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ON ISOMORPHISMS OF CHAIN GEOMETRIES

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0.INTRODUCTION. Isomorphisms (automorphisms) of chain geometries over a ring L were studied mainly for a skewfield [2],[3],[8] alternating field [9], a commutative algebra (see the survey in [4]) a local ring [7] or an algebra of characteristic $\neq 2$,[1]- in most cases by means of harmonic quadruples, cross rations - or (in the last example) generalized harmonic quadruples.

Here instead of this are studied isomorphisms of chain geometries over any K-algebra A of finite dimension with $|K|>3$ (local algebra of finite dimension with $|K| \geq 3$).

It is proved that any fundamental isomorphism is generated by some K-Jordan isomorphism, moreover all antiisomorphisms (and for A a local algebra all K-Jordan isomorphisms) generate isomorphims of chain geometries. No use of crossratios or (generalized) harmonic quadruples is made; the proof of the structure theorem in section 2 is in a geometric way and follows ideas of [8], whereas the proof of section 3.6 also uses methods of [9]. The last section represents a series of examples for proper K-Jordan isomorphisms generating fundamental isomorphisms of chain geometries, thus answering a question of [1].

We have examples of local algebras which belong to isomorphic chain geometries but are neither isomorphic nor antiisomorphic. In fact there are examples of commutative algebras in case of characteristic 2.

1.K-JORDAN HOMOMORPHISMS. *General assumption for the whole paper:* Any K-algebra A,B,C... is finite dimensional, associative with 1 and it is $|K| > 3$ - or the K-algebra is local and it is $|K| \geq 3$. It is A* the group of units of A and J(A) the Jacobson radical of A.

1.1. DEFINITION. Let A,B be K-algebra. A map $\sigma: A \to B$ is called

a) K-*Jordan homomorphism,*

b) K-*algebra homomorphism,*

c) K-*algebra antihomomorphism,* if the following hold:

1. σ is K-semilinear (the algebras A,B considered as K-vector spaces) and $1^\sigma = 1$.

2. $\forall u,v \in A$: a) $(uwu)^\sigma = u^\sigma w^\sigma u^\sigma$

b) $(uw)^\sigma = u^\sigma w^\sigma$

c) $(uw)^\sigma = w^\sigma u^\sigma$.

Clearly b) and c) are special kinds of a). We call σ a *proper* K-Jordan homomorphism if σ is neither of kind b) nor of kind c). The map σ is a K-Jordan isomorphism if σ is bijective, and σ and σ^{-1} are K-Jordan homomorphisms, and so on.

For $k \in K$ we define k^σ by $k^\sigma \cdot 1_B = (k \cdot 1_A)^\sigma$, where $1_A, 1_B$ is the unity of A,B respectively.

Easy consequences:

1.2. *For a* K-*Jordan homomorphism* σ *hold:*

$\forall u,v \in A: \quad (u^n)^\sigma = (u^\sigma)^n$

$\forall n \in \mathbb{N} \quad ((uv)^n u)^\sigma = (u^\sigma v^\sigma)^n u$

$(uv + vu)^\sigma = u^\sigma v^\sigma + v^\sigma u^\sigma.$

So for commutative K-algebras of characteristic $\neq 2$ any K-Jordan homomorphism is a K-algebra homomorphism.
(examples of proper K-Jordan isomorphisms of commutative K-algebras of characteristic 2 see below, 4.3.2).

1.3. *For* a K-*Jordan homomorphism* σ *hold*

(1) $a \in A^* \Rightarrow a^\sigma \in B^*$

(2) $a \in A^*: (a^{-1})^\sigma = (a^\sigma)^{-1}$.

Moreover:

(3) $a,b,\ a-b \in A^* \Rightarrow a^{-1}+(b-a)^{-1} = a^{-1}b(b-a)^{-1} = (a-ab^{-1}a)^{-1}$

In particular

$$a^{-1} + (b-a)^{-1} \in A^*.$$

Proof. (1) Let m,m' be the minimal polynomials of a, a^σ respectively, $n = \sum_{i=0}^{n} c_i x^i$, $c_i \in K$.

Then $a \in A^*$ implies $c_0 \neq 0$. Moreover for $m^\sigma =: \sum_{i=0}^{n} c_i^\sigma x^i$ now $m' | m^\sigma$. Therefore $x \nmid m'$, which means $a^\sigma \in B^*$.

Now (2) follows with (1) from

$$a^\sigma = (aa^{-1}a)^\sigma = a^\sigma (a^{-1})^\sigma a^\sigma$$

and (3) is the "Hua identity" slightly varied.

1.4. PROPOSITION. *Let* $\sigma: A \to B$ *be a* K-*semilinear map with* $1^\sigma.=1$. *The following statements are equivalent:*

(a) σ *is a* K-*Jordan homomorphism*

(b) $\forall x,y \in A$: $xy = 1 \Longrightarrow x^\sigma y^\sigma = 1$

(c) $A^{*\sigma} \subseteq B^*$ *and* $\forall a,b \in A^*$ *with* $b-a \in A^*$:

$$((a^{-1} + (b-a)^{-1})^{-1})^\sigma \Rightarrow ((a^\sigma)^{-1} + (b^\sigma - a^\sigma)^{-1})^{-1} .$$

Proof. (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) follows from 1.4 (1) and (2), and (c) makes sense by 1.3 (3). (c) $\Rightarrow$ (a): For $N = J(A)$ let $A \to A/N$; $x \mapsto \bar{x}$ be the natural epimorphism. Now A/N is the ring-direct sum of simple algebras, say $A_1,\dots,A_r$, where A_v up to isomorphism is a full matrix ring over some skewfield D_v with K in its center. If $(t_k^{(v)})$ is a K-base of D_v and $(e_{ij}^{(v)})$ is the usual D_v-base of $A_v(e_{ij}^{(v)}$: the matrix with entry one in the crosspoint of the i-th row and the j-th column and 0 elswhere), then $C = \{t_k^{(v)} e_{ij}^{(v)} | v = 1,\dots,r\}$ is a K-base of A/N and

$$G = \{x \in A | \bar{x} \in C\} \cup N$$

certainly is a set of generators of A as K-vector space.

(1) For different $x,y \in G$ there are $r,s,t \in K^*$ such that for $a=1+rx$, $b = s\cdot 1+ty$ now a,b and $b-a \in A^*$ holds and moreover $b^{-1} = s'\cdot 1+t'y'$ for $s',t' \in K^*$ and $\bar{y}'=\bar{y}$. In the same way for different $x,y,z \in G$ there is $r,r',s,t \in K^*$ such that for $a=1+rx+r'z$ and $b=s\cdot 1+ty$ again $a,b,b-a \in A^*$ holds.

E.g. for $\bar{x} = e_{ij}^{(v)}$, $\bar{z} = e_{ji}^{(v)}$, $i \neq j$, choose $r=1\neq r'$; for $\bar{x}=e_{ij}^{(v)}$,

$\bar{y} = e_{ji}^{(v)}$, $\bar{z} = e_{kk}^{(v)}$ choose $r=r'\neq -1$ and $s\neq -1$, $(1+r)$ and $t=1$ (here $|K|>3$ is needed; this case does not occur for A a local algebra).

(2) Now from 1.3(3) we may deduce $(ab^{-1}a)^\sigma = a^\sigma(b^\sigma)^{-1}a^\sigma$ and obtain step by step - first using $b=s\,1$ - for any three different $x,y,z\in G$

(i) $(x^2)^\sigma = (x^\sigma)^2$

(ii) $(xz+zx)^\sigma = x^\sigma z^\sigma + z^\sigma x^\sigma$

(iii) $(xyx)^\sigma = x^\sigma y^\sigma x^\sigma$

(iv) $(xyz+zyx)^\sigma = x^\sigma y^\sigma z^\sigma + z^\sigma y^\sigma x^\sigma$.

(3) For $u,w\in A$ exist $x_i,y_j\in G$, $u_i,w_j\in K$ such that $u=\sum_i u_i x_i$, $w=\sum_j w_j y_j$. Therefore from (iii) and (iv) we obtain

$$(uwu)^\sigma = u^\sigma w^\sigma u^\sigma.$$

The following two lemmas without direct connection to the preceding are needed for the next section. Recall that a pair $(c,d)\in A\oplus A$ is *unimodular*, if there are $a',b'\in A$ with $aa'+bb'=1$.

1.5. LEMMA. *Let* $(c,d) \in A \oplus A$ *be unimodular. Then* $cb-d$ *is a unit for suitable* $b\in A$.

Proof. It suffices to prove the assertion for a direct summand of $A/_{J(A)}$. So w.l.o.g. we may assume $A=\mathrm{End}_D(U)$, where D is a skewfield and U is a (finite-dimensional) D-vector space, $rc=\dim_D(\mathrm{im}\ c)$. Now (c,d) unimodular implies $\ker c \cap \ker d = \{0\}$ and $rc+rd\geq \dim_D U$.

Consider the following direct sums of $U: U=U_1\oplus U_2=W_1\oplus W_2$ with $\ker c\leq U_1$ and $U_2=\ker d$ and $\mathrm{im}\ d=W_2$. Now one may choose $b\in\mathrm{End}_D(U)$

with ker cb = U_1 and im cb = W_1. Therefore cb induces a D-linear isomorphism $U_2 \to W_1$, and d induces a linear isomorphism $U_1 \to W_2$.

We show that cb-d is an epimorphism: Let $v \in U$, $u = w_1 + w_2$ with $w_i \in W_i$. There are unique $u_i \in U_i$ with $u_2 cb = w_1$ and $-u_1 d = w_2$, thus

$$(u_1+u_2)(cd-d) = v.$$

1.6. LEMMA: *Let* $0 \neq c \in A$. *There is* $a \in A^*$ *such that* $ac-t \in A^*$ *for any* $t \in K$, $t \neq 0,1$. *If* c *is not of the form* $(k,0)$ *for* $A = K \oplus A_1$, *then there are* $a, a' \in A^*$ *with* $ac-t \in A^*$, $a'c-s \in A^*$ *and* $ac-t \neq a'c-s$ *for any* $s,t \in K \setminus \{0,1\}$. *For* A *a local algebra also* ac *or* ac-1 *is a unit.*

Proof. We can choose $a \in A^*$ in such a way, that for p=ac in $A/_{J(A)}$ the image $\bar{p}$ is an idempotent, namely a projection for any direct summand of $A/_{J(A)}$. Therefore for $t \in K$ now $(\bar{p}-t)(\bar{p}+(t-1)) = t(1-t)$, that is, for $t \neq 0,1$ now p-t is a unit.

If c is not of the special type mentioned above, it is not difficult to find $a' \in A^*$, such that the second assertion is fulfilled.

The third assertion follows from the simple fact that now $A/_{J(A)}$ is a skewfield and so the idempotent $\bar{p}$ is 0 or 1.

2. THE STRUCTURE THEOREM. For $(a,b) \in V = A \oplus A$ define $[a,b] = A(a,b) = \{(ra,rb) \mid r \in A\}$ and $\mathbb{P}(A) = \{[a,b] \mid (a,b) \text{ unimodular}\}$; we call $\mathbb{P}(A)$ the projective line over A, the elements of which are called points. We consider $GL_2(A)$ as the group of invertible (2,2)-Matrices with entries in A and $PGL_2(A)$ as the group of permutations of $\mathbb{P}(A)$ induced by $GL_2(A)$ by means of the operations $[x,y] \to [x',y']$, where $(x',y') = (x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$. Two points [a,b] and $[c,d]$ are *distant* iff $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A)$ holds. Let be

$$k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \{[sa+tc,\ sb+td] \mid (0,0) \neq (s,t) \in K^{(2)}$$

and

$$\mathcal{K}(A) = \{ k\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{pmatrix} a & d \\ c & d \end{pmatrix} \in GL_2(A)\},$$

where the elements of $\mathcal{K}$ are the *chains* (over A), and $\Sigma(K,A) = (\mathbb{P}(A), \mathcal{K}(A))$ is the *chain geometry over* A. Any three points mutually distant are contained in exactly one chain.

An isomorphism of chain geometries $\alpha : \Sigma(K,A) \to \Sigma(K,B)$ is a bijection $\alpha: \mathbb{P}(A) \to \mathbb{P}(B)$, such that α and α^{-1} map any chain onto a chain. The isomorphism is *fundamental*, if moreover $[1,0]^{\alpha} = [1,0], [0,1]^{\alpha} = [0,1]$ and $[1,1]^{\alpha} = [1,1]$ hold. So a fundamental automorphism of $\Sigma(K,A)$ simply is an automorphism fixing the points $[1,0]$, $[0,1]$ and $[1,1]$.

Let $\Gamma(K,A)$ be the group of automorphisms of $\Sigma(K,A)$ and $F(K,A)$ the group of fundamental automorphisms of $\Sigma(K,A)$. Since $PGL_2(A)$ is a subgroup of $\Sigma(K,A)$ acting transitively on triples of points mutually distant, we have the following trivial relation:

$$\Gamma(K,A) = PGL_2(A) \cdot F(K,A).$$

We are interested in the study of fundamental isomorphisms. Since $[a,b]$ is adjacent to $[0,1]$, iff a is a unit, any fundamental isomorphism $\alpha: \Sigma(K,A) \to \Sigma(K,B)$ induces a bijection $\sigma: A \to B$ defined by $[1,b^{\sigma}] = [1,b]^{\alpha}$ $\forall b \in A$.

We call $\sigma: A \to B$ *induced* by the fundamental isomorphism α.

2.1. **PROPOSITION.** *The map* $\sigma: A \to B$ *induced by a fundamental isomorphism* $\alpha: \Sigma(K,A) \to \Sigma(K,B)$ *already determines* α. (So we also may say α is generated by σ.)

Proof. (i) The point $[c,d]$ is distant to $[1,b]$, iff $cb-d$ is a unit: $[1,b]$ and $[c,d]$ distant $\Longrightarrow \exists x,y \in A: (x,xb)+(yc,yd)=(0,1) \Rightarrow x = -yc \Longrightarrow y(d-cb) = 1 \Longrightarrow cb - d \in A^*$. Conversely let $cb-d \in A^*$ and $(x,y) \in [1,b] \cap [c,d]$. Then $x=zc$ and $xb=y=zd$ so $z(cb-d) = 0$, therefore $z = 0$, $(x,y) = (0,0)$. From $[1,b] \cap [c,d] = 0$ now we conclude by reasons of K-dimension $[1,b]+[c,d] = V$, so $\begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \in GL_2(A)$.

(ii) Given a point $[c,d]$ by lemma 1.5 and (i) there is some $b \in A$, such that $[1,b]$ and $[c,d]$ are distant Now for any $a \in A^*$ with the properties of lemma 1.6 the chain $k=k\begin{pmatrix} 1 & b \\ ac & ad \end{pmatrix}$ contains the points $[c,d]$, $[1,b]$ and at least two further points $[1,b']$, $[1,b'']$. Therefore k^α is determined by $b^\sigma, b'^\sigma, b''^\sigma$. But there are at least two different chains of this kind (see below), thus $[c,d]$ is determined as intersection of the images of the chains, which are determined by σ.

The existence of two chains of this kind follows directly from lemma 1.6 if c is not of the form $(k,0)$ for $A=K \oplus A_1$. Else we have $d = (d_1,d_2)$ with $d_2 \neq 0$ and so we may find $a,a' \in A^*$ with $[ac-s, ad-bs] \neq [a'c-t, a'd-bt]$ for any $s,t \in K \setminus \{0,1\}$.

In the affine space $\mathbb{A}$ belonging to A considered as K-vector space, let $L = \{Ka+b \mid a \in A^*, b \in A\}$ be a set of distinguished lines of $\mathbb{A}$ containing full parallel classes.

2.2. **LEMMA:** **1.** Every line s of $\mathbb{A}$ not in L lies in a plane E of $\bar{A}$ such that only lines of at most two directions [1] of E do not belong to L, and in E there are at least lines in three directions [1] belonging to L.

2. Let σ be a permutation of the pointset of A such that σ and σ^{-1} map lines of L onto lines of L and fix 0. Then σ is a K-semilinear map on A.

Proof. 1. Using a translation we may assume $0 \in s$, so $s=Kc$, $c \notin A^*$. Now by Lemma 1.6 for suitable $a \in A^*$ the plane $E'=K+Kac$ has the properties of the assertion, thus also $E=a^{-1}E'$ which contains s.

2. σ and σ^{-1} by their properties map the planes E described under 1 onto planes. Moreover σ and σ^{-1} map triangles of E onto triangles since at least one side of the triangle belongs to L. So σ and σ^{-1} map all lines of E onto lines. Thus σ and σ^{-1} map any line of A onto a line. As is well-known, then σ is a semilinear map, since σ fixes 0.

2.3. **PROPOSITION.** *Any fundamental isomoprhism induces a K-Jordan isomorphism.*

Proof. Let $\alpha:\Sigma(K,A) \to \Sigma(K,B)$ be a fundamental isomorphism inducing the map $\sigma: A \to B$ by means of $[1,b^\sigma] = [1,b]^\alpha$. We have $0^\sigma = 0$ and $1^\sigma = 1$. Moreover we have

$$\begin{aligned} b \in A^* &\iff [1,0] \text{ and } [1,b] \text{ distant} \\ &\iff [1,0] = [1,0]^\alpha \text{ and } [1,b^\sigma] = [1,b]^\alpha \text{ distant} \\ &\iff b^\sigma \in B^* \end{aligned}$$

(1) direction = parallel class

so $A^{*\sigma} = B^*$.

In the affine space $\bar{A}$ belonging to A as K-vector space, for $a \in A^*$ the line $s=Ka+b$ is the trace of the chain $k\begin{pmatrix} 0 & 1 \\ a^{-1} & a^{-1}b \end{pmatrix}$ containing [0,1]. So lemma 2.2 can be applied to σ and we obtain that σ is a K-semilinear map. The same is true for the map $\tau: A \to B$ defined by $[c^\tau,1] = [c,1]^\alpha$. Therefore for a,b, $b-a \in A^*$ because of $(x^{-1})^\sigma = (x^\tau)^{-1}$ ... and $(x^{-1})^\tau = (x^\sigma)^{-1}$ for any $x \in A^*$ we obtain

$$\begin{aligned} ((a^{-1})+(b-a)^{-1})^\sigma &= ((a^{-1}+(b-a)^{-1})^\tau)^{-1} \\ &= ((a^{-1})^\tau + ((b-a)^{-1})^\tau)^{-1} \\ &= ((a^\sigma)^{-1} + (b^\sigma - a^\sigma)^{-1})^{-1}. \end{aligned}$$

Thus by proposition 1.4 now σ (and σ^{-1} too) are K-Jordan homomorphisms.

Let be J(K,A) the group of all K-Jordan automorphisms of A and $J_0(K,A)$ the subgroup of all K-Jordan automorphisms generating (fundamental) automorphisms of $\Sigma(K,A)$.

2.4. THEOREM. *The groups* F(K,A) *and* $J_0(K,A)$ *are isomorphic.*

The proof follows immediately from 2.1. and 2.3.

Using results of the following section and 1.2 we obtain

2.5. COROLLARY. 1. $J_0(K,A)$ *contains all K-algebra automorphisms and antiautomorphisms of* A.

2. *It is* $J_0(K,A) = J(K,A)$ *in the following cases:*

a) A *is a local algebra*

b) A *is semisimple*

c) A *is commutative of characteristic* $\neq 2$

In case c) J(K,A) wholly consists of automorphisms; if A is simple, J(K,A) wholly consists of auto- and antiautomorphisms. Also for A an alternative division ring $J_0(K,A) = J(K,A)$ is valid, see [9].

3. FUNDAMENTAL ISOMORPHISMS GENERATED BY K-JORDAN ISOMORPHISMS.

3.1. *Any* K-*algebra isomorphism* σ *generates a fundamental isomorphism, namely the map* $[a,b]\to[a^\sigma,b^\sigma]$.

From duality theorem in a free A-module of rank 2 we deduce the following

3.2. **LEMMA.** Let $a,b,a'b'\in A$ with $a'a+b'b = 1$. Then there is a unimodular pair (a_0,b_0) such that

$$(a,b)^\perp = \{(x,y)\in A\oplus A \mid xa+yb=0\} = [a_0,b_0].$$

3.3. **PROPOSITION.** *Any* K-*algebra antiisomorphism* $\eta: A\to B$ *generates a fundamental isomorphism* $\alpha: \Sigma(K,A)\to\Sigma(K,B)$.

Proof. We define $\alpha: \mathbb{P}(A)\to\mathbb{P}(B)$ as follows. Given $[a,b]\in\mathbb{P}(A)$, there are $a',b'\in A$ with $aa'+bb' = 1$.Then for $a^\eta,b^\eta,a'^\eta,b'^\eta$ we may apply Lemma 3.2 to obtain a unimodular pair $(a_0,b_0)\in B\oplus B$ with $a_0b^\eta-b_0a^\eta= 0$. Then let be $[a,b]^\alpha = (b^\eta,-a^\eta)^\perp = [a_0,b_0]$. In fact this map is welldefined and $[1,b]^\alpha = [1,b^\eta]$.

For $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in GL_2(A)$ also is $\begin{pmatrix} d^\eta & b^\eta\\ -c^\eta & -a^\eta\end{pmatrix}\in GL_2(B)$. Let $\begin{pmatrix} a_0 & b_0\\ c_0 & d_0\end{pmatrix} =$

$= \begin{pmatrix} d^\eta & b^\eta \\ -c^\eta & -a^\eta \end{pmatrix}^{-1}.$

Then $k\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\alpha = k\begin{pmatrix} a_o & b_o \\ c_o & d_o \end{pmatrix}$:

Namely from the identities

$$a_o b^\eta - b_o a^\eta = 0 = c_o d^\eta - d_o c^\eta$$

$$a_o d^\eta - b_o c^\eta = 1 = c_o b^\eta - d_o a^\eta$$

we obtain for $(0,0) \neq (s,t) \in K^{(2)}$

$$(s^\eta a_o - t^\eta c_o)(sb+td)^\eta - (s^\eta b_o - t^\eta d_o)(sa+tc)^\eta = 0,$$

so

$$[sa+tc,\ sb+td]^\alpha = [s^\eta a_o - t^\eta c_o,\ s^\eta b_o - t^\eta d_o].$$

3.4 **COROLLARY.** *Let* $A = A_1 \oplus \ldots \oplus A_n$ *and* α *a* K-*Jordan automorphism such that the restriction* σ_i *of* σ *to* A_i *is a* K-*algebra automorphism or antiautomorphism of* A_i *for* $i=1,\ldots,n$. *Then* σ *generates a fundamental automorphism* σ *of* $\Sigma(K,A)$.

Proof: By 3.1 and 3.3 σ_i generates a fundamental automorphism α_i of $\Sigma(K,A_i)$. Now using a categorial argument (see [5]) we have $\Sigma(K,A) = \prod_{i=1}^{n} \Sigma(K,A_i)$ where $\mathbb{P}(A) = \prod_{i=1}^{n} \mathbb{P}(A_i)$ is the cartesian product, and $\alpha = \prod_{i=1}^{n} \alpha_i$ is the fundamental automorphism of $\Sigma(K,A)$ inducing σ on A.

3.5. **REMARK.** For a semisimple algebra A with simple components $A_1,\ldots A_n$ any K-Jordan, automorphism of A up to a K-algebra automorphism is of the form described in 3.4 (see [1], this is valid also for char. 2!). So Corollary 3.4 may be applied:

Any K-Jordan automorphism of the semisimple algebra A generates a fundamental automorphism of $\Sigma(K,A)$

3.6. PROPOSITION. *For local algebras* A,B *any* K-*Jordan isomorphism* $\sigma : A \to B$ *generates a fundamental isomorphism* $\alpha : \Sigma(K,A) \to \Sigma(K,B)$.

Proof. Since A is local, any point of $\mathbb{P}(A)$ is either of the form [a,1] or of the form [1,b]. Define α by $[a,1]^\alpha = [a^\sigma,1]$ and $[1,b]^\alpha = [1,b^\sigma]$. For $a \in A^*$ this map is unambiguous, since $(a^{-1})^\sigma = (a^\sigma)^{-1}$ holds, see also [7].

We have to show, that α (and α^{-1}) map any four points of a chain ("cocatenal points") onto cocatenal points. W.l.o.g. the first three points are of the form [1,a],[1,b],[1,c], with $a-b$, $a-c$, $b-c \in A^*$; the fourth point let have the general form [x,y], where x=1 or y=1. Now we use the following criterion:

(*) If for a suitable $\gamma \in PGL_2(A)$ we have $[1,a]^\gamma = [0,1]$, $[1,b]^\gamma = [1,0]$, $[1,c]^\gamma = [1,1]$ and $[x,y]^\gamma = [x',y']$, then $y' = kx'$ holds for some $k \in K^*$, iff the four points are cocatenal.

For this we may take

$$[x,y]^\gamma = [(xa-y)((a-b)^{-1}-(a-c)^{-1}),\ (xb-y)(a-b)^{-1}].$$

For a point [x,y] distant to [1,a] by (i) in the proof of 2.1 is $(xa-y) \in A^*$, so we may reduce to

$$[x,y]^\gamma = [(a-b)^{-1} - (a-c)^{-1},\ (a-b)^{-1}-(xa-y)^{-1}x\ ]\text{, thus}$$

(i) $[1,z]^\gamma = [(a-b)^{-1} - (a-c)^{-1},\ (a-b)^{-1} - (a-z)^{-1}]$,

(ii) $[z,1]^{\gamma} = [(a-b)^{-1}-(a-c)^{-1}, \quad (a-b)^{-1}+z+\sum_{i=1}^{N}(za)^i z] \quad , \quad z\in J(A),$

where the last identity comes from the fact, that za is nilpotent, say $(za)^{N+1}=0$, and therefore $(1-za)^{-1} = 1 + \sum_{i=1}^{n}(za)^i$.

Now using $a^\sigma, b^\sigma, c^\sigma, z^\sigma$ instead of a,b,c,z by the properties of σ (see 1.2, 1.3), we have

$$(a-b)^{-1}-(a-z)^{-1} = k((a-b)^{-1}-(a-c)^{-1}), \quad k\in K$$

$$\Longleftrightarrow \quad (a^\sigma-b^\sigma)^{-1}-(a^\sigma-z^\sigma)^{-1}=k^\sigma((a^\sigma-b^\sigma)^{-1}-(a^\sigma-c^\sigma)^{-1}),$$

$$(a-b)^{-1} + z + \sum_{i=1}^{N}(za)^i z = k((a-b)^{-1}-(a-c)^{-1}), \quad k\in K$$

$$\Longleftrightarrow \quad (a^\sigma-b^\sigma)^{-1} + z^\sigma + \sum_{i=1}^{N}(z^\sigma a^\sigma)^i z^\sigma = k\ ((a^\sigma-b^\sigma)^{-1}-(a^\sigma-c^\sigma)^{-1}),$$

which means

$$[1,a],\ [1,b],\ [1,c],\ [x,y] \quad \text{cocatenal}$$
$$\Longleftrightarrow [1,a]^\alpha,\ [1,b]^\alpha,\ [1,c]^\alpha,\ [x,y]^\alpha \quad \text{cocatenal}.$$

4. PROPER K-JORDAN ISOMORPHISMS GENERATING FUNDAMENTAL ISOMORPHISMS.

Known examples of proper K-Jordan isomorphisms generating fundamental isomorphisms are those mentioned in Cor. 3.4 (see [1] with the question on pg. 367). We give here another class of examples, since we are still far away from a general theory.

Let $A=B\oplus M$ (direct sum as K-vector space), where B is a subalgebra of A and M is a two sided ideal with $M \subseteq N = J(A)$. Let I be the two sided ideal of A generated by all $x\in A$ such that $Mx\neq M$. We ask $I\neq M$. (Since the local case already is handled, we may suppose $|K|>3$.)

4.1. LEMMA. *Let* σ *be a* K-*Jordan isomorphism* $A \to C$ *such that*

$$\forall a \in A \quad \forall b \in B: \quad (ab)^\sigma = a^\sigma b^\sigma, \quad (ba)^\sigma = b^\sigma a^\sigma$$

$$\forall m_1, m_2 \in M \quad \exists m_3 \in M; \; m_1^\sigma m_2^\sigma = m_3^\sigma$$

is valid. Then σ *generates a fundamental isomorphism* $\alpha : \Sigma(K,A) \to \Sigma(K,C)$.

Proof: (i) Any point of $\mathbb{P}(A)$ is of the form $[a_0,b]$ or $[a,b_0]$ for $a_0,b_0 \in B \setminus I$: Let $[a,b] \in \mathbb{P}(A)$. Then either a or b does not belong to I. W.l.o.g. let be $a \in A \setminus I$, $a=a_0+a_1$ for $a_0 \in B$ and $a_1 \in M$. There is $m \in M$ for which $ma=a_1$ holds. Using $1-m \in A^*$, for $c=(1-m)b$ we obtain $[a,b]=[a_0,c]$.

(ii) Define α by $[a_0,b]^\alpha = [a_0^\sigma, b^\sigma]$, $a_0 \in B \setminus I$,

$$[a,b_0]^\alpha = [a^\sigma, b_0^\sigma], \; b_0 \in B \setminus I.$$

This is well defined: Let $[a_0,b]=[u_0,v]$, $u_0 \in B$. There is $c \in A^*$ with $ca_0 = u_0$, $cb = v$. So for $c=c_0+c_1$, $c_0 \in B$, $c_1 \in M$, we have $u_0 = (c_0+c_1)a_0 = c_0a_0 + c_1a_0$ with $c_0a_0 \in B$ and $c_1a_0 \in M$ with $Ma_0=M$, from which follows $c_1=0$. Therefore $u_0^\sigma = (c_0a_0)^\sigma = c_0^\sigma a_0^\sigma$

$$v^\sigma = (c_0b)^\sigma = c_0^\sigma b^\sigma$$

and so $[u_0,v]^\sigma = [a_0,b]^\alpha$

On the other hand let $[a_0,b] = [u,v_0]$, $v_0 \in B$ and $z \in A^*$ with $za_0 = u$, $zb = v_0$. Then since $b^\sigma = (z^{-1}v_0)^\sigma = (z^\sigma)^{-1}v_0^\sigma$ we have

$$[u,v_0]^\alpha = [u^\sigma, v_0^\sigma] = [z^\sigma a_0^\sigma, z^\sigma b^\sigma] = [a_0^\sigma, b^\sigma] = [a_0,b]^\alpha$$

too.

(iii) α (and α^{-1}) map any four cocatenal points onto cocatenal points:

Supposing $|K|>3$ we may assume the four points in the form

$$\underline{\underline{a}} = [a_o, a_1+a_2], \ldots, \underline{\underline{d}} = [d_o, d_1+d_2],$$

$a_o, a_1, \ldots, d_o, d_1 \in B$, $a_2, \ldots, d_2 \in M$.

Looking on the criterion (*) in the proof of 3.6 and reading modulo M shows, that also $[a_o, a_1], \ldots, [d_o, d_1]$ are cocatenal.

So there is some $\gamma \in GL_2(B)$, inducing $\beta \in PGL_2(A)$, which maps the points $a, \underline{\underline{b}}; \underline{\underline{c}}$ to $[0,1]$, $[1,0]$ and $[1,1]$, say $\gamma = \begin{pmatrix} g_1 & g_2 \\ h_1 & h_2 \end{pmatrix}$, and there are units a,b,c,d from B such that for $k \in K$ holds

$$[a_o, a_1+a_2]^\beta = [a_2h_1; a+a_2h_2] = [a'h_1, 1-a'h_2]$$

$$[b_o, b_1+b_2]^\beta = [b+b_2h_1, b_2h_2] = [1-b'h_1, b'h_2]$$

$$[c_o, c_1+c_2]^\beta = [c+c_2h_1, c+c_2h_2] = [1-c'h_1, 1-c'h_2]$$

$$[d_o, d_1+d_2]^\beta = [d+d_2h_1, kd+d_2h_2] = [1-d'h_1, k-d'h_2].$$

In the same way one may deduce for β' induced by $\gamma^\sigma = \begin{pmatrix} g_1^\sigma & g_2^\sigma \\ h_1^\sigma & h_2^\sigma \end{pmatrix}$:

$$\underline{\underline{a}}^{\alpha\beta'} = [a'^\sigma h_1^\sigma, 1-a'^\sigma h_2^\sigma]$$

$$\ldots$$

$$\underline{\underline{d}}^{\alpha\beta'} = [1-d'^\sigma h_1^\sigma, k^\sigma - d'^\sigma h_2^\sigma].$$

Now we may consider $\underline{\underline{a}}^\beta, \ldots, \underline{\underline{d}}^\beta$, $\underline{\underline{a}}^{\alpha\beta'}, \ldots, \underline{\underline{d}}^{\alpha\beta'}$ as points of $\Sigma(K, K \oplus M)$, $\Sigma(K, K \oplus M^\sigma)$ respectively, which are chain geometries over local algebras, where by 3.6 α operates as fundamental isomorphism.

So it suffices to show

$\underline{\underline{a}}^{\alpha\beta'} = \underline{\underline{a}}^{\beta\alpha},\ldots\underline{\underline{d}}^{\alpha\beta'} = \underline{\underline{d}}^{\beta\alpha}$.

We calculate this for d with $(1-d'h_1)^{-1} = 1 + \sum_{i=1}^{N}(d'h_1)^i$:

$$\underline{\underline{d}}^{\beta\alpha}=[1,(1+\sum_{i=1}^{N}(d'h_1)^i(k-d'h_2))^\sigma]$$
$$= [1,k^\sigma + k^\sigma \sum_{i=1}^{N}((d'h_1)^\sigma)^i-(d'^\sigma+\sum_{i=1}^{N}(d'^\sigma h_2^\sigma)^i d'^\sigma)h_2^\sigma\ ]$$
$$=[1,1+\sum_{i=1}^{N}(d'^\sigma h_1^\sigma)^i(k^\sigma - d'^\sigma h_2^\sigma)|$$
$$= [1-d'^\sigma h_1^\sigma\ ,\ k^\sigma - d'^\sigma h_2^\sigma\ ]= \underline{\underline{d}}^{\alpha\beta'}.$$

4.2. CONSTRUCTION OF A CLASS OF EXAMPLES.

For the construction we need (i) a D-algebra B with epimorphism $B \to D$; $b\to\bar{b}$, where D itself is a commutative local K-algebra, (ii) a D-left-module M with a D-admissible(2) associative multiplication such that $m^2 = 0\ \ \forall m\in M$.

(i) *First construction:* Let C be any D-algebra and construct the algebra $B = D \oplus C$ by the multiplication $(d_1,c_1)(d_2,c_2)=(d_1d_2, d_1c_2+d_2c_1+c_1c_2)$. The epimorphism $B\to D$ is given by $(d,c)\mapsto d$.

Second construction: For a finite group G let be B=DG the group algebra over D with epimorphism $\sum_{g\in G} d_g g \mapsto \sum_{g\in G} d_g$.

(ii) Let M=U⊕W be the direct sum of D-left modules U and W. U free with base $u_1,\ldots,u_r$ and $w_{ij}\in W$ for $1 \leq i < j \leq r$.

For $m = \sum_{i=1}^{r} x_iu_i+w$, $m' = \sum_{j=1}^{r} y_ju_j+w'$, $w,w'\in W$ define $mm' = \sum_{1\leq i<j\leq r}(x_iy_j-x_jy_j)w_{ij}$.

(2) $\forall d\in D\ \ \forall m,m'\in M : (dm)m' = d(mm') = m(dm')$.

Clearly this multiplication is D-admissible, furthermore $m^2=0$ $\forall m \in M$ and $m_2 m_1 = - m_1 m_2$. Since $M^3=0$ this multiplication also is associative.

(iii) Let $A = B \oplus M$ where for $b \in B$ and $m \in M$ the multiplication is defined by $bm = \bar{b}m = mb$. Using the maximal ideal N of D the ideal I of A according to 4.1 may be described by $I = C \oplus M$, where C is the kernel of the epimorphism $B \to D/N$; $b \mapsto \bar{b}+N$. To obtain a K-Jordan isomorphism σ we construct another algebra $A'=B'\oplus M'$, where B and B' are isomorphic as D-algebras by means of an isomorphism σ_1 compatible with the homomorphisms $B \to D$ and $B' \to D$:

$$\forall b \in B : \bar{b} = \overline{b^{\sigma_1}},$$

whereas M and M' only are isomorphic as D-modules, say by means of a D-isomorphism σ_2. We show, that $\sigma := \sigma_1 \oplus \sigma_2$ is a Jordan isomorphism: Let $a = b+c$ be a unit of A for $b \in B$ and $c \in M$. Then b is a unit in B and $\bar{b}$ is a unit in D. So we can write $a=b+\bar{b}m$ with $m \in M$ and see $a^{-1}=b^{-1}-\bar{b}^{-1}m$. Now $(a^{-1})^{\sigma} = (b^{\sigma})^{-1} - (\overline{b^{\sigma}})^{-1} m^{\sigma} = (a^{\sigma})^{-1}$. Moreover $(bm)^{\sigma} = (\bar{b}m)^{\sigma} = \bar{b}m^{\sigma} = \overline{b^{\sigma}} m^{\sigma} = b^{\sigma} m^{\sigma}$.

4.3.REMARKS: 1. For $A = B \oplus M$ in the preceding example a K-Jordan automorphism $\sigma = \sigma_1 \oplus \sigma_2$ may be obtained putting $\sigma_1 = id_B$ and σ_2 any D-linear automorphism of M (as D-module). Here also in general are many proper K-Jordan automorphisms.

2. Let B be commutative. For characteristic $\neq 2$ we may construct an antiisomorphism $\eta = \eta_1 \oplus \eta_2$ of $A = B \oplus M$ by $\eta_1 = id_B$ and $\eta_2 : M \to M$; $m \mapsto -m$. For characteristic 2 the constructions in 4.2, 4.3,1 give examples of proper K-Jordan isomorphisms (or automorphisms) of commutative algebras.

3. For B a local algebra, A itself is local. Now consider this for the most simple case B=D=K: All K-algebras $A=K\oplus M$ with $m^2=\acute{0}$ for all $m\in M$ and fixed K-dimension d of M are Jordan-isomorphic and so give isomorphic chain geometries (so-called Laguerre geometries, see [3]). These algebras are kinematic, and their Hotje-representation is given by a quadric Q where the chains are those plane sections of Q which contain 3 non-collinear points, but do not contain any line (see [6]). The quadric Q is defined in the projective space belonging to the K-vector space $V = K \oplus K \oplus A = E \oplus M$, where $E = K\oplus K+K$ by $xy=a\bar{a}$ for $(x,y)\in V$; or better with $a=z+m$ by $xy=z^2$ for $(x,y,z,m)\in V$, namely $(x,y,z)\in E$ and $m\in M$.

This is a big cone over a conic, and its shape is entirely independent of the multiplication on M. The automorphisms of the quadric fixing one such conic and three points on it are induced by the semilinear invertible mappings on V which fix M and the canonical base of E. But these mappings now on A induce all the K-Jordan automorphisms, see also [10].

REFERENCES

[1] C.BARTOLONE and F.BARTOLOZZI, *Topics in Geometric algebra over rings, in:* Rings and Geometry, E .R.Kaya, P.Plaumann, K.Strambach, 1985, 353-389.

[2] W.BENZ, Zur Geometrie der Körpererweiterungen, *Canad.J.Math.* 21, 1969, 1097-1122.

[3] W.BENZ, *Vorlesungen über Geometrie der Algebren,* Berlin-Heidelberg - New York, 1973.

[4] W.BENZ, H.-J.SAMAGA, H.SCHAEFFER, *Cross Ratios and a Unifyin Treatment of Reeller Zug, in* Geometry - von Staudt's Point of View, Ed.P.Plaumann, K.Strambach, 1981, 127-150.

[5] A.HERZER, Die Kategorie der Kettengeometrien, *Results in Mathematics* 12, 1987, 278-288.

[6] H.HOTJE, Einbettung gewisser Kettengeometrien in projektive Räume, *J.Geometry* 5/1, 1974, 85-94.

[7] B.V.LIMAYE, N.B.LIMAYE, The Fundamental Theorem for the Projective Line over Non-Commutative Local Rings, *Arch.Math.* 28,1977, 102-109.

[8] H.MÄURER,R.METZ,W.NOLTE, Die Automorphismengruppe der Mobius-Geometrie einer Körpererweiterung, *Aequationes Math.* 21,1980, 110-112.

[9] H.SCHAEFFER,Zur Möbius-Geometrie über Alternativkörper,*Geometriae Dedicata,* 10, 1981, 183-189.

[10] M.WERNER, Zur Darstellung der Automorphismengruppe einer kinematischen Kettengeometrie im Quadrikenmodell, *Journal of Geometry* 19, 1982, 146-153.

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