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ON SOME SEMIGROUPS WITHOUT INCREASING ELEMENTS

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Sommario. *In questo lavoro si studia (§1) il legame esistente tra la decomposizione di* Szép *e le relazioni di* Green.

Si descrive (§2) la Γ*-decomposizione di un semigruppo di tipo* T*-finito. Infine (§3) si determinano alcuni teoremi relativi ai semigruppi nucleari sinistri, generalizzando risultati precedenti.*

INTRODUCTION. In [12] J. Szép introduced a particular decomposition, $D_L(S)$, of a semigroup S and used it in the study of the structure of a finite semigroup.

Afterwards, F. Migliorini and J. Szép [7], B. Piochi [10], R. Scozzafava [11] and the present authors [3] have studied classes of also infinite semigroups by using such a decomposition. F. Migliorini and J. Szép, in [8], introduce the Γ-decomposition of S, Γ(S), which is a refinement of Szép's decomposition.

In this work we continue the study of such decompositions, in particular for some semigroups, which are without left increasing elements.

In section 1 we determine the connexion between the decomposition $D_L(S)$ and Green's relations on S and we prove that a sufficient condition for every component S_i of the decomposition $D_L(S)$

(i=0,2,4,5) to be a union of $\mathcal{L}$-classes is for the semigroup S to be without left increasing elements (*). This condition is necessary if S is regular or if $S_5 \neq \emptyset$.

In section 2 we determine the Γ-decomposition of the semigroups of T-finite type and prove that the groupbound semigroups belong to this class.

In [5] one defines the condition P_R and P_L, and proves that a semigroup is g.b. if and only if satisfies $P_L \wedge P_R$, and that every left separative semigroup that satisfies P_R is a disjoint union of groups. We prove also that every left separative semigroup S that satifies P_R is an orthogroup with E(S) left regular band.

In section 3, we note that every left kernel semigroup is without left increasing elements, as for semigroups of T-finite type, and we extend a theorem of Szép, on the existence of at most a maximal left kernel subsemigroup in a finite semigroup, to the case of a infinite semigroup. Moreover, for an arbitrary semigroup, we prove the existence of at most a maximal left kernel subsemigroup generated by g.b. elements.

We assume the reader to be familiar with the standard notation of semigroup theory.

We would like to express our gratitudine to professor F. Migliorini for his useful hints in the preparation of this work.

(*) An element a of S is called *left* [*right*] *increasing* if $\exists T \subset S \ni' aT = S$ $[Ta = S]$

1. - Let S be a semigroup without annihilators different from zero. Recall the disjoint decomposition of S introduced by Szép in [12]:

$$D_L(S) = \{S_0, S_1, S_2, S_3, S_4, S_5\}$$

$$D_R(S) = \{D_0, D_1, D_2, D_3, D_4, D_5\}$$

where

$$S_0 = \{a \in S / \ aS \subset S \wedge \exists x \in S,\ x \neq 0 \ \ni' \ ax = 0\}$$

$$S_1 = \{a \in S / \ aS = S \wedge \exists y \in S,\ y \neq 0 \ \ni' \ ay = 0\}$$

$$S_2 = \{a \in S - (S_0 \cup S_1) / \ aS \subset S \wedge \exists x_1, x_2 \in S,\ x_1 \neq x_2 \ \ni' \ ax_1 = ax_2\}$$

$$S_3 = \{a \in S - (S_0 \cup S_1) / \ aS = S \wedge \exists y_1, y_2 \in S,\ y_1 \neq y_2 \ \ni' \ ay_1 = ay_2\}$$

$$S_4 = \{a \in S - (S_0 \cup S_1 \cup S_2 \cup S_3) / \ aS \subset S\}$$

$$S_5 = \{a \in S - (S_0 \cup S_1 \cup S_2 \cup S_3) / \ aS = S\}$$

and where D_i $(i=0,1,\ldots,5)$ is defined in a similar manner, and multiplication by the element a is on the right rather than on the left. The subsets S_i and D_i $(i=0,\ldots,5)$, if not empty, are subsemigroups of S.

Recall, furthermore, the decomposition $\Gamma(S) = \{C_{ij}\}_{i,j}$, where $C_{ij} = S_i \cap D_j$ $(i,j=0,\ldots,5)$ which was introduced by F. Migliorini and J. Szép in [8]. The subsets C_{ij}, if they are not empty, are subsemigroups of S.

Connexions between these decompositions and those derived from

Green's relations $\mathcal{L}, \mathcal{R}, \mathcal{H}$ have been studied in the works of B.Piochi [10] and F. Migliorini [6] for some types of semigroups. The following theorem extends those connexions to the case of any semigroups.

THEOREM 1.1. *Let* S *be a semigroup and let* $a\in S$, *then:*

$$a\in S_0 \cup S_1 \Rightarrow L_a \subseteq S_0 \cup S_1 \quad [a\in D_0 \cup D_1 \Rightarrow R_a \subseteq D_0 \cup D_1]$$

$$a\in S_2 \cup S_3 \Rightarrow L_a \subseteq S_2 \cup S_3 \quad [a\in D_2 \cup D_3 \Rightarrow R_a \subseteq D_2 \cup D_3]$$

$$a\in S_4 \cup S_5 \Rightarrow L_a \subseteq S_4 \cup S_5 \quad [a\in D_4 \cup D_5 \Rightarrow R_a \subseteq D_4 \cup D_5] .$$

Proof. Let $a\in S_0 \cup S_1$ and let $b\in L_a$, then there exists an $x\in S'$ such that $b=xa$ and there exists a $y\in S$, $y\neq 0$, such that $ay=0$ and hence $by=xay=x0=0$, therefore $b\in S_0 \cup S_1$. Let $a\in S_2 \cup S_3$ and $b\in L_a$, then there exist an $x\in S^1$ such that $b=xa$ and $y_1,y_2\in S$, $y_1\neq y_2$ such that $ay_1= ay_2$ so that $by_1 = xay_1 =xay_2 = by_2$.

Moreover $b \notin S_0 \cup S_1$, since, otherwise, $L_b \subseteq S_0 \cup S_1$: hence $a\in S_0 \cup S_1$. Thus $b\in S_2 \cup S_3$. If $a\in S_4 \cup S_5$, then $L_a \subseteq S_4 \cup S_5$.

Indeed, if $b \in L_a \cap (S-(S_4 \cup S_5))$, then $L_b \subseteq S-(S_4 \cup S_5)$ and hence $a\in S-(S_4 \cup S_5)$, against the assumption that $a\in S_4 \cup S_5$.

Similarly for the $\mathcal{R}$-classes.

It is, in general, not true that $a\in S_i$ $[D_i]$ implies $L_a \subseteq S_i$ $[R_a \subseteq D_i]$ $(i=0,\dots,5)$; indeed, if S is a regular semigroup and $a\in S_3$ and $x\in V(a)$, then $xa\in L_a$, but $xa\notin S_3$ because xa is an idempotent element of S.

However the following theorems hold:

THEOREM 1.2. *If S is a semigroup, then:*

$$S_1 = S_3 = \emptyset \Rightarrow \forall a \in S_i : L_a \subseteq S_i, \quad i = 0,2,4,5 .$$

Proof. The implication for i = 0,2 is an obvious consequence of Theorem 1.1. If $a \in S_5$ and $b \in L_a$, then there exists $y \in S^1$ such that $a=yb$.

Let us assume, *ab absurdo*, that $b \in S_4$, and then

$$S = aS = ybS \subseteq yS:$$

therefore $y \in S_5$, so that $a = yb \in S_5S_4 \subseteq S_4$, which contradicts the assumption. It follows from Theorem 1.1 and from what has just been proved that if $a \in S_4$ then $L_a \subseteq S_4$.

THEOREM 1.3. *If S is a semigroup and $S_5 \neq \emptyset$, then*

$$L_a \subseteq S_5 \quad \forall a \in S_5 \Rightarrow S_1 = S_3 = \emptyset .$$

Proof. Assume $S_1 \cup S_3 \neq \emptyset$ and let $a \in S_1 \cup S_3$ and $e = e^2 \in S_5$, then there exists an $x \in S$ such that $ax=e$. As $e \mathscr{L} xe$, then $xe \in S_5$; therefore $e=e^2=axe \in (S_1 \cup S_3)S_5 \subseteq S_1 \cup S_3$ which is a contradiction, since $S_1 \cup S_3$ contains no idempotent element.

COROLLARY 1.4. *If S is a semigroup and $S_5 \neq \emptyset$, then:*

$$S_4 = \emptyset \Rightarrow S_1 = S_3 = \emptyset .$$

THEOREM 1.5. *If* S *is a regular semigroup, then:*

$$S_1 = S_3 = \emptyset \iff \forall a \in S_i : L_a \subseteq S_i, \quad i=0,2,4,5 .$$

Proof. The condition is necessary by Theorem 1.2.

Let us prove that it is also sufficient.

Assume, *ab absurdo*, $S_1 \cup S_3 \neq \emptyset$ and let $s \in S_1 \cup S_3$ and $x \in V(s)$; then $sx \mathscr{L} x$. But $x \in S_4$ [see [7], Th.1.4] and $sx \notin S_4$, against the hypothesis.

2. - A *semigroup* S, without annihilators different from zero, is said to be *of T-finite type* (*) if $D_L(S) = \{S_0, S_2, S_5\}$, $D_R(S) = \{D_0, D_2, D_5\}$.

J. Szép in [12], R. Scozzafava in [11], F. Migliorini and J.Szép in [7], B. Piochi in [10], and the present authors in [3] have singled out several classes of semigroups of T-finite type. In the sequel we shall give a wide class of semigroups of T-finite type.

A semigroup S is said to satisfy the condition P_L [P_R] if for each $x \in S$ there exists a positive integer n such that

$$Sx^n = Sx^{n+i} \quad [x^nS = x^{n+i}S]$$

for all $i \in N$.

It is proved in [5], Th. 1.1, that a semigroup fulfils $P_R \wedge P_L$

(*) We choosed this word because the traslations of this semigroup are of finite type, indeed they are injective if and only if they are surjective.

if and only if at least a power of an arbitrary element of S is in a subgroup of S.

Such a *semigroup* is called *groupbound* (g.b.).

Periodic semigroups and completely regular semigroups provide examples of g.b. semigroups.

THEOREM 2.1. *Every* g.b. *semigroup is of T-finite type.*

Proof. Let $a \in S_1 \cup S_3$. As S is a g.b. semigroup, then there exists an $e \in E(S)$ and $n \in N$ such that $a^n \in H_e$; therefore the completely regular element a^n is left magnifying for S against theorem 1.11 of [2]. Let $a \in S_4$, $a^n \in H_e$ (for some $e \in E(S)$ and $n \in N$). Because of theorem 1.4 of [7], every inverse of the completely regular element a^n is in $S_1 \cup S_3$, which contradicts the first part of the proof.

Similary one proves that $D_1 = D_3 = D_4 = \emptyset$.

COROLLARY 2.2. *Let* S *be a* g.b. *semigroup and let* $D_L(S) = \{S_0, S_2, S_5\}$, $D_R(S) = \{D_0, D_2, D_5\}$ *and* $\Gamma(S) = \{C_{ij}\}_{i,j=0,2,5}$, *then all non-empty* S_i, D_i *and* C_{ij} $(i,j=0,2,5)$ *are* g.b. *semigroups.*

Recall [5] *that every left separative semigroup that satisfies* P_R *is a disjoint union of groups. Furthermore, the following theorem holds.*

THEOREM 2.3. *A left separative semigroup* S *that satifies* P_R *is an orthogroup* (i.e. S *is a completely regular semigroup, with* E(S) *a subsemigroup), with* E(S) *a left regular band.*

Proof. Let S be a left separative semigroup that satisfies P_R; by Th. 1.2 of [5], S is completely regular.

Let $a,x,y \in S$ be such that $a = axa$, $a = aya$, $ay = ya$, then $a(xa) = a(ya)$ implies $xa = ya$, since S is left separative. Thus

$$ya^2 = (ya)a = (ay)a = a$$

$$a = axa = ax(ya^2) = ax^2a^2$$

and, because of Th. 4 of [9], S is an orthogroup.

Let $a,x,z \in S$ be such that $a = aza$, then it follows from $ax=(aza)x$ that $xa = zaxa$, so that $Saxa = Sxa$ and hence, by Th. 9 of [1], E(S) is a left regular band.

Corollary 1.3 of [5] follows from Th. 2.3 and its dual.

In the following theorem we determine the Γ-decomposition of the semigroups of T-finite type.

THEOREM 2.4. *Let* S *be a semigroup of* T-*finite type; then the following holds:*

i) *if* $1 \in S$, *then* $S_5 = D_5 = G$ (*where* G *is a group*), $C_{25}=C_{52}=C_{50} = C_{05} = \emptyset$ *and* $S = C_{00} \cup C_{02} \cup C_{20} \cup C_{22} \cup G$ *where* C_{ij} $(i,j=0,2)$ *and* G *are unions of* $\mathcal{H}$-*classes of* S;

ii) *if* $1 \notin S$, *then, if* $S_5 \neq \emptyset$, *one has, either,*

a) $C_{52} = S_5$, $C_{25} = C_{50} = C_{05} = C_{55} = \emptyset$

and

$$S = C_{00} \cup C_{02} \cup C_{20} \cup C_{22} \cup S_5$$

or

b) $D_2 \cup D_5 = \emptyset$, $C_{50} = S_5$ *and* $S = C_{00} \cup C_{20} \cup S_5$;

dually, if $D_5 \neq \emptyset$, *then one has either*

c) $C_{25} = D_5$, $C_{52} = C_{50} = C_{05} = C_{55} = \emptyset$ *and*

$S = C_{00} \cup C_{02} \cup C_{20} \cup C_{22} \cup D_5$

or

d) $S_2 \cup S_5 = \emptyset$, $C_{05} = D_5$ *and* $S = C_{00} \cup C_{02} \cup D_5$

while, if $D_5 = S_5 = \emptyset$, *one has*

e) $S = C_{00} \cup C_{02} \cup C_{20} \cup C_{22}$;

in every case, C_{ij} $(i,j = 0,2)$, S_5 *e* D_5 *are unions of* $\mathcal{H}$*-classes of* S.

Proof.

If $1 \in S$, by Theorem 1.2 and because of (1.xiv) of [8], $C_{25} = \emptyset$, $C_{52} = C_{50} = C_{05} = \emptyset$ and $S_5 = D_5 = G$, where G is a group; moreover, by Theorem 1.1, C_{ij} $(i,j=0,2)$ and G are unions of $\mathcal{H}$-classes of S.

If $1 \notin S$, $S_5 \neq \emptyset$ and S is without left annihillators different from zero, then, by Theorem 1.2 of [8] and by (1.iv) of [8], $D_5 = \emptyset$. Therefore $C_{50} = C_{05} = C_{25} = C_{55} = \emptyset$. Moreover $C_{52} = S_5 \cap D_2 = S_5$; indeed if $z \in S_5 - D_2$ exists, it belongs to D_0 $(S = D_0 \cup D_2)$. Thus there exists $y \neq 0$ such that $yz = 0$. Now, since S_5 is a right group, if $\hat{z}$ is the unit of the $\mathcal{H}$-class of z, then for every $s \in S$ one has $ys=y\hat{z}s=yzz^{-1}s = 0$ where z^{-1} is the inverse of z in H_z. Therefore $S=D_0$. Then there follows $C_{50} = S_5 \neq \emptyset$, i.e. a contradiction.

If $1 \notin S$, $S_5 \neq \emptyset$ and S has left annihillators different from zero, and if there is an $x \in S$, $x \neq 0$ such that $xS = \{0\}$ one has $S=D_0$, and hence $S = C_{00} \cup C_{20} \cup C_{50}$. Moreover, since $D_0 = S$, then

$$C_{50} = S_5$$

The points c) and d) can be proved dually.

If $S_5 = D_5 = \emptyset$, e) is trivial. Furthermore, by Theorem 1.1, C_{ij} $(i,j=0,2)$, S_5 and D_5 are, in every case, the union of $\mathcal{H}$-classes of S.

3. - A semigroup S is said to be *left kernel* if $sS \subset S$, for every $s \in S$. A *right kernel* semigroup is defined dually.

A semigroup is called *kernel* if it is both a left and right kernel.

We recall that every left kernel semigroup is without left increasing (or magnifying) elements, and therefore every component of Szép's decomposition is union of $\mathcal{L}$-classes, as for semigroups of T-finite type.

In [12] Szép proved that every finite semigroup has at most a maximal (left, right) kernel subsemigroup.

Here we extend this result.

A semigroup S is said *quasi-regular* if every element of S has some regular power.

THEOREM 3.1.

Every semigroup S contains at most a maximal quasi-regular left [right] kernel subsemigroup.

Proof. Let S be a semigroup and let F, F' be two different maximal quasi-regular left kernel subsemigroups of S.

If V is the semigroup generated by F and F', then

$$xV \subset V \qquad \forall x \in F \cup F' \qquad (1)$$

In fact, if $x \in F \cup F'$ and $xV = V$, then $x \in V_1 \cup V_3 \cup V_5$, where V_i (i=1,3,5) are the components of $D_L(V)$. Now $x \notin V_5$: indeed if $x \in V_5$, since $x \in F$ (analogously if $x \in F'$) and F is quasi-regular, $\exists n \in N$ and $\exists y \in F$ such that $x^n = x^n y x^n$.

Now $yx^n \mathscr{L}^V x^n$; but, if e is the unity of the $\mathscr{H}$-class of x^n, $e \mathscr{L}^V x^n$, and therefore $yx^n \mathscr{L}^V e$. By Theorem 1.1 $e \in V_5$ and since yx^n is idempotent, $yx^n \in V_5$, and then $yx^n e = e$.

Moreover $yx^n = yx^n{}_c$, and therefore $e \in F$, but this is impossible, otherwise $eF = F$.

Moreover $x \notin V_1 \cup V_3$. In fact if $x \in V_1 \cup V_3$, since $\exists n \in N$ and $\exists y \in F$ $\ni'$ $x^n = x^n y x^n$, then $x^n y S = x^n S = S$ and hence $x^n y \in V_5$, which contradicts what was said above.

Now if a is an element of V, it follows from (1) that $aV \subset V$

$\forall a \in V$, i.e. V is a left kernel subsemigroup of S, which contradicts the maximality of F.

THEOREM 3.2.

If S *is a semigroup generated by* g.b. *elements, then*

$$S_1 = S_3 = S_4 = \emptyset .$$

Proof. Assume that there exists a left increasing element a of a semigroup generated by g.b. elements. If $a=a_1\ldots a_n$, where a_i $(i = 1,\ldots,n)$ are g.b. elements and if $T \subset S$ is such that $aT = S$, then, since the g.b. elements are not left increasing $S=aT=a_1\ldots a_n T \subset \subset S$, a contradiction.

Moreover, if a is a g.b. element of S_4, then $\exists n \in N$ such that $a^n \in H_e$, where e is an idempotent of S. Thus $a^n \mathscr{L} e$, and therefore, by Theorem 1.2, $e \in S_4$, a contradiction. So $S_4 = \emptyset$, because $S_0 \cup S_2 \cup S_5$ is a subsemigroup of S.

THEOREM 3.3.

Every semigroup S *contains at most a maximal left kernel subsemigroup, generated by* g.b. *elements*.

Proof. Let F and F' be two different maximal left kernel subsemigroups of a semigroup S which are generated by g.b. elements.

If V denote the semigroup generated by F and F', then $aV \subset V$, $\forall a \in F \cup F'$ (1). In fact, if $a \in F[F']$, by Theorem 3.2, $\exists y \in F \subseteq V$, $y \neq 0$ $\ni'$ $ay = 0$ o $\exists y_1, y_2 \in F \subseteq V$, $y_1 \neq y_2$ $\ni'$ $ay_1 = ay_2$, and therefore $a \in V_0 \cup V_2$, Thus, it follows from (1) that V is a left kernel, against the maximality of $F[F']$.

REFERENCES

[1] F.CATINO: "Λ-*right congruences and classification of orthogroups*". Ist.Mat.Univ.Siena, R. 44 (1981).

[2] F.CATINO - F.MIGLIORINI: "*Elementi quasi-accrescitivi ed accrescitivi nei semigruppi*". Ist. Mat. Univ. Siena R.60 (1982).

[3] F.CATINO - M.G.MURCIANO: "*Sulla struttura dei semigruppi periodici*". Ist. Mat. Univ. Lecce, Q. 16 (1981).

[4] J.M.HOWIE: "*An introduction to Semigroups Theory*". Acad.Press (1976).

[5] B.L.MADISON - T.K.MUKHERJEE - M.K.SEN: "*Periodic properties of groupbound semigroup*". Sem. Forum Vol. 22 (1981) 225-234.

[6] F.MIGLIORINI: "*Sulla* Γ-*decomposizione dei semigruppi regolari*" (in preparazione).

[7] F.MIGLIORINI - J. SZÉP : "*On a special decomposition of regular semigroups*". Acta Sci. Math. (Szeged), 40 (1978) 121-128.

[8] F.MIGLIORINI - J. SZÉP : "*On* Γ-*decomposition of semigroups*". Ist. Mat. Univ. Siena, R. 45 (1981).

[9] M.PETRICH: "*The structure of completely regular semigroups*". Trans. Am. Mat. Soc., 189, (1974) 211-236.

[10] B.PIOCHI: "*Sulla decomposizione di Szép dei semigruppi generati da idempotenti*". Ist. Mat. Univ. Siena, R. 38 (1981).

[11] R. SCOZZAFAVA: "*Remarks on Szép decomposition of semigroups*". Ist. Mat. Univ. Lecce, Q. 10 (1978).

[12] J.SZÉP: "*On the structure of finite semigroups III*". Dpt.Math., K.Marz Univ. Econ. Budapest, (1973), N° 3.

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