1}T^{\mathbb{C}}H^{2m}$ with respect to metric Q. This proves that σ is a holomorphic section of β. By the similar argument one can show that $\rho = \bar{\partial}\Phi\Lambda\ldots\Lambda\partial\Phi^{m-1}$ is an anti-holomorphic section of $\Lambda^{m-1}\phi^{-1}T^{\mathbb{C}}H^{2m}$. Let $p\in U$ be a zero of F. W.l.o.g. assume the complex coordinate z is centred on p, and that U is so small that there are no other zeros in U. Then we have $F(z) = (\Phi\Lambda\sigma\Lambda\rho)(z) = \Phi(z)\Lambda z^k\sigma_1(z)\Lambda\bar{z}^r\rho_1(z)$ for some integers r,k and $\rho_1 \neq 0, \sigma_1 \neq 0$ on some neighbourhood V of p. Set $\theta = \Phi\Lambda\sigma_1\Lambda\rho_1$ and $f_1 = \Pi\circ(*\theta)$. Now observe that $f_1 = f$ on $V\setminus\{p\}$ and f_1 is continuous at p. Thus f_1 is holomorphic on V. Setting $f(p) = f_1(p)$ gives desired result.

LEMMA 1.4: *Let* f_u, f_v *be defined by using charts* (U,z), (V,w) *with* $U \cap V \neq \emptyset$ respectively. Then $f_u = f_v$ *on* $U \cap V$.

Proof. Observe that $\partial^k \Phi_u / \partial z^k = (\partial^k \Phi_v / \partial w^k)(dw/dz)^k$ + terms in $(\partial^r \Phi_v / \partial w^r)$ with $r < k$. Therefore we get $F_u = (F_v)(dw/dz)^m |dw/dz|^q$ where $q = 1+2+\ldots+(m-1)$. Then the result follows.

This lemma allow us to define $\not f : M \to \mathbb{C}P^{2m}$ globally such that $\not f(p) = f_u(p)$ for some local coordinate chart (U,z).

Proposition 1.5: $\not f$ is totally Q-isotropic i.e. for a local lift η of $\not f$, $Q(\partial^k \eta, \partial^k \eta) = 0$; $0 \leq k \leq m-1$.

Proof: We shall frequently make use of the isometry $\delta : H^{2m} \to H^{2m}$ defined by $\delta((x_1,\ldots,x_{2m},r)) = (-x_1,\ldots,-x_{2m},r)$. Set $\tilde{\Phi} = \delta \circ \Phi$ and $\nu(p) = \text{span}\{\bar{\partial}\tilde{\Phi}(p) \ldots, \bar{\partial}^m \tilde{\Phi}(p)\}$. W.l.o.g. we work at points where $F \neq 0$. Observe that $\forall a,b \in \nu$ we have $Q(a,b) = 0$ and therefore to get the result it is sufficient to prove that $\partial^k *F \in \nu$ for $0 \leq k \leq m-1$. For let $\varepsilon = \Phi\Lambda\ldots\Lambda\partial^m\Phi$ then $\bar{\partial}^k F = (\varepsilon\Lambda\bar{\partial}\Phi\Lambda \ldots$ $\ldots\Lambda\bar{\partial}^{m-k-1}\Phi\Lambda\bar{\partial}^{m-k+1}\Phi\Lambda\ldots\Lambda\partial\ldots\Lambda\bar{\partial}^m\Phi)$ + terms in $(\varepsilon\Lambda\bar{\partial}\Phi\Lambda\ldots\Lambda\bar{\partial}^{m-k}\Phi\Lambda\bar{\partial}^{r_2}\Phi\Lambda\ldots$ $\ldots\Lambda\bar{\partial}^{r_k}\Phi)$ where $\{r_2,\ldots,r_k\} \subseteq \{3,\ldots,(2m-2)\}$ with $r_2<\ldots<r_k$ and $k = 1,\ldots,(m-1)$. (Convention: $A\Lambda\partial^0 A = A$, $A\Lambda\partial A\Lambda\ldots\Lambda\partial^r A\Lambda\ldots\Lambda\partial^m A =$ $= A\Lambda\partial A\Lambda\ldots\Lambda\partial^r A$ if $r \geq m$). Now consider the case where $k = 0$ i.e. $\bar{\partial}^0 F = F = (\varepsilon\Lambda\bar{\partial}\Phi\Lambda\ldots\Lambda\bar{\partial}^{m-1}\Phi)$. Observe that $\langle *\bar{F}, \partial^s\Phi\rangle = 0$; $s=0,\ldots,m$. So $Q(*F, \partial^s\tilde{\Phi}) = 0$. On the other hand $\mathbb{C}^{2m+1} = \nu(p) \oplus \bar{\nu}(p) \oplus \mathbb{C}\,\tilde{\Phi}(p)$. Therefore $*F \in \nu$. If $k \neq 0$, by the same argument one gets $\partial^k *F =$

$= \sum_{r=1}^{n} A_r; A_r \in \nu$ for some integer n. So $\partial^k * F \in \nu$.

REMARK 1.6: 1) $\mathcal{f}_{m-1} = \mathrm{span}(*F, \partial * F, \ldots, \partial^{m-1} * F) = \mathrm{span}\{\bar\partial\tilde\Phi, \ldots, \bar\partial^m\tilde\Phi\}$.
So $\mathcal{f}_{m-1} + \bar{\mathcal{f}}_{m-1} = T^c_{\tilde\Phi}H^{2m}$. *Since* Q *is positive definite on* $T^c_{\tilde\Phi}H^{2m}$, *we have that* Q *is positive definite on* $\mathcal{f}_{m-1} + \bar{\mathcal{f}}_{m-1}$.

2) *Note that* $\mathcal{f}$ *is full by the construction of* $\mathcal{f}$ *and fullness of* ϕ.

Section 2:

In the last section we manufactured a holomorphic map $\mathcal{f} : M \to \mathbb{C}P^{2m}$ from a harmonic one $\phi : M \to H^{2m}$ under certain conditions. Now we shall produce a harmonic map from the holomorphic one.

Let $Z : M \to \mathbb{C}P^{2m}$ be a full totally Q-isotropic holomorphic map such that Q is positive definite on $Z_{m-1}(p)$ for every $p \in M$ and $\eta : U \to \mathbb{C}^N \setminus 0$ be the local holomorphic lift of Z. Where U is an open subset of M. Set

$$G = G^\eta_u = \eta \wedge \ldots \eta^{m-1} \wedge \bar\eta \wedge \ldots \wedge \bar\eta^{m-1} \quad \text{where } \eta^k = \partial^k \eta \text{ and } \bar\eta^k = \bar\partial^k \bar\eta .$$

Again we shall work on the dense open set of points where $G^\eta_U \neq 0$.

LEMMA 2.1: 1) $*G$ *is real if* m *is even, pure imaginary if* m *is odd.*

2) $Q(*G, *\bar G) < 0$.

Proof. 1) Obvious.

2) Choose a Q-orthonormal basis $e_1, \ldots, e_{2m}$ for $Z_{m-1} + \bar Z_{m-1}$

and set $\omega = \sum_{k=1}^{2m} Q(b,\bar{e}_k)e_k$ where $b = (0,\ldots,0,1) \in \mathbb{C}^{2m+1}$ and $\psi = b-\omega$. One can see that $*G = \lambda\widetilde{\psi}$, $\lambda : U \to \mathbb{C}\setminus 0$ and $Q(\widetilde{\psi},\overline{\widetilde{\psi}}) < Q$. Therefore $Q(*G,*\bar{G}) < 0$.

For $*G = (h_1,\ldots,h_{2m},h)$, set $\chi_u = \chi_u^\eta = a(*G/\alpha)$ where $a = \{1$ if m is even and $h > 0$, -1 if m is even $h < 0$, i if m is odd and $ih > 0$, $-i$ if m is odd and $ih < 0\}$ and $\alpha = (-Q(*G,*\bar{G}))^{\frac{1}{2}}$.

We see that $Q(\chi_u,\chi_u) = -1$ therefore the map $\chi_u : U\setminus\Omega_1 \to \mathbb{R}^{2m+1}$ where $\Omega_1 = \{x\in U : G(x) = 0\}$ has image in H^{2m}.

LEMMA 2.2: χ_u *is independent of the choice of the local lifts used to construct it.*

Proof. Let ξ be another lift, then away from zeros of ξ and η, $\xi = \mu\eta$ where $\mu : U \to \mathbb{C}\setminus 0$. Therefore $\xi^k = \mu\eta^k$ + terms in η^r, $r < k$; hence $G_u^\xi = |\mu|^{2m} G_u^\eta$. So we get $\chi_u^\xi = \chi_u^\eta$.

LEMMA 2.3: χ_u *may be extended uniquely to a smooth map on* U.

Proof: Let $G(p) = 0$ for some $p\in U$ with its local lift η.W.l.o.g assume that the complex coordinate z is centered on p, and that U is so small that there are no other zeros in U. Set $\tau = \eta\wedge\ldots\wedge\eta^{m-1}$. By the argument used in Lemma 1.3, τ is a holomorphic section of the bundle $U\times\Lambda^m\mathbb{C}^{2m+1}$. Therefore $G(z) = z^r\tau_1(z)\wedge\bar{z}^r\overline{\tau_1(z)}$ where $\tau_1(0)\neq 0$ and $r \in \mathbb{Z}$. So $G(z) = |z|^{2r}\tau_1(z)\wedge\overline{\tau_1(z)}$. Define $\chi_u(0) = a(\tau_2(0)/\beta)$ where $\tau_2 = \tau_1 \wedge \bar{\tau}_1$, and $\beta = (-Q(\tau_2(0),\overline{\tau_2(0)}))^{\frac{1}{2}}$ then $Q(\chi_u(0),\chi_u(0)) = -1$.

Proposition 2.4: Let $\Gamma_u : U \to H^{2m}$ defined as $\chi_u(p) = (j \circ F_U)(p)$ where $j : H^{2m} \hookrightarrow R^{2m+1} \hookrightarrow \mathbb{C}^{2m+1}$ is the inclusion map then Γ_u is a harmonic map.

Proof: Observe that $\langle \partial\bar{\partial}\chi_u, \eta^k \rangle = 0,\ 0 \leq k \leq m-1$ if and only if

$\partial\bar{\partial}\chi_u \perp Z_{m-1} + \bar{Z}_{m-1}$ if and only if $\partial\bar{\partial}\chi_u = \lambda\chi_u$ where $\lambda : U \to R$

is smooth. Consider

$$\partial\bar{\partial}\chi_u = a\{\partial\bar{\partial}(1/\alpha)\,\theta + \partial(1/\alpha)\bar{\partial}\theta + \bar{\partial}(1/\alpha)\bar{\partial}\theta + (1/\alpha)\bar{\partial}\partial\theta\} \quad \text{where}$$

$\theta = *G$ and $\alpha = (-Q(\theta,\theta))^{\frac{1}{2}}$. We see that $\langle \bar{\partial}\partial\chi_u, \eta^k \rangle = 0,\ 0 \leq k \leq m-2$. Thus it remains to show that $\langle \bar{\partial}\partial\chi_u, \eta^{m-1} \rangle = 0$, where $\eta^k = \partial^k \eta$.

But

$$\langle \bar{\partial}\partial\chi_u, \eta^{m-1} \rangle = a\bar{\partial}\ \{(1/\alpha)\ \langle \partial\theta, \eta^{m-1} \rangle \ldots\ 1° \quad \text{and}$$

$$\langle \partial\theta, \eta^{m-1} \rangle = *((-1)^{m-1}\eta\Lambda\ldots\Lambda\eta^m\Lambda\ \bar{\eta}\Lambda\ \ldots\Lambda\bar{\eta}^{m-1}) \in \mathbb{C}\ \ldots\ 2°.$$

On the other hand $\eta^m \in \text{span}\{\tilde{\chi}_u, Z_{m-1}$ since $Q(\eta^m, \bar{\gamma}) = 0$ for every $\gamma \in \bar{Z}_{m-1}$ by totally Q-isotropy of Z and positive definiteness of Q on $\bar{Z}_{m-1}$ for all $p \in M$. So we have

$$\eta^m = \gamma - Q(\tilde{\chi}_u, \eta^m)\tilde{\chi}_u \ \ldots \qquad (3°)$$

for some $\gamma \in Z_{m-1}$. By replacing η^m with 3° in 2° we get

$$<\partial\theta, \eta^{m-1}> = (-1)^{2m+1} Q(\tilde{\chi}_u, \eta^m)(*(\eta\Lambda\ldots\Lambda\eta^{m-1}\Lambda\bar{\eta}\Lambda\ldots\Lambda\bar{\eta}^{m-1}\Lambda\tilde{\chi}_u)) =$$

$$-Q(\tilde{\chi}_u, \eta^m) <\theta, \tilde{\chi}_u> = -Q(\tilde{\chi}_u, \eta^m) aQ(\theta,\theta)/\alpha \ \ldots \ 4° .$$

By substituting 4° in 1° we get

$$<\bar{\partial}\partial\chi_u, \eta^{m-1}> = a^2\bar{\partial}Q(\tilde{\chi}_u, \eta^m) \ \ldots \ 5°$$

But we have by 3° that $Q(\eta^m, \eta^m) = Q(\tilde{\chi}_u, \eta^m)^2$ and so

$$\bar{\partial}Q(\eta^m, \eta^m) = \bar{\partial}Q(\chi_u, \eta^m)^2 = 0 \quad \text{by the}$$

holomorphicity of η. That is to say, $<\partial\bar{\partial}\chi_u, \eta^{m-1}> = 0$. Hence the result follows.

LEMMA 2.5: *Let* χ_u *(respectively* χ_v*) be defined by using charts* (U,z) *(respectively* (V,w) *with* $U \cap V \neq \emptyset$ *then* $\chi_u = \chi_v$ *on* $U \cap V$.

Proof. Let η_u, η_v be the local lifts expressed by using charts (U,z), (V,w) respectively. Then $\eta_v^k = \eta_u^k (dz/dw)^k$ + terms in η_u^r with $r < k$. So $G_v = G_u |dz/dw|^q$ where $q = 1+2+\ldots+(m-1)$, hence

$\chi_u = \chi_v$ on $U \cap V$.

This Lemma allows us to define χ on M globally by $\chi(p) = \chi_u(p)$ for some neighbourhood U of p. Hence $\Gamma : M \to H^{2m}$ given by $\chi = j \circ \Gamma$ is defined on the whole of M and harmonic, where $j : H^{2m} \to R^N$ is the inclusion map.

REMARK 2.6: 1) Γ *is* Q-*isotropic* i.e. $Q(\partial^s \chi, \partial^k \chi) = 0$ *for all* $s+k>0$. This can be shown by using a similar argument to that used in proposition (1.5).

2) Γ is full. This follows from the fact that Z is full.

Now let $\mathscr{A}$ be the set of all full totally Q-isotropic holomorphic maps $Z : M \to \mathbb{C}P^{2m}$ such that Q is positive definite on $Z_{m-1}(p)$ for all $p \in M$ and τ be the set of all full Q-isotropic harmonic maps $\phi : M \to H^n$. Then consider the transformation $\nu : \mathscr{A} \to \tau$ and $g : \tau \to \mathscr{A}$ where $\nu(\phi) = \mathfrak{f}$ and $g(Z) = \Gamma$ are obtained by the processes described in section (1) and (2) respectively.

THEOREM 2.7: *The transformations* ν *and* g *defined above are inverse and thus give a* 1 : 1 *correspondence between* $\mathscr{A}$ *and* τ

Proof: All we need to prove is that: a°) g is a left inverse of ν, hence g is surjective. b°) g is injective.

a°) Let $\mathfrak{f} = \nu(\phi)$ with its local lift $\eta = *N$ where $N = \Phi \wedge \ldots \wedge \partial^m \Phi \wedge \bar{\partial} \Phi \wedge \ldots \wedge \bar{\partial}^{m-1} \Phi$. We see that, as in the proof of proposition

(1.5), $\Phi \perp \eta^k$, $0 \le k \le m-1$. But $\chi \perp \eta^k$ if $g(\not f) = \Gamma$. So $\chi \| \Phi$. By the fact that $Q(\Phi,\Phi) = Q(\chi,\chi)$ and their last component have the same sign, we get $\phi = \Gamma$.

b°) Suppose that $\not f, \Xi \in \tau$ with $g(\not f) = g(\Xi) = \Gamma$.

Let η,ξ be local holomorphic lifts of $\not f,\Xi$ respectively. Choose a lift on an open set $U \subset M$ such that, at all points of U, $\xi(p)\wedge\ldots\wedge\xi^{m-1}(p) \neq 0$. $g(\not f) = g(\Xi)$ implies that η is in the $\Xi_{m-1}+\bar{\Xi}_{m-1}$ so

$$\eta = \sum_{k=0}^{m-1} \alpha_k \xi^k + \sum_{k=0}^{m-1} \beta_k \bar{\xi}^k \quad \ldots \; 1° .$$

By the fact that $\bar{\partial}\eta = 0$, $\xi^m = (\sum_0^{m-1} \mu_k \bar{\xi}^k) + Q(\bar{\xi}^m,\tilde{\chi})\tilde{\chi}$ and $\xi,\ldots,\xi^{m-1},\bar{\xi},\ldots,\bar{\xi}^{m-1},\tilde{\chi}$ are linearly independent we get

$$\left.\begin{array}{ll} \bar{\partial}\alpha_k = 0 & \text{for } 0 \le k \le m-1 \\ \bar{\partial}\beta_o + \mu_o\beta_{m-1} = 0 & \\ \bar{\partial}\beta_k + \beta_{k-1} + \mu_k\beta_{m-1} = 0 & \text{for } 1 \le k \le m-1 \\ \beta_{m-1}Q(\bar{\xi}^m,\tilde{\chi}) = 0. & \end{array}\right\} \ldots \; 2°$$

As it is shown in proposition (2.4), $Q(\bar{\xi}^m,\tilde{\chi})$ is holomorphic so it has only isolated zeros. [$Q(\bar{\xi}^m,\tilde{\chi})$ cannot be identically zero by the fullness of Ξ]. So we have

$$\beta_{m-1} = 0 \quad \ldots \; 3°.$$

Equations 2° and 3° give us $\beta_o = \ldots = \beta_{m-1} = 0$.

Hence 1° becomes

$$\eta = \sum_{0}^{m-1} \alpha_k \xi^k \quad \ldots \; 4^\circ .$$

But $Q(\partial\eta,\partial\eta) = \alpha_{m-1}^2 Q(\xi^m,\xi^m) = 0$. This implies that $\alpha_{m-1} = 0$. So equation 4° becomes $\eta = \sum_{0}^{m-2} \alpha_k \xi^k$. Repeating this process we get $\alpha_{m-1} = \ldots = \alpha_{m-j} = 0 \quad 1 \leq j \leq m-1$. Therefore $\eta = \alpha_0 \xi$. This proves $\not= \Xi$ thus g is injective. This completes the proof.

A special case:

Note that if $m = 1$, (H^{2m},Q) may be regarded as a Hermitian manifold of complex dimension one. Let $\emptyset : M \to H^2$ be a smooth map from a connected Riemann surface M. Then we may decompose the $\mathbb{C}$-linear extension

$$D^c : T^cM \times C(\emptyset^{-1}T^cH^2) \to C(\emptyset^{-1}T^cH^2)$$

of the connection D on $\emptyset^{-1}TH^2$ into connections:

$$D' : C(\emptyset^{-1}T^cH^2) \overset{\rho}{\to} T'M \times C(\emptyset^{-1}T^cH^2) \xrightarrow{D^c} C(\emptyset^{-1}T^cH^2) \overset{\Pi}{\to} C(\emptyset^{-1}T'H^2)$$

$$D'' : C(\emptyset^{-1}T^cH^2) \overset{\mu}{\to} T''M \times C(\emptyset^{-1}T^cH^2) \xrightarrow{D^c} C(\emptyset^{-1}T^cH^2) \overset{\Pi}{\to} C(\emptyset^{-1}T'H^2)$$

where $\rho(\sigma) = (\partial/\partial z,\sigma), \mu(\sigma) = (\partial/\partial\bar{z},\sigma)$ and Π is the projection along $\emptyset^{-1}T''H^2$. Here $\sigma \in C(\emptyset^{-1}T^cH^2)$.

Now set $\Phi = j \circ \emptyset$ where $j : H^2 \to R^3 \to \mathbb{C}^3$ is the inclusion map. Let $\tilde{Q}$ be the associated Hermitian form on $\emptyset^{-1}T^cH^2$, i.e.

$\tilde{Q}(K,L) = Q'(K,\bar{L})$ for $K,L \in C(\emptyset^{-1}T^cH^2)$ where Q' is induced from the metric Q on T^cH^2. Write $D'\emptyset = \Pi(\partial\Phi)$, $D''\emptyset = \Pi(\bar{\partial}\Phi)$.

Definition: [8]. The map $\emptyset$ said to be *complex* $\tilde{Q}$-*isotropic* if $\tilde{Q}(D'^k\emptyset, D''^r\emptyset) = 0$ for $k,r \geq 1$.

Now we have

(2.8) The following statements are equivalent

a) $\emptyset$ is Q-isotropic

b) $\emptyset$ is complex $\tilde{Q}$-isotropic

c) $\emptyset$ is $\pm$ holomorphic and therefore harmonic.

Proof. Assume (a), one can show that

$$Q(\partial^k\Phi, \partial^k\Phi) = 2\tilde{Q}(D'^k\emptyset, D''^k\emptyset) = 0 \quad \text{for all} \quad k \geq 1 \ldots\ldots (1°)$$

by the Q-isotropy of $\emptyset$. So it easily follows from (1°) that $\tilde{Q}(D'^k\emptyset, D''^r\emptyset) = 0$ for all $k,r \geq 1$, hence (b) holds.

Assume (b). We first show

$$Q(\partial^k\Phi, \Phi = 0 \quad \text{for all } k \geq 1 \ldots\ldots (2°).$$

To do this, we use induction on k. Note that $Q(\partial\Phi,\Phi) = 0$, so suppose now that it is true for $1 \leq k \leq \beta$ for some $\beta \geq 1$ then $(D^c_{\partial/\partial z})^k\ \emptyset = \partial^k\Phi$. On the other hand

$$0 = \partial Q(\partial^\beta\Phi,\Phi) = Q(\partial^{\beta+1}\Phi,\Phi) + Q(\partial^\beta\Phi,\partial\Phi) \ldots\ldots (3°).$$

But $Q(\partial^{\beta}\Phi, \partial\Phi) = \tilde{Q}(D'^{\beta}\emptyset, D''\emptyset) + \tilde{Q}(D''^{\beta}\emptyset, D'\emptyset) = 0$ by (b).
Therefore using (3°) we see that $Q(\partial^{\beta+1}\Phi, \Phi) = 0$, so (2°) follows.
Thus $\partial^{k}\Phi = (D^{c}_{\partial/\partial z})^{k}\ \emptyset$ for all $k \geq 1$. By complex $\tilde{Q}$-isotropy of $\emptyset$, this implies Q-isotropy of $\emptyset$. Hence (a) holds.

To show the equivalence of (b) and (c), it is enough to show that (b) implies (c) since (b) follows from (c) immediately. Observe that $\tilde{Q}(D'\emptyset, D''\emptyset) = 0$ by the complex Q-isotropy. So at each point of M, either $D'\emptyset = 0$ or $D''\emptyset = 0$ since $\emptyset^{-1}T'H^{2}$ is of complex dimension one. It follows that either $D'\emptyset$ or $D''\emptyset$ is identically zero on M since M is connected.

This gives (c).

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