Note di Matematica Vol. VIII - n. 1, 45-56(1988)

INTERPOLATIVE CONSTRUCTIONS FOR OPERATOR IDEALS

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The problem from which this article originated is the following: given an operator T: $E \to F$ between Banach spaces belonging simultaneously to two operator ideals, $\mathcal{A}$ and $\mathcal{B}$ say, when is it possible to find a decomposition $T = A \cdot B$, where $A \in \mathcal{A}$ and $B \in \mathcal{B}$, or at least $A \in \dot{\mathcal{A}}$ and $B \in \ddot{\mathcal{B}}$, with $\dot{\mathcal{A}}$ and $\ddot{\mathcal{B}}$ being associated with $\mathcal{A}$ and $\mathcal{B}$ in a specific sense? It was shown by S. Heinrich [2] that such a decomposition is always possible, with $\mathcal{A} = \dot{\mathcal{A}}$ and $\mathcal{B} = \ddot{\mathcal{B}}$, if $\mathcal{A}$ and $\mathcal{B}$ are uniformly closed, $\mathcal{A}$ is surjective, and $\mathcal{B}$ is injective.

Heinrich's arguments are based on a simple interpolation technique which appears to be strongy related to certain general constructions with opertor ideals that were successfully applied in a seemingly different context in recent years (ref. [8], [5], and [4]-[7], [1]). We intend to investigate the fundamentals of such constructions and their interpolation-theoretic background in this paper, with emphasis on the impact to the factorization problem. Applications will be given for ideals generated by s-number sequences and to type p and cotype q operators.

1. PRELIMINARIES

We shall use standard notation and terminology for Banach spaces. As for the theory of operator ideals, we shall essentially follow A. Pietsch's monograph [9].

$\mathcal{A}, \mathcal{B}$ ... will always denote operator ideals, and $E, F, G, \ldots$ are used to denote Banach spaces. If E is a Banach space, then E^* will be its dual, and B_E will be the unit ball of E.

Given an ideal $\mathcal{A}$, its *injective hull*, $\mathcal{A}^i$, is defined via

$$T \in \mathcal{A}^i(E, F) :\Leftrightarrow J_F T \in \mathcal{A}(E, F_\infty);$$

here F_∞ is the (injective) Banach space $\ell_\infty(B_{F^*})$, and J_F: $F \to F_\infty$ is the canonical isometric embedding. Dually, the *surjective hull*, $\mathcal{A}^s$, of $\mathcal{A}$ is defined via

$$T \in \mathcal{A}^s(E, F) :\Leftrightarrow T Q_E \in \mathcal{A}(E_1, F);$$

here E_1 is the (surjective) Banach space $\ell_1(B_E)$, and Q_E: $E_1 \to E$ is the canonical metric surjection. $\mathcal{A}$ is said to be *injective (surjective)* if $\mathcal{A} = \mathcal{A}^i$ (if $\mathcal{A} = \mathcal{A}^s$) holds.

Finally, we write $\overline{\mathcal{A}}$ for the ideal of all operators $T : E \to F$ which satisfy $\lim_{n\to\infty} ||T - T_n|| = 0$ for some sequence (T_n) in $\mathcal{A}(E, F)$. If $\mathcal{A} = \overline{\mathcal{A}}$, then $\mathcal{A}$ is said to be *closed*. More generally, if $\mathcal{A}$ and $\mathcal{B}$ are quasi-normed ideals such that (component-wise) $\mathcal{A} \subset \mathcal{B}$, then $\overline{\mathcal{A}}^{\mathcal{B}}$ is used to denote the ideal of all operators in $\mathcal{B}$ which can be obtained as limits of sequences from $\mathcal{A}$ in the quasi-norm of $\mathcal{B}$.

Let us start by recalling a simple and well-known characterization of the injective resp. surjective hull of an ideal (cfr. e.g. [9]):

(1) Let $T \in \mathcal{L}(E, F)$ be given.

(a) T belongs to $\mathcal{A}^i$ if and only if there exist G and $S \in \mathcal{A}(E, G)$ such that $\|Tx\| \leq \|Sx\| \forall x \in E$.

(b) T belongs to $\mathcal{A}^s$ if and only if there exist G and $S \in \mathcal{A}(G, F)$ such that $T(B_E) \subset S(B_G)$.

Less known, but equally simple to verify, is the following:

(2) Let $T \in \mathcal{L}(E, F)$ be given.

(a) T belongs to $\overline{\mathcal{A}}^i$ if and only if, for every $\epsilon > 0$, there are G_ϵ and $S_\epsilon \in \mathcal{A}(E, G_\epsilon)$ such that $\|Tx\| \leq \|S_\epsilon x\| + \epsilon \cdot \|x\| \forall x \in E$.

(b) T belongs to $\overline{\mathcal{A}}^s$ if and only if, for every $\epsilon > 0$, there are G_ϵ and $S_\epsilon \in \mathcal{A}(G_\epsilon, F)$ such that $T(B_E) \subset S_\epsilon(B_{G_\epsilon}) + \epsilon \cdot B_F$.

The presence of a complete (ideal) quasi-norm yields a considerable simplification:

(3) Suppose $\mathcal{A}$ admits a complete quasi-norm. Let $T \in \mathcal{L}(E, F)$ be given.

(a) T belongs to $\overline{\mathcal{A}}^i$ if and only if there are G, $S \in \mathcal{A}(E, G)$, and a function $N : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\|Tx\| \leq N(\epsilon) \cdot \|Sx\| + \epsilon \cdot \|x\| \forall x \in E, \forall \epsilon > 0$.

(b) T belongs to $\overline{\mathcal{A}}^s$ if and only if there are G, $S \in \mathcal{A}(G, F)$, and a function $N : \mathbb{R}_+ \to \mathbb{R}_+$ such that $T(B_E) \subset N(\epsilon) \cdot S(B_G) + \epsilon \cdot B_F, \forall \epsilon > 0$.

(a) of (2) and (3) is proved in [3], the corresponding statements (b) are due to T. Terzioğlu (unpublished) and can be proved in a dual fashion. Of course, (3) can be generalized as follows:

(4) Let $\mathcal{A}$ an $\mathcal{B}$ be complete quasi-normed ideals such that $\mathcal{A} \subset \mathcal{B}$. Let $T \in \mathcal{L}(E, F)$ be given.

(a) T belongs to $(\overline{\mathcal{A}}^{\mathcal{B}})^i$ if and only if there are G, H, $S \in \mathcal{A}(E, G)$, $R \in \mathcal{B}(E, H)$, and a function $N\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that, $\forall x \in E, \forall \epsilon > 0 : \|Tx\| \leq N(\epsilon) \cdot \|Sx\| + \epsilon \cdot \|Rx\|$.

(b) T belongs to $(\overline{\mathcal{A}}^{\mathcal{B}})^s$ if and only if there are G, H, $S \in \mathcal{A}(G, F)$, $R \in \mathcal{B}(H, F)$, and a function $N : \mathbb{R}_+ \to \mathbb{R}_+$ such that, $\forall \epsilon > 0 : T(B_E) \subset N(\epsilon) \cdot S(B_G) + \epsilon \cdot R(B_H)$.

It is not our intention to strive for still more generality. However, starting from (4), we shall study in some detail in the next section the effect of imposing certain growth conditions on our function $N(\epsilon)$.

The above concepts have been gradually developed in the context of weakly compact operators on $C(K)$-space in particular. About 1977 it became clear that if X is an L_∞-space (or the disk algebra, ...), then $\mathcal{W}(X, \cdot) = \overline{\Pi_p}^i(X, \cdot) = \overline{\Gamma_2}^i(X, \cdot)$ (cfr. [8], [3]). Here we denote (as usual) by $\mathcal{W}$, Π_p, and Γ_2 the ideals of all weakly compact operators, of all p-summing operators $(1 \leq p < \infty)$, and of all L_2-factorable operators, respectively. More recently, it was shown that $\mathcal{W}(X,\cdot) = \overrightarrow{\Gamma_2}^{i}(X,\cdot)$ even holds if X is a C^*-algebra, or only a JB^*-triple. As a consequence, one gets for example that every reflexive quotient of such a space is automatically super-reflexive and of type 2, which not only gives another proof but also a considerable generalization of a well-known theorem of Rosenthal [13] about quotients of L_∞ resp. subspaces of L_1. See [5], [4], and [1] for further details. Furtheron, it was shown in [6] that among the Banach spaces Y with GL-local unconditional structure, the condition $\mathcal{L}(L_1,Y) = \overrightarrow{\Gamma_2}^{i}(L_1,Y)$ characterizes precisely the super-reflexive ones.

2. INTERPOLATIVE CONSTRUCTIONS

One way of quantifying the results mentioned at the end of the preceding section is by analyzing the effect of assuming that the function $N(\epsilon)$ appearing in (3) and (4) is majorized by a function of the form $C \cdot \epsilon^{-r}$, for fixed C, $r > 0$. This leads to the following concepts.

Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals. Fix $0 < \theta < 1$, and put $r = \dfrac{\theta}{1-\theta}$. Given an operator $T\colon E \to F$, write

$$T \in (\mathcal{A},\mathcal{B})_\theta(E,F)$$

if there are G, H, $S \in \mathcal{A}(E, G)$, and $R \in \mathcal{B}(E, H)$ such that

$$(*) \qquad ||Tx|| \in \epsilon^{-r} \cdot ||Sx|| + \epsilon \cdot ||Rx|| \quad \forall x \in E, \quad \forall \epsilon > 0 .$$

Similarly, we write

$$T \in (\mathcal{A},\mathcal{B})^\theta(E,F)$$

if there are G, H, $S \in \mathcal{A}(G, F)$, and $R \in \mathcal{B}(H, F)$ such that

$$(**) \qquad T(B_E) \subset \epsilon^{-r} \cdot S(B_G) + \epsilon \cdot R(B_H) \quad \forall \epsilon > 0 .$$

It is easily seen that (*) is equivalent with

$$(*') \qquad ||Tx|| \leq c \cdot ||Sx||^{1-\theta} \cdot ||Rx||^\theta \quad \forall x \in E,$$

where c is a constant depending only on θ ($c = \left(\dfrac{1-\theta}{\theta}\right)^\theta + \left(\dfrac{\theta}{1-\theta}\right)^{1-\theta}$ in fact). This reminds of what is called «intermediate space of J-type θ» in interpolation theory. Similarly, (**) bears features reminiscent of the so-called «intermediate spaces of K-type θ». We refer to [14] for this and for further results from interpolation theory to be used later.

It is easily seen that $(\mathcal{A}, \mathcal{B})_\theta$ and $(\mathcal{A}, \mathcal{B})^\theta$ are operator ideals. If $\mathcal{A}$ and $\mathcal{B}$ are given (ideal) quasi-norms α and β, respectively, then we obtain quasi-norms on $(\mathcal{A},\mathcal{B})_\theta$ and $(\mathcal{A},\mathcal{B})^\theta$ by taking the infimum over all expressions of the form $\alpha(S)^{1-\theta} \cdot \beta(R)^\theta$, with S and R admissible in (*) resp. in (**). It is to these quasi-norms we refer in the next proposition:

(5) Let $\mathcal{A}$ and $\mathcal{B}$ be operator ideals, and let $0 < \theta < \eta < 1$.

(a) $(\mathcal{A}, \mathcal{B})_\theta$ is injective, and $(\mathcal{A}, \mathcal{B})^\theta$ is surjective.

(b) $(\mathcal{A} \cap \mathcal{B})^i \subset (\mathcal{A}, \mathcal{B})_\theta \subset (\mathcal{A}, \mathcal{B})_\eta \subset (\mathcal{A} + \mathcal{B})^i$ and $(\mathcal{A} \cap \mathcal{B})^s \subset (\mathcal{A}, \mathcal{B})^\theta \subset (\mathcal{A}, \mathcal{B})^\eta \subset (\mathcal{A} + \mathcal{B})^s$.

(c) If $\mathcal{A} \subset \mathcal{B}$, then $(\mathcal{A}, \mathcal{B})_\theta \subset (\overline{\mathcal{A}}^{\mathcal{B}})^i$ and $(\mathcal{A}, \mathcal{B})^\theta \subset (\overline{\mathcal{A}}^{\mathcal{B}})^s$.

(d) If $\mathcal{A}$ and $\mathcal{B}$ admit complete quasi-norms, then the same is true for the ideals $(\mathcal{A}, \mathcal{B})_\theta$ and $(\mathcal{A}, \mathcal{B})^\theta$, and all the inclusions in (b) and (c) become contractive embeddings with respect to these canonical quasi-norms.

(e) If $\mathcal{A}$ and $\mathcal{B}$ are ultra-stable (power-stable), then so are $(\mathcal{A}, \mathcal{B})_\theta$ and $(\mathcal{A}, \mathcal{B})^\theta$.

See [9] for the definition of ultra-stability. Power-stability refers in the same manner to stability with respect to ultra-powers of the operators in question.

In an obvious way, the two types of ideals introduced above are essentially dual to each other. Possible obstructions are due to the usual problems which one encounters when one passes to second adjoints. Recall from [9], that given any ideal $\mathcal{A}$, the dual ideal, $\mathcal{A}^d$, is defined via $T \in \mathcal{A}^d(E, F) :\Leftrightarrow T^* \in \mathcal{A}(F^*, E^*)$. Similarly, we write $\mathcal{A}^r$ for the ideal defined via $T \in \mathcal{A}^r(E, F) :\Leftrightarrow e_F T \in \mathcal{A}(E, F^{**})$; here e_F denotes the canonical embedding $F \to F^{**}$.

(6) Suppose $\mathcal{A} \subset \mathcal{A}^{dd}$ and $\mathcal{B} \subset \mathcal{B}^{dd}$ (so $S \in \mathcal{A}$ implies $S^{**} \in \mathcal{A}$, etc.). Then $(\mathcal{A}^d, \mathcal{B}^d)_\theta = [(\mathcal{A}, \mathcal{B})^\theta]^d$ and $[(\mathcal{A}^d, \mathcal{B}^d)^\theta]^r = [(\mathcal{A}, \mathcal{B})_\theta]^d$.

Proof. We only prove the second statement, the first one can be settled similarly. Let $T \in [(\mathcal{A}^d, \mathcal{B}^d)^\theta]^r(E, F)$ be given, and let $S \in \mathcal{A}^d(G, F^{**})$ and $R \in \mathcal{B}^d(H, F^{**})$ be such that $e_F T(B_E) \subset \epsilon^{-r} \cdot S(B_G) + \epsilon \cdot R(B_H)$, $\forall \epsilon > 0$, where $r = \dfrac{\theta}{1-\theta}$. By polarization, we get $||T^* y^*|| \leq \epsilon^{-r} \cdot ||S^* e_{F^*} y^*|| + \epsilon \cdot ||R^* e_{F^*} y^*||$ $\forall y^* \in F^*, \forall \epsilon > 0$. As $S^* e_{F^*} \in \mathcal{A}$ and $R^* e_{F^*} \in \mathcal{B}$ we conclude $T^* \in (\mathcal{A}, \mathcal{B})_\theta(F^*, E^*)$, that is, $T \in [(\mathcal{A}, \mathcal{B})_\theta]^d(E, F)$.

Suppose conversely that T^* belongs to $(\mathcal{A}, \mathcal{B})_\theta$. Let $S \in \mathcal{A}(F^*, G)$ and $R \in \mathcal{B}(F^*, H)$ be such that $||T^* y^*|| \leq \epsilon^{-r} \cdot ||S y^*|| + \epsilon \cdot ||R y^*||$ $\forall y^* \in F^*, \forall \epsilon > 0$. By our hypothesis, $S^{**} \in \mathcal{A}$ and $R^{**} \in \mathcal{B}$, so that $S^* \in \mathcal{A}^d$ and $R^* \in \mathcal{B}^d$. Again polarization leads to $e_F T \in (\mathcal{A}^d, \mathcal{B}^d)^\theta$, so that $T \in [(\mathcal{A}^d, \mathcal{B}^d)^\theta]^r$. ∎

An interesting case occurs when $\mathcal{B}$ is the ideal $\mathcal{L}$ of all operators. We shall simplify our notation in this case and write

$$\mathcal{A}_\theta := (\mathcal{A}, \mathcal{L})_\theta \quad \text{and} \quad \mathcal{A}^\theta := (\mathcal{A}, \mathcal{L})^\theta.$$

So, if in addition $\mathcal{A} \subset \mathcal{A}^{dd}$, we get from (6)

$$(\mathcal{A}^d)_\theta = (\mathcal{A}^\theta)^d \quad \text{and} \quad [(\mathcal{A}^d)^\theta]^r = (\mathcal{A}_\theta)^d.$$

Another result to be used later is the following one:

(7) For all ideals $\mathcal{A}$ and $\mathcal{B}$ and all $0 < \theta < 1$ we have

$$[(\mathcal{A},\mathcal{B})_\theta]^s = [(\mathcal{A},\mathcal{B})^\theta]^i.$$

Proof . (a) Let $T \in (\mathcal{A}, \mathcal{B})^\theta(E, F)$ be given. Then $TQ_E(B_{E_1}) \subset \epsilon^{-r} \cdot S(B_G) + \epsilon \cdot R(B_H)$ for all $\epsilon > 0$, with suitably chosen operators $S \in \mathcal{A}(G, F)$ and $R \in \mathcal{B}(H, F)$.

Let now $\epsilon > 0$ be fixed. Given $x \in B_E$, let e_x be the corresponding standard unit vector in $E_1 = \ell_1(B_E)$. We may choose $g_x \in B_G$ and $h_x \in B_H$ such that $TQe_x = \epsilon^{-r} \cdot Sg_x + \epsilon \cdot Rh_x$. Now $e_x \mapsto g_x$ and $e_x \mapsto h_x$ define operators $U : E_1 \to G$ resp. $V : E_1 \to H$ such that $||U||, ||V|| \leq 1$ and $TQ_E = \epsilon^{-r} \cdot SU + \epsilon \cdot RV$. It follows that TQ_E belongs to $(\mathcal{A}, \mathcal{B})_\theta(E_1, F)$, so that $T \in [(\mathcal{A}, \mathcal{B})_\theta]^s(E, F)$.

(b) Using the extension property of the space $\ell_\infty(I)$, one proves in a similar manner that $(\mathcal{A}, \mathcal{B})_\theta$ is contained in $[(\mathcal{A}, \mathcal{B})^\theta]^i$.

(c) The rest is by manipulating properly with injective and surjective hulls. In fact, from (a) and (b) we get

$$\begin{aligned}[(\mathcal{A},\mathcal{B})_\theta]^s &= [(\mathcal{A},\mathcal{B})_\theta]^{is} = [(\mathcal{A},\mathcal{B})_\theta]^{si} \subset [(\mathcal{A},\mathcal{B})^\theta]^{isi} = \\ &= [(\mathcal{A},\mathcal{B})^\theta]^i = [(\mathcal{A},\mathcal{B})^\theta]^{si} = \\ &= [(\mathcal{A},\mathcal{B})^\theta]^{is} \subset [(\mathcal{A},\mathcal{B})_\theta]^{sis} = [(\mathcal{A},\mathcal{B})_\theta]^s. \end{aligned}$$ ■

We mention in passing that $\bigcup_{0<\theta<1} \mathcal{A}_\theta$ and $\bigcup_{0<\theta<1} \mathcal{A}^\theta$ will in general not coincide with $\overrightarrow{\mathcal{A}}^i$ resp. $\overrightarrow{\mathcal{A}}^s$, even if $\mathcal{A}$ admits a complete quasi-norm. $\bigcup_{0<\theta<1} \mathcal{A}_\theta$ carries a complete quasi-norm if and only if $\mathcal{A}_\theta = \mathcal{A}_{\tilde{\theta}}$ for some $0 < \tilde{\theta} < 1$ and all $\tilde{\theta} \leq \theta < 1$ (independent of components) if and only if $\mathcal{A}_\theta = \overrightarrow{\mathcal{A}}^i$ for all $\tilde{\theta} \leq \theta < 1$; see [7]. Analogously for $\bigcup_{0<\theta<1} \mathcal{A}^\theta$.

More concrete examples are as follows: suppose X is a reflexive Banch space having no finite cotype. Then we get, using [6],

$$\mathcal{L}(L_\infty, X) = \overrightarrow{\Pi_2}^i(L_\infty, X) \neq \bigcup_\theta (\Pi_2)_\theta(L_\infty, X) = \bigcup_{p<\infty} \Pi_p(L_\infty, X),$$

and for the Schatten-von Neumann classes $\mathcal{S}_p$ on a Hilbert space H one has

$$\mathcal{K}(H, H) = \overrightarrow{\Pi_2}^i(H, H) \neq \bigcup_\theta (\Pi_2)_\theta(H, H) = \bigcup_{p<\infty} \mathcal{S}_p(H, H),$$

$\mathcal{K}$ being the ideal of all compact operators.

We pass now to extensions of the classes $\mathcal{S}_p$ to operator ideals defined by various s-number sequences in the sense of Pietsch [9]. Given any (pseudo) s-number sequence $s = (s_n(\cdot))$ for operators, we denote by $\mathcal{S}_p^{(s)}$ the class of all operators T between Banach spaces such that $(s_n(T))_n \in \ell_p$, $0 < p < \infty$. For convenience, we shall also put $\mathcal{S}_\infty^{(s)} = \mathcal{L}$.

(8) Suppose s is additive, or equivalent to an additive (pseudo) s-number sequence. Let $0 < p, q \leq \infty$ and $0 < \theta < 1$ be given. Define r by $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$. Then $(\mathcal{S}_p^{(s)}, \mathcal{S}_q^{(s)})_\theta \subset \mathcal{S}_r^{(s)}$ if s is injective, and $(\mathcal{S}_p^{(s)}, \mathcal{S}_q^{(s)})^\theta \subset \mathcal{S}_r^{(s)}$ if s is surjective.

We recall that our terminology is as in [9].

Proof. We restrict ourselves to proving the first statement, the proof for the second one is even slightly simpler.

Let then $T \in (\mathcal{S}_p^{(s)}, \mathcal{S}_q^{(s)})_\theta(E, F)$ be given. Let $S \in \mathcal{S}_p^{(s)}(E, G)$ and $R \in \mathcal{S}_q^{(s)}(E, H)$ be such that

$$(*) \qquad ||Tx|| \leq \epsilon^{-\alpha} \cdot ||Sx|| + \epsilon \cdot ||Rx|| \quad \forall x \in E, \quad \forall \epsilon > 0,$$

where $\alpha = \theta \cdot (1-\theta)^{-1}$. Since we are dealing with injective ideals, we may suppose $F = \ell_\infty(I)$, I a suitably chosen set. We proceed as in [3], §20, 7.

Fix $\epsilon > 0$ and define $A\colon X \to G \oplus_1 H\colon x \mapsto (\epsilon^{-\alpha} \cdot Sx, \epsilon \cdot Rx)$, $G \oplus_1 H$ being the ℓ_1-direct sum of G and H. A is linear and continuos. By virtue of (*), a well-defined map $B\colon A(E) \to F$ is obtained by setting $B(Ax) := Tx, \forall x \in E$. B is linear and continuous, with $||B|| \leq 1$. Since $F = \ell_\infty(I)$, B is the restriction of some $\widetilde{B} \in \mathcal{L}(G \oplus_1 H, F)$ with $||\widetilde{B}|| \leq 1$. Define now operators $U\colon E \to F\colon x \mapsto (\epsilon^{-\alpha} \cdot Sx, 0)$ and $V\colon E \to F\colon x \mapsto \widetilde{B}(0, \epsilon \cdot Rx)$. Note that U is in $\mathcal{S}_p^{(s)}$, V is in $\mathcal{S}_q^{(s)}$, and $T = U + V$. Further, since s is additive, $s_{2m-1}(T) \leq s_m(U) + s_m(V)$, so that $s_{2m-1}(T) \leq s_m(S)^{1-\theta} \cdot s_m(R)^\theta$ for all m. But this implies $T \in \mathcal{S}_r^{(s)}$, because of

$$\begin{aligned}\Sigma s_m(T)^r &\leq 2 \cdot \Sigma s_{2m-1}(T)^r \leq 2 \cdot \Sigma s_m(S)^{r(1-\theta)} \cdot s_m(R)^{r\theta} \\ &\leq 2 \cdot (\Sigma s_m(S)^p)^{r(1-\theta)/p} \cdot (\Sigma s_m(R)^q)^{r\theta/q}.\end{aligned}$$

∎

Similar inclusions can be established for other types of ideals, notably those of p-summing operators, or for operators with certain type and cotype properties. See [6] and [7] for the first case, and the next section for type and cotype.

For operators on Hilbert spaces, all s-number sequences coincide and the corresponding ideals $\mathcal{S}_p^{(s)}$ just yield the Schatten-von Neumann classes $\mathcal{S}_p$. In that case, (8) can be improved:

(9) Let H be a Hilbert space. Given $0 < p, q \leq \infty$ and $0 < \theta < 1$, define r again by $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$. Then

$$\mathcal{S}_r(H, H) = (\mathcal{S}_p, \mathcal{S}_q)_\theta(H, H) = (\mathcal{S}_p, \mathcal{S}_q)^\theta(H, H).$$

Proof. One inclusion follows from (8), the other from Hölder's inequality. Let $T \in \mathcal{S}_r(H,H)$ be given. Then $T = \Sigma\lambda_n x_n \otimes y_n$, where $(\lambda_n) \in \ell_r$ and (x_n) and (y_n) are orthonormal sequences in H. We may assume $\lambda_n \geq 0 \ \forall n \in N$. By $S := \Sigma\lambda_n^{r/p} x_n \otimes y_n$ and $R := \Sigma\lambda_n^{r/q} x_n \otimes y_n$, we obtain operators $S \in \mathcal{S}_p(H,H)$ and $R \in \mathcal{S}_q(H,H)$. Since for each $x \in H$

$$\begin{aligned} ||Tx||^2 &= \Sigma\lambda_n^2 |(x|x_n)|^2 = \\ &= \Sigma\lambda_n^{2r(1-\theta)/p} \cdot |(x|x_n)|^{2(1-\theta)} \cdot \lambda_n^{2r\theta/q} \cdot |(x|x_n)|^{2\theta} \leq \\ &\leq (\Sigma\lambda_n^{2r/p} \cdot |(x|x_n)|^2)^{1-\theta} \cdot (\Sigma\lambda_n^{2r/q} \cdot |(x|x_n)|^2)^{\theta} \leq \\ &\leq (||Sx||^{1-\theta} \cdot ||Rx||^{\theta})^2, \end{aligned}$$

we conlcude that T is in $(\mathcal{S}_p, \mathcal{S}_q)_\theta(H, H)$.

Our hypothesis also gives $T^* \in \mathcal{S}_r(H, H)$, hence $T^* \in (\mathcal{S}_p, \mathcal{S}_q)_\theta(H, H)$. Since we are working in a Hilbert space, $T \in [(\mathcal{S}_p, \mathcal{S}_q)_\theta]^d(H, H) = (\mathcal{S}_p, \mathcal{S}_q)^\theta(H, H)$ follows, by (6). ∎

Remark: Interpolation between operator ideals can also be defined component-wise: given two complete quasi-normed ideals $\mathcal{A}$ and $\mathcal{B}$, one may define e.g. $(\mathcal{A}, \mathcal{B})_{\theta,q}$ (real method) via

$$(\mathcal{A},\mathcal{B})_{\theta,q}(E,F) := (\mathcal{A}(E,F),\mathcal{B}(E,F))_{\theta,q}(0 < \theta < 1, 1 \leq q \leq \infty)$$

for all E and F. It can be shown that

$$(\mathcal{A},\mathcal{B})^i_{\theta,1} \subset (\mathcal{A},\mathcal{B})_\theta \subset (\mathcal{A},\mathcal{B})^i_{\theta,\infty}$$

and

$$(\mathcal{A},\mathcal{B})^s_{\theta,1} \subset (\mathcal{A},\mathcal{B})^\theta \subset (\mathcal{A},\mathcal{B})^s_{\theta,\infty}$$

hold.

3. THE PRODUCT PROBLEM

If $\mathcal{A}$ and $\mathcal{B}$ are ideals, then $T \in \mathcal{L}(E, F)$ is said to belong to $\mathcal{A} \cdot \mathcal{B}(E, F)$ if there are G and $S \in \mathcal{A}(G, F)$, $R \in \mathcal{B}(E, G)$ such that $T = S \cdot R$. $\mathcal{A} \cdot \mathcal{B}$ is again an ideal, called the product of $\mathcal{A}$ and $\mathcal{B}$.

It was shown by S. Heinrich [2] that

$$\mathcal{A} \cap \mathcal{B} = \mathcal{A} \cdot \mathcal{B}$$

holds whenever $\mathcal{A}$ and $\mathcal{B}$ are closed, $\mathcal{A}$ is surjective, and $\mathcal{B}$ is injective. It is open if and how these requirements can be relaxed. We may generalize, however, Heinrich's result in a somewhat different direction. His arguments, which are based on elementary interpolation techniques, fit quite nicely with the concepts developed in the previous sections.

Let $T\colon E \to F$ be any operator. Put $X := E/\ker T$, and denote by $Q\colon E \to X$ the canonical quotient map and by $\widetilde{T}\colon X \to F$ the unique map such that $\widetilde{T} \cdot Q = T$. As $\widetilde{T}$ is an injective bounded operator, we may consider (X, F) as an interpolation couple. Let $0 < \theta < 1$ be given, and let G be any intermediate space of (X, F) which is simultaneously of J-type θ and of K-type θ. Examples are provided by the real interpolation spaces, $(X, F)_{\theta,q}$, or by the complex interpolation spaces, $[X, F]_\theta$; cf. [14]. Let $S : X \to G$ and $R : G \to F$ be the corresponding canonical operators, so that $R \cdot S = \widetilde{T}$.

Let now $\mathcal{A}$ and $\mathcal{B}$ be again any ideals, and suppose that T belongs to $\mathcal{A}^s \cap \mathcal{B}^i$. It is then easily checked that SQ is in $(\mathcal{B}^i)_{1-\theta} = \mathcal{B}_{1-\theta}$, because G is of J−type θ, and that R is in $(\mathcal{A}^s)^\theta = \mathcal{A}^\theta$, because G is of K−type θ. So we have:

(10) For all ideals $\mathcal{A}$ and $\mathcal{B}$, we have
$$\mathcal{A}^s \cap \mathcal{B}^i \subset \bigcap_{0<\theta<1} \mathcal{A}^\theta \cdot \mathcal{B}_{1-\theta}.$$

The latter intersection is in turn contained in $\overline{\mathcal{A}}^s \cdot \overline{\mathcal{B}}^i$, so that Heinrich's result appears if we require $\mathcal{A} = \overline{\mathcal{A}}^s$ and $\mathcal{B} = \overline{\mathcal{B}}^i$. According to what was remarked after the proof of (7), the formally weaker condition $\mathcal{A} = \mathcal{A}^\theta$ and $\mathcal{B} = \mathcal{B}_\eta$ for some $0 < \theta, \eta < 1$ suffices.

There are cases in which we have equality in (10), even in a stronger sense. Entropy numbers (cf. [9]) yield a pseudo s-number sequence which is injective (up to equivalence), surjective, and additive. Let us write $\mathcal{S}_p^{(e)}$ for the corresponding ideals, $0 < p < \infty$. Then we get
$$\mathcal{S}_p^{(e)} \subset (\mathcal{S}_p^{(e)})^\theta \cdot (\mathcal{S}_p^{(e)})_{1-\theta} = (\mathcal{S}_p^{(e)}, \mathcal{S}_\infty^{(e)})^\theta \cdot (\mathcal{S}_p^{(e)}, \mathcal{S}_\infty^{(e)})_{1-\theta}$$
$$\subset \mathcal{S}^{(e)}_{\frac{p}{1-\theta}} \cdot \mathcal{S}^{(e)}_{\frac{p}{\theta}} \subset \mathcal{S}_p^{(e)}$$

(with $\mathcal{S}_\infty^{(e)} = \mathcal{L}$), where the last inclusion is due to the so-called multiplicativity of entropy numbers. Hence
$$\mathcal{S}_p^{(e)} = (\mathcal{S}_p^{(e)})^\theta \cdot (\mathcal{S}_p^{(e)})_{1-\theta} \quad \forall 0 < p < \infty, \quad \forall 0 < \theta < 1.$$

See [10] for a corresponding multiplication formula for the ideals $\mathcal{S}_p^{(e)}$.

But for other injective and surjective ideals the situation may just be the opposite. It follows from [6] that if we choose $0 < \theta < 1$ and $p \neq 2$ with $\max\left\{\frac{2}{1+\theta}, \frac{2}{2-\theta}\right\} \leq p \leq \min\left\{\frac{2}{\theta}, \frac{2}{1-\theta}\right\}$ and if we let $Q\colon \ell_1 \to \ell_p$ be a surjective operator and $J\colon \ell_p \to \ell_\infty$ an isomorphic embedding, then JQ belongs to $(\Gamma_2)^\theta(\Gamma_2)_{1-\theta}$. But of course, JQ does not belong to Γ_2, by our choice of p. So we get $\Gamma_2 \neq (\Gamma_2)^\theta \cdot (\Gamma_2)_{1-\theta}$. A closed graph argument which goes back to Grothendieck shows that from this even follows that Γ_2 is properly contained in $\bigcap_{0<\theta<1} (\Gamma_2)^\theta \cdot (\Gamma_2)_{1-\theta}$.

Let us return to the situation of (10). If we put

$$\widehat{\mathcal{A}} := \bigcup_{0<\theta<1} \mathcal{A}^\theta$$

and

$$\widetilde{\mathcal{B}} := \bigcup_{0<\theta<1} \mathcal{B}_\theta,$$

then $\widehat{\mathcal{A}}$ is a surjective ideal, and $\widetilde{\mathcal{B}}$ is an injective ideal. But in general, neither of these ideals will carry a complete ideal quasi-norm. We have

(11) For all $\mathcal{A}$ and $\mathcal{B}$ and all $0 < \theta < 1$,

$$\widehat{\mathcal{A}} = \widehat{\mathcal{A}}^\theta \quad \text{and} \quad \widetilde{\mathcal{B}} = \widetilde{\mathcal{B}}_\theta,$$

hence

$$\widehat{\mathcal{A}} \cap \widetilde{\mathcal{B}} = \widehat{\mathcal{A}} \cdot \widetilde{\mathcal{B}}.$$

Proof. Given $T \in \widetilde{\mathcal{B}}_\theta(E, F)$, let G and $S \in \widetilde{\mathcal{B}}(E, G)$ be such that $\|Tx\| \leq \|Sx\|^{1-\theta}$ $\forall x \in B_E$. Let $0 < \eta < 1$ be such that $S \in \mathcal{B}_\eta(E, G)$. Correspondingly, for some H and $R \in \mathcal{B}(E, H)$ we get $\|Sx\| \leq \|Rx\|^{1-\eta}$ $\forall x \in B_E$. Thus $\|Tx\| \leq \|Rx\|^{1-\theta-(1-\theta)\eta}$ $\forall x \in B_E$, i.e., $T \in \mathcal{B}_{\theta+(1-\theta)\eta} \subset \widetilde{\mathcal{B}}$. Similarly for $\widehat{\mathcal{A}}$, and the claim follows. ■

Similarly, if we put $\underset{\wedge}{\mathcal{A}} := \bigcap_{0<\theta<1} \mathcal{A}^\theta$ and $\underset{\sim}{\mathcal{B}} := \bigcap_{0<\theta<1} \mathcal{B}^\theta$, then we obtain a surjective resp. injective ideal, and by the same kind of arguments one is led to a relation like $\underset{\wedge}{\mathcal{A}} \cdot \underset{\sim}{\mathcal{B}} = \bigcap_{0<\theta<1} \underset{\wedge}{\mathcal{A}}^\theta \cdot \underset{\sim}{\mathcal{B}}_{1-\theta}$. Again, neither $\underset{\wedge}{\mathcal{A}}$ nor $\underset{\sim}{\mathcal{B}}$ will in general carry a complete ideal quasi-norm.

We like to conclude by an application to operators which are of certain type and cotype. Recall that $T\colon E \to F$ is said to be of (Rademacher) *type* p, $1 < p \leq 2$, if there is a constant $K > 0$ such that

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) \cdot Tx_i \right\|^2 dt \right)^{\frac{1}{2}} \leq K \cdot \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}}$$

for all finite collections of vectors $x_1, \ldots, x_n \in X$. Here r_i is the i-th Rademacher function. These operators form an injective and surjective ideal, $\mathcal{T}_p$. We also put $\mathcal{T}_1 := \mathcal{L}$.

Similarly, T is said to be of *cotype* q, $2 \leq q < \infty$, if there is a constant $K > 0$ such that

$$\left(\sum_{i=1}^{n} \|Tx_i\|^q\right)^{\frac{1}{q}} \leq K \cdot \left(\int_0^1 \left\|\sum_{i=1}^{n} r_i(t) \cdot x_i\right\|^2 dt\right)^{\frac{1}{2}}$$

for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in X$. These operators also form an ideal, $\mathcal{C}_q$, which is injective, but not surjective. Again we put $\mathcal{C}_\infty := \mathcal{L}$.

Details on these notions can be found in [12] and [9].

(12) Given $0 < \theta < 1$ and $1 \leq p_1, p_2 \leq 2 \leq q_1, q_2 \leq \infty$, define p by $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$.

Let E and F be Banach spaces.

(a) $(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})_\theta(E, F) \subset \mathcal{C}_q(E, F)$.

(b) If F is K-convex, then $(\mathcal{T}_{p_1} \cdot \mathcal{T}_{p_2})^\theta(E, F) \subset \mathcal{T}_p(E, F)$.

Proof. We omit the proof of (a) which is hardly more than hitting properly the definitions by Hölder's inequality. Let us pass to the proof of (b), and let $T \in (\mathcal{T}_{p_1}, \mathcal{T}_{p_2})^\theta(E, F)$ be given. Well-known results on type and cotype together with (6) yield $T^* \in (\mathcal{C}_{p_1^*} \cdot \mathcal{C}_{p_2^*})_\theta(F^*, E^*)$. Here we write $r^* = \frac{r}{r-1}$ for $1 \leq r \leq \infty$. Now (a) implies $T^* \in \mathcal{C}_{p^*}(F^*, E^*)$, which yields $T \in \mathcal{T}_p(E, F)$, see [11], since F is K-convex. ∎

To reach the conclusion in (b), it would have been sufficient to require that T is of the form $T = T_1 \cdot T_2$ with $T_2 \in (\mathcal{T}_{p_1}, \mathcal{T}_{p_2})^\theta(E, G)$ and T_1 a K-convex operator from G into F; see again [11].

In particular, we get

$$\mathcal{C}_q(E,F) \subset (\mathcal{C}_q)_{1-\theta}(E,F) \subset \mathcal{C}_{\frac{q}{\theta}}(E,F)$$

for $2 \leq q < \infty$ and $0 < \theta < 1$, and similarly

$$\mathcal{T}_p(E,F) \subset (\mathcal{T}_p)^\theta(E,F) \subset \mathcal{T}_{\frac{p}{1+\theta(p-1)}}(E,F)$$

for $0 < \theta < 2$, $1 < p \leq 2$, and K-convex spaces F. Therefore we get from an application of (10):

(13) If E and F are Banach spaces, F K-convex, and $T \in \mathcal{L}(E, F)$ is of type $1 < p \leq 2$ and of cotype $2 \leq q < \infty$, then there are, for every $0 < \theta < 1$, a Banach space G and operators $R \in \mathcal{L}(G, F)$ of type $\frac{p}{1+\theta(p-1)}$ and $S \in \mathcal{L}(E, G)$ of cotype $\frac{q}{\theta}$ such that $T = R \cdot S$.

Of particular interest is the case $p = q = 2$, where we get

$$(\mathcal{T}_2 \cap \mathcal{C}_2)(E,F) \subset \bigcap_{0<\theta<1} (\mathcal{T}_{\frac{2}{1+\theta}} \cdot \mathcal{C}_{\frac{2}{\theta}})(E,F),$$

at least if F is K-convex.

We do not know if the resulting inclusions can be improved further. Is $\mathcal{T}_2 \cap \mathcal{C}_2 \subset \mathcal{T}_{\frac{4}{3}} \cdot \mathcal{C}_4$, for example, best possible?

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Received March 8, 1988.

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