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A NOTE ON CONTROLLABILITY OF CERTAIN NONLINEAR SYSTEMS

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Abstract. *Sufficient conditions for global and local controllability of certain types of nonlinear time-varying systems with implicit derivative are given. The results obtained extend previous results through the notions of condensing map and measure of noncompactnes of a set.*

INTRODUCTION. The aim of this note is to introduce the notion and the techniques of condensing maps for studying the controllability of nonlinear control systems. The results we obtain extend the work reported in ([2]). Such methods seem to be crucial when the regularity of the system does not allow the use of geometrical methods for nonlinear control systems.

1. NOTATIONS, DEFINITIONS, PRELIMINARY RESULTS

We first summarize some facts concerning condensing maps: for definitions and results about the measure of non-compactness and related topics the reader is referred to the paper of C.Dacka ([2]).

DEFINITION 1.1. *Let* X *be a subset of a Banach space. An operator* $T : X \to X$ *is called condensing if for any bounded subset* $E \subset X$, *with* $\mu(E)\neq 0$, *we get* $\mu(T(E)) < \mu(E)$, *when* $\mu(E)$ *denotes the measure of noncompactness of the set* E *, whose properties are delineated in* ([2]).

We observe that, as a consequence of the properties of μ, if an operator T is the sum of two operators, a compact one and a condensing one, then T itself is a condensing operator.

Remark 1.1. We want to note that, if the operator $W: X \to X$ satisfies the condition $|W(x)-W(y)| \leq k|x-y|$, $x,y \in X$, $0 \leq k < 1$, then the operator W is a *μ-contractive operator* with constant k, that is $\mu(T(E)) \leq k\mu(E)$, for $E \subset X$, E bounded. In this case W has the fixed point property, according to Darbo's theorem ([3]). However, when W satisfies the condition $|W(x)-W(y)| < |x-y|$, then it is not possible to say that W is a condensing map and in general W will not admit a fixed point ([1],[4]). The fixed point property holds in the condensing case ([5]).

As in ([2]), let θ denote the modulus of continuity of a bounded set and let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be a right continuous, non-decreasing function such that, for $r > 0$, $\omega(r) < r$.

LEMMA 1.1. *Let* $X \subset C^o(I,\mathbb{R}^n)$ *and* β *and* γ *be functions defined on* $[0,t_1-t_o]$ *such that* $\lim_{s\to o} \beta(s) = \lim_{s\to o} \gamma(s) = 0$.

If a mapping $T : X \to C^o(I,\mathbb{R}^n)$ *is given such that it maps bounded sets into bounded sets and, with* ω *as above is such that, for any* $x \in X$,

$$\theta(T(x),h) < \omega(\theta(x,\beta(h))) + \gamma(h) \quad \textit{for all} \quad h \in [0,t_1-t_o],$$

then T *is a condensing mapping.*

Proof. The above inequality implies that for any bounded set E

$$\theta(T(E),h) < \omega(\theta(E,\beta(h))) + \gamma(h) \qquad \text{and furthermore}$$

$$\theta_o(T(E)) \leq \omega(\theta_o(E)) < \theta_o(E) \qquad \text{for } \theta_o(E) \neq 0.$$

Finally it is possible to show ([8]) that for any bounded and equicontinuous set $E \subset C^1(I,\mathbb{R}^n)$ the following relation holds

$$\mu_{C^1}(E) \equiv \mu_1(E) = \mu(DE) \equiv \mu_{C^0}(DE)$$

where $DE = (Dx : x \in E)$.

2. THE MAIN RESULT

Let us consider the nonlinear control system

$$(1) \qquad x'(t)=A(t)x(t)+B(t)u(t)+f(t,x(t),x'(t),u(t))$$

where $x(t)$ is an nx1 state vector, $u(t)$ is a kx1 input vector, $t \to A(t)$ is an nxn matrix, $t \to B(t)$ is an nxk matrix and $f(t,x(t),x'(t),u(t)))$ is an nx1 vector-valued function. Assume that

i)the entries $a_{ij}(t)$ of $A(t)$, $i,j = 1,\dots,n$, are continuous functions;

ii)the entries $b_{im}(t)$ of $B(t)$, $i=1,\dots,n$, $m=1,\dots,k$, are continuous functions;

iii)the function $(t,x,y,u) \to f(t,x,y,u)$, $(t,x,y,u) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^k$ is continuous.

It is well-know that, under conditions i)-iii), the solutions of (1) with $x(t_o) = x_o$ are given by

$$x(t) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,s)B(s)u(s)ds + \int_{t_o}^{t} \Phi(t,s)f(s,x(s),x'(s),u(s))ds$$

where $\Phi(t,t_o)$ is the transition matrix for the system

$x'(t) = A(t)x(t)$ with $\Phi(t_o,t_o) = Id.$

Let us put

$$G(t_o,t) = \int_{t_o}^{t} \Phi(t_o,s)B(s)B'(s)\Phi'(t_o,s)ds$$

where the prime indicates the transpose matrix.

The main result concerning the global controllability of the system (1) is given in the following theorem.

THEOREM 2.1. *Suppose that conditions* i)-iii) *hold for the system* (2) *and assume the additional conditions:*

iv)
$$\limsup_{|x|\to\infty} \frac{|f(t,x,y,u)|}{|x|} = 0 \ ;$$

v) *there exists a continuous, non-decreasing function* $\omega:\mathbb{R}^+\to\mathbb{R}^+$, *with* $\omega(r) < r$, *such that for all* $(t,x,y,u) \in \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^k$

$$|f(t,x,y,u) - f(t,x,z,u)| < \omega(|y-z|);$$

vi) *the symmetric matrix* $G(t_o,t_1)$ *is nonsingular, alternatively, for*

a) *some* $t_1 > t_o$

b) *all* t_o *and all* $t_1 > t_o$.

Then the nonlinear control system (1) is completely controllable at t_o if condition a) holds, or is completely controllable if condition b) holds.

Proof. Let us introduce the following pair of operators:

$T_1 : Dom(T_1) \subset C^o(I,\mathbb{R}^k) \to C^o(I,\mathbb{R}_k)$ defined as

$T_1(u,x)(t) = - B'(t)\Phi'(t_o,t)G^{-1}(t_o,t_1)$ ∘

$$\circ\left(\int_{t_o}^{t_1} \Phi(t_o,s)f(s,x(s),x'(s),u(s))ds+x_o-\dot{\Phi}(t_o,t_1)\right)$$

$T_2 : \mathrm{Dom}(T_2) \subset C^1(I,\mathbb{R}^n) \to C^1(I,\mathbb{R}^n)$ defined as

$$T_2(u,x)(t) = \Phi(t,t_o)x_o + \int_{t_o}^{t} \Phi(t,s)B(s)T_1(u,x)(s)ds +$$

$$+ \int_{t_o}^{t} \Phi(t,s)f(s,x(s),x'(s),T_1(u,x)(s))ds$$

Define now the nonlinear tranformation

$$T : \mathrm{Dom}(T) \subset C^o(I,\mathbb{R}^k)xC^1(I,\mathbb{R}^n) \to C^o(I,\mathbb{R}^k)xC^1(I,\mathbb{R}^n) \quad \text{as}$$

$$T(u,x)(t) = (T_1(u,x)(t),\ T_2(u,x)(t)).$$

We remark that because of the continuity of all the functions involved T is a continuous operator. Moreover, it is easy to see that, by direct differentiation with respect to t, a fixed point for the operator T gives rise to a control u and a corresponding function x=x(u) solution of the nonlinear control system (1) satisfying $x(t_o) = x_o$, $x(t_f) = x_f$.

Let $\eta=(u^\circ,x^\circ) \in C^o(I,\mathbb{R}^k)xC^1(I,\mathbb{R}^n)$,

$$\zeta = (u,x) \neq (0,0) \in C^o(I,\mathbb{R}^k)xC^1(I,\mathbb{R}^n)$$

and consider the equation

$$(2) \qquad \eta = \zeta - \lambda\, T(\zeta) \qquad \lambda \in [0,1].$$

This equation can be equivalently written as

$$(3) \qquad u = u^\circ + \lambda T_1(u,x) \quad ,$$

(4) $$x = x^\circ + \lambda T_2(u,x).$$

Condition iv) says that for any $\varepsilon > 0$ there exists $R>0$ such that if $|x| > R$ then $|f(t,x,y,u)| < \varepsilon|x|$. Then from (3) we get

(5) $$|u| \leq |u^\circ| + k_1 + |B|\ |\Phi|^2|G^{-1}|\varepsilon\bar{t}|x|,$$

where

$$\bar{t} = t_1 - t_o \text{ and } k_1 = |B|\ |\Phi|\ |G^{-1}|(|x_o|+|\Phi|\ |x|).$$

From this inequality and from (4), by applying the Gronwall lemma, we obtain

$$|x| \leq (\ |x^\circ|+|\Phi|\ |x_o|+|T_1(u,x)|\ |\Phi|\ |B|\bar{t})\exp(|\Phi|\varepsilon\bar{t}) \leq$$

$$\leq (\ |x^\circ|+|\Phi|\ |x_o|+(k_1+|B|\ |\Phi|^2|G^{-1}|\varepsilon\,\bar{t}|x|)|\Phi|\ |B|\bar{t})\ \exp(|\Phi|\varepsilon\bar{t}).$$

We want to note that

$$\frac{d}{dt}(T_2(u,x)(t)) = A(t)\Phi(t,t_o)x_o + B(t)T_1(u,x)(t) +$$

$$+ \int_{t_o}^{t} A(t)\Phi(t,s)B(s)T_1(u,x)(s)ds + f(t,x(t),x'(t),T_1(u,x)(t)) +$$

$$+ \int_{t_o}^{t} A(t)\Phi(t,s)f(s,x(s),x'(s),T_1(u,x)(s))ds =$$

$$= A(t)T_2(u,x)(t) + B(t)T_1(u,x)(t) + f(t,x(t),x'(t),T_1(u,x)(t)).$$

An additional application of the Gronwall lemma gives

(7) $$T_2(u,x) \leq (|B|\,T_1(u,x)|\bar{t} + \bar{t}|x|)\exp(A_o),$$

where $$A_o = \int_{t_o}^{t_1} |A(t)|dt.$$

Differentiating with respect to t, we obtain from (4)

$$x' = (x^\circ)' + \lambda \frac{d}{dt}(T_2(u,x)(t))$$

and that gives

(8) $|x'| \le |(x^\circ)'| + |\frac{d}{dt}(T_2(u,x)| \le |(x^\circ)'| + |A||T_2(u,x)| +$

$+|B||T_1(u,x)| + |\varepsilon||x| + |\beta| \le |(x^\circ)'| + |T_1(u,x)|[|A||B|\bar{t}\circ\exp(A_o) + |B|\ ] +$

$+\ |x|(\bar{t}\varepsilon\exp(A_o) + \varepsilon) =$

$= |(x^\circ)'| + k_2 + |x|[\ |B^2||\Phi|^2|G^{-1}|\bar{t}\varepsilon(|A|\bar{t}\circ\exp(A_o)+1) + (\varepsilon\bar{t}\circ\exp(A_o+\varepsilon)]$,

where

$$k_2 = k_1[|B|(|A|\exp(A_o)(t_1-t_o) + 1].$$

From (5) we get

$$|u| - |B|\ |\Phi|^2|G^{-1}|\varepsilon\bar{t}|x| \le |u^\circ| + k_1$$

and from (6), (7) and (8)

$|x|(\exp(-|\Phi|\varepsilon\bar{t}) - |B|^2|\Phi|^3|G^{-1}|\bar{t}\circ\varepsilon\circ\bar{t}) \le k_3 + |x^\circ|$,

where

$$k_3 = |\Phi||x_o| + k_1|B|\ |\Phi|\bar{t}$$

and

$|x'| - |x|[|B|^2|\Phi|^2|G^{-1}|\varepsilon\bar{t}(|A|\bar{t}\circ\exp(A_o)+1) + (\varepsilon\bar{t}\circ\exp(A_o) + \varepsilon\ )] \le$

$\le k_2 + |(x^\circ)'|$.

Taking the sums of all the above quantities we obtain

$|u| - |x|[|B||\Phi|^2|G^{-1}|\varepsilon\bar{t} - \exp(-|\Phi|\varepsilon\bar{t}) + |B|^2|\Phi|^3|G^{-1}|\bar{t}\circ\varepsilon\circ\bar{t} +$

$$+ \quad [|B|^2|\Phi|^2|G^{-1}|\,\varepsilon\bar{t}(|A|\bar{t}\circ\exp(A_o)+1) + \varepsilon\bar{t}\circ\exp(A_o)+\varepsilon\,] + |x'| =$$

$$= |u| - \tau|x| + |x'| \leq |u^\circ| + k_1+k_3+|x^\circ|+k_2+|(x^\circ)'|,$$

where

$$\tau = |B||\Phi|^2|G^{-1}|\varepsilon\bar{t}[1+|B||\Phi|\bar{t}+|B|(|A|\bar{t}\circ\exp(A_o)+1) +$$

$$+ \varepsilon + \varepsilon\bar{t} \circ \exp(A_o)] - \exp(-|\Phi|\varepsilon\bar{t}).$$

Then, for suitable positive constants k_4, k_5, k_6 we can write

$$|u|-(\varepsilon k_4+\exp(-\varepsilon k_5))|x|+|x'| \leq |u^\circ| + |x^\circ| + |(x^\circ)'| + k_6.$$

So we divide by $|u|+|x|+|x'|$ and, by the arbitrariness of ε, we get the existence of a sufficiently large ball $S \subset C^o(I\times\mathbb{R}^k)\times C^1(I\times\mathbb{R}^n)$ such that

$$(9) \qquad |\zeta - \lambda T(\zeta)| > 0 \qquad \text{for} \quad \zeta = (u,x) \in \partial S.$$

Now we want to show that T is a condensing map. To this aim we note that $T_1 : C^o(I,\mathbb{R}^k) \to C^o(I,\mathbb{R}^k)$ is a compact operator (from Ascoli-Arzelà theorem) and then, if E is a bounded set, $\mu(T_1(E)) = 0$. Then it will be enough to show that T_2 is a condensing operator. To do this we will use the modulus of continuity argument and the fact that

$$\mu_1(T_2(E)) = \mu(DT_2(E)) = \frac{1}{2}\theta_o(DT_2(E)).$$

Let us consider the modulus of continuity of $DT_2(u,x)(t)$:

$$|DT_2(u,x)(t) - DT_2(u,x)(t')| \leq |A(t)T_2(u,x)(t) - A(t')T_2(u,x)(t')|+$$

$$+ |B(t)T_1(u,x)(t) - B(t')T_1(u,x)(t')| +$$

$$+ |f(t,x(t),x'(t),T_1(u,x)(t))-f(t',x(t'),x'(t'),T_1(u,x)(t'))| .$$

For the first two terms of the right-hand side of the inequality we may give the upper estimate $\beta_0(t-t')$, with $\lim_{t\to t'}\beta_0(t-t') = 0$, and it may be chosen independent of the choice of (u,x). For the third term we can give the following estimate:

$$|f(t,x(t),x'(t),T_1(u,x)(t))-f(t',x(t'),x'(t'),T_1(u,x)(t'))| \le$$

$$\le |f(t,x(t),x'(t),T_1(u,x)(t))-f(t,x(t),x'(t'),T_1(u,x)(t)))|+$$

$$+ |f(t,x(t),x'(t'),T_1(u,x)(t))-f(t',x(t'),x'(t'),T_1(u,x)(t'))|.$$

For the first term we have the upper estimate $\omega(|x'(t)-x'(t')|)$, whereas for the second term we may find estimate $\beta_1(|t'-t|)$ with $\lim_{t\to t'}\beta_1(t'-t) = 0$.

Such estimates β_0 and β_1 arise from the fact that the quantities $T_1(u,x)(t),x(t),x'(t)$ remain bounded for $t\in[t_0,t_1]$ and from the fact that f is uniformly continuous on compacta. Therefore

$$\theta(DT_2(u,x)(h)) \le \omega(\theta(DE,h)) + \beta(h),$$

$$\beta(h) = \beta_1(h) + \beta_2(h).$$

Making use of Lemma 1.1 we get $\theta_0(DT_2(E)) < \theta_0(DE)$. Hence from

$$2\mu_1(T_2(E)) = 2\mu(DT_2(E)) = \theta_0(DT_2(E)) < \theta_0(DE) =$$

$= 2\mu(DE) = 2\mu_1(E)$ it follows that $\mu_1(T(E))<\mu_1(E)$.

Thus the existence of a fixed point of the operator T is a consequence of the following fixed point theorem due to B.J.Sadowski ([5]):

"Let J be the unit interval of the real line, $S \subset X$ be a bounded, closed, convex set. Let $H : J \times S \to X$ be an operator such that for any $\lambda \in J$, $H(\lambda, \circ) : S \to X$ is condensing. If, for any $\lambda \in J$ and any $x \in \partial S$ (the boundary of S) it happens that $x = H(\lambda,x)$, then $H(1,x)$, has a fixed point".

Then there exist functions $u^+ \in C^o(I,\mathbb{R}^k)$, $x^+ \in C^1(I,\mathbb{R}^n)$ such that

$$T(u^+,x^+) = (u^+,x^+) \quad \text{or, equivalently that}$$

$$u^+(t) = T_1(u^+,x^+)(t), \quad x^+(t) = T_2(u^+,x^+)(t).$$

Remark 2.1. The results obtained concern the global complete controllability because of the hypothesis that the function f is defined on all $\mathbb{R}x\mathbb{R}^n x\mathbb{R}^n x\mathbb{R}^k$. If we suppose that f is defined on a subset D of such a space, say

$$D = \{(u,x,y,t) : |u| \leq 1, |x| \leq p, |y| \leq r, t \in [t_o,t_1]\},$$

where $p > 0$, $r > 0$, and the conditions on f, x_o,x_1 are such that the constant k_6 in the proof of Theorem 2.1 satisfy the inequality $k_6 < 1 + r + p$, then the controllability results hold relatively to the set D. In such a case we will talk of local complete controllability.

AN EXAMPLE

We give now an example of application of the above result to the following nonlinear control system

$$x_1' = x_1 - x_2 + (\cos t)u_1 + (\sin t)u_2 + \frac{\log|x_1|}{\sqrt{1+u_1^2+u_2^2}} + \operatorname{arctg} x_1',$$

$$x_2' = x_1 + x_2 - (\sin t)u_1 + (\cos t)u_2 + \frac{1}{\sqrt{1+u_2^2+u_2^2}} \log|x_2|$$

We have

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$f = \left(\frac{\log|x_1|}{\sqrt{1+u_1^2+u_2^2}} + \operatorname{arctg} x_1' \ , \ \frac{\log|x_2|}{\sqrt{1+u_1^2+u_2^2}} \right)' .$$

Then $G(0,t_1) = \int_0^{t_1} \Phi(0,t)B(t)B'(t)\,\Phi'(0,t)dt$ is given by

$$G(0,t_1) = \frac{1}{2}(\exp(2t_1) - 1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and it is nonsingular for all $t_1 \neq 0$.

Furthermore

$$|f(t,x,y,u) - f(t,x,z,u)| = |\operatorname{arctg} y_1 - \operatorname{arctg} z_1| < \operatorname{arctg}|y_1 - z_1|$$

if $y_1 \neq z_1$

and

$\lim\limits_{|x|\to\infty} \dfrac{|f(t,x,y,u)|}{|x|} = 0$. So the hypotheses of Theorem 2.1 are satisfied.

Finally we want to remark that the conditions of the main result of [2] do not apply in our example;

Remark 2.2.The conditions of Theorem 2.1 can be weakened, as established in the following result whose proof is straightforward:

THEOREM 2.2. *If* f *satisfies the hypotheses of Theorem 2.1. but the condition* iv) *is replaced by*

iv)' $|f(t,x,y,u)| \leq \alpha(t)|x| + \beta(t)$ $\alpha,\beta \in C^{o}(I,\mathbb{R}^{+})$;

iv)" $\tau < 0$, *where the real number* τ *is defined by*

$$\tau = |B||\Phi|^{2}G^{-1}|\alpha_{o}[1+|B||\Phi|\bar{t} + |B|(|A|\bar{t}\circ\exp(A_{o})+1) + $$
$$+ \alpha + \alpha_{o}\circ\exp(A_{o})] - \exp(-|\Phi|\alpha_{o}),$$

where $\alpha_{o} = \int_{t_o}^{t_1} \alpha(t)dt$, $\beta_{o} = \int_{t_o}^{t_1}\beta(t)dt$ and, as before $,\bar{t}=t_1-t_o$, then the nonlinear control system (1) is completely controllable at t_o if condition a) holds, or completely controllable if condition b) holds.

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