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ON A CLASS OF SYMMETRIC DESIGNS *

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Abstract. *Non-degenerate quadrics in* $PG(d,2)$, *where* d *is odd and* $d \geq 3$, *are used to construct a class of* $(2^{d+1}, 2^d + \varepsilon 2^{\frac{d-1}{2}}, 2^{d-1} + \varepsilon 2^{\frac{d-1}{2}})$ *-designs, where* $\varepsilon = \pm 1$. *The constructed designs are shown to be isomorphic to the designs with these parameters considered by Kantor (1975). The set of values taken by the number of absolute points of a polarity of these designs is shown to be* $\{0, 2^d, 2^{d+1}\}$.

1. INTRODUCTION

There is a simple procedure for converting $(4u^2, 2u^2 + \varepsilon u, u^2 + \varepsilon u)$-designs $(\varepsilon = \pm 1)$ into $(4u^2 - 1, 2u^2 - 1, u^2 - 1)$-designs (see Section 2). When this process is applied the constructed Hadamard design must possess a special tactical decomposition with two point and block classes. Provided a $(4u^2 - 1, 2u^2 - 1, u^2 - 1)$-design admits such a tactical decomposition the conversion procedure can be applied in reverse. We use the existence of non-degenerate quadrics in $PG(d,2)$, d odd and $d \geq 3$, to show that the symmetric design $PG_{d-1}(d,2)$ admits the sort of tactical decomposition required for the reverse application of the conversion procedure referred to above. The $(2^{d+1}, 2^d + \varepsilon 2^{\frac{d-1}{2}}, 2^{d-1} + \varepsilon 2^{\frac{d-1}{2}})$-designs thus constructed are then shown to be isomorphic to the designs with these parameters considered by Block [1], Kantor [6] and others. As a consequence we show that the set of values taken by the number of absolute points of polarities of these $(2^{d+1}, 2^d + \varepsilon 2^{\frac{d-1}{2}}, 2^{d-1} + \varepsilon 2^{\frac{d-1}{2}})$-designs is $\{0, 2^d, 2^{d+1}\}$.

2. HADAMARD AND RELATED DESIGNS

Consider a (v,k,λ)-design $\mathscr{D} = (\mathscr{P}, \mathscr{B}, \mathscr{I})$. Let (P_0, B_0) be a flag of $\mathscr{D}$, $\mathscr{P}_1$ (resp. $\mathscr{B}_1$) be the set of points (blocks) of $\mathscr{D}$ non-incident with $B_0(P_0)$, and $\mathscr{P}_2(\mathscr{B}_2)$ be the set of points (blocks) of $\mathscr{D}$ incident with $B_0(P_0)$ and different to $P_0(B_0)$. It is straightforward to verify that $\mathscr{P}_0 = \{P_0\}, \mathscr{P}_1, \mathscr{P}_2, \mathscr{B}_0 = \{B_0\}, \mathscr{B}_1, \mathscr{B}_2$ is a tactical decomposition ([2], p. 7 and p. 17) of $\mathscr{D}$ with incidence matrices

$$C = D = \begin{bmatrix} 1 & 0 & 1 \\ 0 & k-\lambda & k-\lambda \\ k-1 & \lambda & \lambda-1 \end{bmatrix}$$

and $|\mathscr{P}_0| = |\mathscr{B}_0| = 1, |\mathscr{P}_1| = |\mathscr{B}_1| = v - k$ and $|\mathscr{P}_2| = |\mathscr{B}_2| = k - 1$.

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Next, define an incidence structure $\overline{\mathscr{D}} = (\overline{\mathscr{P}}, \overline{\mathscr{B}}, \overline{\mathscr{I}})$, where $\overline{\mathscr{P}} = \mathscr{P} \setminus \mathscr{P}_0, \overline{\mathscr{B}} = \mathscr{B} \setminus \mathscr{B}_0$ and $\overline{\mathscr{I}} = \left[\bigcup_{i=1}^{2} (\mathscr{I} \cap (\mathscr{P}_i \times \mathscr{B}_i)\right] \cup \left[\bigcup_{i \neq j} (\mathscr{I}^c \cap (\mathscr{P}_i \times \mathscr{B}_j)\right]$, where $\mathscr{I}^c$ is the complement of $\mathscr{I}$. It is straightforward to show that $\mathscr{P}_1, \mathscr{P}_2, \mathscr{B}_1, \mathscr{B}_2$ is a tactical decomposition of $\overline{\mathscr{D}}$ with incidence matrices

$$\overline{C} = \overline{D} = \begin{bmatrix} k - \lambda & v - 2k + \lambda \\ k - \lambda - 1 & \lambda - 1 \end{bmatrix}$$

For $\overline{\mathscr{D}}$ to be a tactical configuration we require the two column sums of $\overline{C}$ to be equal. This occurs if and only if $v = 4(k - \lambda)$. But a (v, k, λ)-design satisfying $v = 4(k - \lambda)$ must have parameters $v = 4u^2, k = 2u^2 + \varepsilon u, \lambda = u^2 + \varepsilon u$, where $u = \sqrt{k - \lambda}$ and $\varepsilon = 1$ or -1. So $\overline{\mathscr{D}}$ is a tactical configuration if and only if $\mathscr{D}$ is a $(4u^2, 2u^2 + \varepsilon u, u^2 + \varepsilon u)$ - design for some $u > 0$ and $\varepsilon = \pm 1$.

In fact, more can be said, for we have

Theorem 1. *Suppose $\mathscr{D}$ is a $(4u^2, 2u^2 + \varepsilon u, u^2 + \varepsilon u)$-design for some $u > 0$ and $\varepsilon = 1$ or -1. Then $\overline{\mathscr{D}}$ is a Hadamard design of order u^2 and $\mathscr{P}_1, \mathscr{P}_2, \mathscr{B}_1, \mathscr{B}_2$, is a tactical decomposition of $\overline{\mathscr{D}}$ with incidence matrices*

$$\overline{C} = \overline{D} = \begin{bmatrix} u^2 & u^2 - \varepsilon u \\ u^2 - 1 & u^2 + \varepsilon u - 1 \end{bmatrix} \tag{1}$$

and

$$|P_1| = |B_1| = 2u^2 - \varepsilon u \quad and \quad |\mathscr{P}_2| = |\mathscr{B}_2| = 2u^2 + \varepsilon u - 1. \tag{2}$$

Proof. It is sufficient to show that $\overline{\mathscr{D}}$ is pairwise balanced with index (of pairwise balance) $u^2 - 1$ since the rest of what we need to show follows immediately from the discussion prior to the statement of the theorem. In order to do this we define the following parameters:

$$\lambda_i(P, Q) = \text{the number of } \mathscr{B}_i \text{ blocks on each of points } P, Q \text{ in } \mathscr{D}, i = 1, 2$$

$$\lambda_i^c(P, Q) = \text{the number of } \mathscr{B}_i \text{ blocks on neither of points } P, Q \text{ in } \mathscr{D}, i = 1, 2$$

and

$$\overline{\lambda}(P,Q) = \textit{the number of blocks on each of points } P,Q \textit{ in } \overline{\mathscr{D}}.$$

Case 1

$$\begin{aligned} & P,Q \in \mathscr{P}_1 \\ & \overline{\lambda}(P,Q) = \lambda_1(P,Q) + \lambda_2^c(P,Q) \\ & \quad = \lambda_1(P,Q) + (2u^2 + \varepsilon u - 1) - 2(u^2 + \varepsilon u) + \lambda_2(P,Q) \\ & \quad = u^2 + \varepsilon u - \varepsilon u - 1 \\ & \quad = u^2 - 1 \end{aligned}$$

Case 2

$$\begin{aligned} & P,Q \in \mathscr{P}_2 \\ & \overline{\lambda}(P,Q) = \lambda_1^c(P,Q) + \lambda_2(P,Q) - 1 \\ & \quad = (2u^2 - \varepsilon u) - 2u^2 + \lambda_1(P,Q) + \lambda_2(P,Q) - 1 \\ & \quad = u^2 - 1 \end{aligned}$$

Case 3

$$\begin{aligned} & P \in \mathscr{P}_1, Q \in \mathscr{P}_2 \\ & \overline{\lambda}(P,Q) = (u^2 - \lambda_1(P,Q)) + (u^2 + \varepsilon u - 1 - \lambda_2(P,Q)) \\ & \quad = 2u^2 + \varepsilon u - 1 - (\lambda_1(P,Q) + \lambda_2(P,Q)) \\ & \quad = u^2 - 1 \end{aligned}$$

Consider a $(4u^2-1, 2u^2-1, u^2-1)$ -design $\overline{\mathscr{D}} = (\overline{\mathscr{P}}, \overline{\mathscr{B}}, \overline{\mathscr{I}})$ which possesses a tactical decomposition $\mathscr{P}_1, \mathscr{P}_2, \mathscr{B}_1, \mathscr{B}_2$ with incidence matrices given by (1) and order of sets given by (2). Define an incidence structure $\mathscr{D} = (\mathscr{P}, \mathscr{B}, \mathscr{I})$ by $\mathscr{P} = \overline{\mathscr{P}} \cup \{P_0\}, \mathscr{B} = \overline{\mathscr{B}} \cup \{B_0\}$ (where $P_0 \neq B_0$ and $P_0, B_0 \notin \overline{\mathscr{P}} \cup \overline{\mathscr{B}}$) and $\mathscr{I} = \bigcup_{i=1}^{4} \mathscr{I}_i$, where

$$\mathscr{I}_1 = \bigcup_{i=1}^{2} (\overline{\mathscr{I}} \cap (\mathscr{P}_i \times \mathscr{B}_i)),$$

$$\mathscr{I}_2 = \bigcup_{i=j} (\overline{\mathscr{I}}^c \cap (\mathscr{P}_i \times \mathscr{B}_j)),$$

$$\mathcal{T}_3 = \{(P_0, B) : B \in \mathcal{B}_2\},$$

and

$$\mathcal{T}_4 = \{(P, B_0) : P \in \mathcal{P}_2 \cup \{P_0\}\}.$$

Arguments of a similar nature to those that established Theorem 1 yield

Theorem 2. *$\mathcal{D}$ is a $(4u^2, 2u^2 + \varepsilon u, u^2 + \varepsilon u)$-design.*

Remark 1. Theorem 2 can actually be proved under the weaker hypothesis that $\mathcal{P}_1, \mathcal{P}_2, \mathcal{B}_1, \mathcal{B}_2$, is simply a block tactical decomposition ([5], p. 44), or simply a point tactical decomposition, with incidence matrix given by (1) and order of sets given by (2) (see [7], pp. 37-38).

We note that, if χ is a polarity of the design $\mathcal{D}$ of Theorem 1 such that $\chi(P_0) = B_0$, then χ restricted to $\overline{\mathcal{P}} \cup \overline{\mathcal{B}}$ is a polarity ($\overline{\chi}$ say) of the Hadamard design $\overline{\mathcal{D}}$ of Theorem 1 such that $\overline{\chi}(\mathcal{P}_i) = \mathcal{B}_i, i = 1, 2$. Conversely, if $\overline{\chi}$ is a polarity of the design $\overline{\mathcal{D}}$ introduced just prior to Theorem 2 such that $\overline{\chi}(\mathcal{P}_i) = \mathcal{B}_i, i = 1, 2$, then χ defined by $\chi(X) = \overline{\chi}(X)$ for all $X \in \overline{\mathcal{P}} \cup \overline{\mathcal{B}}, \chi(P_0) = B_0$ and $\chi(B_0) = P_0$ is a polarity of the design $\mathcal{D}$ of Theorem 2.

Remark 2. Let $a(\chi)$ be the number of absolute points of a polarity χ of a (v, k, λ)-design of square order. It is known ([5], p. 41) that

$$a(\chi) = 2\sqrt{k - \lambda}\,\alpha + k - (v - 1)\sqrt{k - \lambda}$$

for some α such that

$$\frac{v-1}{2} - \frac{k}{2\sqrt{k-\lambda}} \leq \alpha \leq \frac{v-1}{2} + \frac{v-k}{2\sqrt{k-\lambda}}.$$

Using this result we readily have that (a) the number of absolute points of a polarity of a $(4u^2, 2u^2 + \varepsilon u, u^2 + \varepsilon u)$-design must be $2u\alpha + 2u^2 - 4u^3$ for some α such that $2u^2 - u \leq \alpha \leq 2u^2 + u$, and

(b) the number of absolute points of a polarity of a $(4u^2 - 1, 2u^2 - 1, u^2 - 1)$-design must be $2u\overline{\alpha} + 2u^2 - 4u^3 - 1$ for some $\overline{\alpha}$ such that $2u^2 - u + 1 \leq \overline{\alpha} \leq 2u^2 + u$.

3. A CONSTRUCTION USING QUADRICS

We denote the d-dimensional projective space over $GF(q)$ by $PG(d, q)$ and the $\left[\frac{q^{d+1}-1}{q-1}, \frac{q^d-1}{q-1}, \frac{q^{d-1}-1}{q-1}\right]$-design formed by the points and hyperplanes of $PG(d, q)$ by $PG_{d-1}(d, q)$. We also define the *characteristic* of a non-degenerate quadric $\mathcal{Q}$ in $PG(d, q)$, where d is odd and greater than or equal to three, to be 1 if $\mathcal{Q}$ is hyperbolic and -1 if $\mathcal{Q}$ is elliptic.

Next, let $\mathcal{Q}_\varepsilon$ be a non-degenerate quadric of characteristic ε in $PG(d,q)$, where d is odd and $d \geq 3$. Let $\mathscr{P}_2$ (resp. $\mathscr{P}_1$) be the set of points on $\mathcal{Q}_\varepsilon$ (not on $\mathcal{Q}_\varepsilon$), and $\mathscr{B}_2$ (resp. $\mathscr{B}_1$) be the set of tangent (non-tangent) hyperplanes of $\mathcal{Q}_\varepsilon$. Then (see, for example, [4]) $\mathscr{P}_1, \mathscr{P}_2, \mathscr{B}_1, \mathscr{B}_2$, form a tactical decomposition of $PG_{d-1}(d,q)$ with incidence matrices

$$\overline{C} = \overline{D} = \begin{bmatrix} q^{d-1} & q^{d-1} & - & \varepsilon q^{\frac{d-1}{2}} \\ \dfrac{q^{d-1}-1}{q-1} & \dfrac{q^{d-1}-1}{q-1} & + & \varepsilon q^{\frac{d-1}{2}} \end{bmatrix}$$

and $|\mathscr{P}_1| = |\mathscr{B}_1| = q^d - \varepsilon q^{\frac{d-1}{2}}, |\mathscr{P}_2| = |\mathscr{B}_2| = \dfrac{q^d-1}{q-1} + \varepsilon q^{\frac{d-1}{2}}$. When $q = 2$ the symmetric design $PG_{d-1}(d,q)$ is a $(4u^2-1, 2u^2-1, u^2-1)$-design and this tactical decomposition has incidence matrices given by (1) and order of sets given by (2), where $u = 2^{\frac{d-1}{2}}$. From Theorem 2 we have that there is a $(4u^2, 2u^2+\varepsilon u, u^2+\varepsilon u)$-design $D(\mathcal{Q}_\varepsilon, d)$, where $u = 2^{\frac{d-1}{2}}$, constructible from $PG_{d-1}(d,2)$ using $\mathcal{Q}_\varepsilon$. We note that if $\mathcal{Q}$ and $\mathcal{Q}'$ are non-degenerate quadrics in $PG(d,2)$ of the same characteristic, then $\mathcal{Q}$ and $\mathcal{Q}'$ are projectively equivalent, from which it follows that $D(\mathcal{Q},d)$ and $D(\mathcal{Q}',d)$ are isomorphic.

Kantor [6] has given a construction for symmetric designs with the same parameters as $D(\mathcal{Q}_\varepsilon, d)$ which we now outline. Let Q be a non-degenerate quadratic form of index $\dfrac{d+\varepsilon}{2}$ on a $(d+1)$-dimensional vector space V over $GF(2)$, where d is odd and greater than or equal to three, and let f be the associated non-degenerate alternating bilinear form, that is,

$$f(x,y) = Q(x+y) + Q(x) + Q(y)$$

for all $x, y \in V$. Let $B'_0 = \{x \in V : Q(x) = 0\}$ and B'_z be the set $B'_0 + z$. We note that the non-zero vectors of B'_0 form a quadric $\mathcal{Q}_\varepsilon$ of characteristic ε in $PG(d,2)$. Also, let $\mathscr{P} = V$ and $\mathscr{B} = \{B'_z : z \in V\}$. Then $(\mathscr{P}, \mathscr{B}, \mathscr{I})$, where $\mathscr{I}$ is defined by set-theoretical inclusion, is a $(4u^2, 2u^2+\varepsilon u, u^2+\varepsilon u)$-design, where $u = 2^{\frac{d-1}{2}}$. We denote these designs by $T^\varepsilon(d)$.

Theorem 3. *$T^\varepsilon(d)$ is isomorphic to $D(\mathcal{Q}_\varepsilon, d)$.*

Proof. Consider $z \in V \setminus \{\underset{\sim}{0}\}$. Then

$$\begin{aligned} y \in B'_z &\Leftrightarrow \ y = v + z \ for\ some\ v\ such\ that\ Q(v) = 0 \\ &\Leftrightarrow \ Q(y+z) = 0 \\ &\Leftrightarrow \ Q(y) = \begin{cases} f(y,z) & if\ Q(z) = 0, \\ f(y,z) + 1 & if\ Q(z) = 1. \end{cases} \end{aligned}$$

Denoting the hyperplane of $PG(d,2)$ consisting of $x \in V \setminus \{\underset{\sim}{0}\}$ such that $f(x,z) = 0$ by $z^{\perp}$, we infer from this that

(i) if z is a point of $\mathcal{Q}_\varepsilon$, then B'_z contains $\underset{\sim}{0}$, the points which are on both $\mathcal{Q}_\varepsilon$ and $z^{\perp}$ and the points which are on neither of $\mathcal{Q}_\varepsilon$ and $z^{\perp}$, and

(ii) if z is not a point of $\mathcal{Q}_\varepsilon$, then B'_z contains the points which are on precisely one of $\mathcal{Q}_\varepsilon$ and $z^{\perp}$.

Using the fact that, for each point z of $\mathcal{Q}_\varepsilon$, $z^{\perp}$ is the tangent hyperplane of $\mathcal{Q}_\varepsilon$ at z, it is straightforward to verify that the mapping

$$\psi : \begin{array}{l} \underset{\sim}{0} \rightarrow P_0 \\ z \rightarrow z \\ B'_0 \rightarrow B_0 \\ B'_z \rightarrow z^{\perp} \end{array}$$

is an isomorphism from $T^{\varepsilon}(d)$ onto $D(\mathcal{Q}_\varepsilon, d)$.

Remark 3. The designs $T^{\varepsilon}(d)$, in fact, admit many constructions (see, for example, [6]). Included among their rather special properties is the double transitivity of their automorphism groups.

4. POLARITIES

The designs $D(\mathcal{Q}_\varepsilon, d)$ of the last section are $(2^{d+1}, 2^d + \varepsilon 2^{\frac{d-1}{2}}, 2^{d-1} + \varepsilon 2^{\frac{d-1}{2}})$-designs. By Remark 2, the number $a(\chi)$ of absolute points of a polarity χ of a design with these parameters must be $2^{\frac{d+1}{2}}\alpha + 2^d - 2^{\frac{3d+1}{2}}$ for some α such that $2^d - 2^{\frac{d-1}{2}} \leq \alpha \leq 2^d + 2^{\frac{d-1}{2}}$. However, the values that actually occur as the number of absolute points of a polarity of $D(\mathcal{Q}_\varepsilon, d)$ are much more restricted. Denoting $\{a(\chi) : \chi \text{ is a polarity of } D(\mathcal{Q}_\varepsilon, d)\}$ by Λ we have

Theorem 4. $\Lambda = \{0, 2^d, 2^{d+1}\}$.

Proof. Every polarity χ of $D(\mathcal{Q}_\varepsilon, d)$ with $a(\chi) \geq 1$ induces a polarity $\overline{\chi}$ of $PG_{d-1}(d,2)$ with $a(\overline{\chi}) = a(\chi) - 1$. Since polarities of $PG_{d-1}(d,2)$ have either $2^d - 1$ or $2^{d+1} - 1$ absolute points we have that $\Lambda \subseteq \{0, 2^d, 2^{d+1}\}$. Also, every polarity $\overline{\chi}$ of $PG_{d-1}(d,2)$ which interchanges the points of $\mathcal{Q}_\varepsilon$ and the tangent hyperplanes of $\mathcal{Q}_\varepsilon$ induces a polarity χ of $D(\mathcal{Q}_\varepsilon, d)$ with $a(\chi) = a(\overline{\chi}) + 1$. Since $PG_{d-1}(d,q)$ admits a symplectic polarity and also orthogonal polarities with this property we have that $\{2^d, 2^{d+1}\} \subseteq \Lambda$.

To complete the proof we show that $D(\mathcal{Q}_\varepsilon, d)$ admits a polarity with no absolute points. From Theorem 3 it is sufficient to show that $T^{\varepsilon}(d)$ admits such a polarity.

From the construction of $T^\varepsilon(d)$ the set B'_z is a perfect difference set in the additive group of V for each $z \in V$. The mapping

$$\chi_z : \begin{array}{l} \xi \to B'_z + \xi, \xi \in V, \\ B'_z + \eta \to \eta, \eta \in V, \end{array}$$

is a polarity of $T^\varepsilon(d)$ ([2], p. 13). Let $z \notin B'_0$ and suppose ξ is an absolute point of χ_z. Then $\xi \in B'_z + \xi$ which implies $z \in B'_0$, a contradiction. So, if $z \notin B'_0$, then χ_z is a polarity of $T^\varepsilon(d)$ with no absolute points.

Remark 4. There seem to be few self-polar designs $\mathscr{D}$ for which $\{a(\chi) : \chi$ is a polarity of $\mathscr{D}\}$ has been determined.

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