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CODIMENSION TWO PRODUCT SUBMANIFOLDS WITH NON-NEGATIVE CURVATURE

YURIKO Y. BALDIN,*, MARIA HELENA NORONHA *

Abstract. *We prove that if* $f : M = M_1^{n_1} \times M_2^{n_2} \to R^{n+2}$ *is an isometric immersion of a complete, non-compact Riemannian manifold* M *which is a product of non-negatively curved manifolds* $M_1^{n_1}$, $n_i \geq 2$, M_1 *non-flat and irreducible, then either* f *is* n_2*-cylindrical; or* f *is a product of hypersurface immersions with* $M_1 \approx S^{n_1}$ *or* R^{n_1}; *or* f *is* $(n_2 - 1)$*-cylindrical with* $M_1 \approx S^{n_1}$ *or* RP^2 *when* M_1 *is compact, and* $M_1 \approx R^{n_1}$ *when* M_1 *is non-compact.*

1. INTRODUCTION

Let $f : M^n \to R^{n+2}$ be an isometric immersion of $M^n = M_1^{n_1} \times M_2^{n_2}$ a Riemannian product of complete, connected Riemannian manifolds M^{n_i} with $n_i \geq 2$. Moore proved (cf. [6]) that either f is a product of hypersurface immersions or f carries a complete geodesic onto a straight line in R^{n+2}. In this paper we consider such immersion with non-negative sectional curvatures and prove the following:

Theorem. *Let* $f : M^n \to R^{n+2}$ *be an isometric immersion of a complete, non-compact Riemannian manifold with* $K \geq 0$ *and suppose* M *is a Riemannian product* $M_1^{n_1} \times M_2^{n_2}$ *with* $n_i \geq 2$ *and* M_1 *non-flat and irreducible. Then either*

(a) f *is a product of hypersurface immersion with* $M_1^{n_1}$ *homeomorphic to either* S^{n_1} *or* R^{n_1}; *or*

(b) M_2 *is flat and* f *is* n_2*-cylindrical; or*

(c) M_2 *is flat and* f *is* $(n_2 - 1)$*-cylindrical. In this case, if* M_1 *is compact then either* M_1 *is homeomorphic to* S^{n_1} *or to* RP^2; *and if* M_1 *is non-compact then* M_1 *is diffeomorphic to* R^{n_1}.

A well-known result of Cheeger-Gromoll (see [5]) says that M is diffeomorphic to the total space of a vector bundle over a compact submanifold, its soul. It follows from the theorem above that if the soul of a codimension two product submanifold is not trivial it is essentially one of the factors.

We want to observe that the (c) above is the best result we can expect considering the example below:

Example. Let $h : M_1^{n_1} \to R^{n_1+1}$ be a codimension one isometric immersion and consider $M_1 \times R^2 \xrightarrow{h \times id} R^{n_1+3} \xrightarrow{g} R^{n_1+4}$ where g is an isometric immersion. $f = g \circ (h \times id)$ is not in general either a product of immersions nor 2-cylindrical.

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Although the techniques used in this paper do not allow to conclude that in the situation (c) M_1 is always a hypersurface, the proof of lemma 2.2 (see below) shows that we can consider a one dimensional vector bundle over M and define a second fundamental form on this bundle verifying the Gauss equation. This suggests that M_1 may be a hypersurface and then the possibility M_1 be homeomorphic to RP^2 would be ruled out.

2. PRELIMINARIES AND LEMMAS

Let M^n be a complete, non-compact Riemannian manifold with non-negative sectional curvatures ($K \geq 0$). It is well known that if $f : M^n \to R^{n+p}$ is an isometric immersion with the dimension of the first normal space at most two then M^n has non-negative curvature operator ($\rho \geq 0$) (cf. [10]).

The restriction $\overline{f} = f|A$ of f to a soul A of M is an isometric immersion of a compact manifold with the dimension of the first normal space at most two (dim $N_1 \leq 2$), since A is a totally geodesic submanifold of M. To this case the classification of codimension two compact submanifolds with $K \geq 0$ (cf. [1] and [2]) is generalized, since in their proofs the arguments follow from general topological properties of non-negatively curved manifolds combined with the inequality $\sum_{i=1}^{n-1} \dim H_i(M, F) \leq 2$ (see [1]) which depends only on the fact that the second fundamental form of the immersion is contained in a two dimensional subspace of the normal space. We will make use of the following theorem, generalized from Theorem 2.2. of [1]:

Theorem 2.1. *Let $f : M^n \to R^{n+p}, n \geq 3$, be an isometric immersion of a compact manifold with $K \geq 0$ and $K_p > 0$ for a point $p \in M$. Then if* dim $N_1 \leq 2, M^n$ *is simply connected and homeomorphic to S^n.*

In order to state the first lemma we recall (cf. [4]) that the Lie algebra generated by the range of the curvature operator at a point x of a codimension two euclidean submanifold M is one of the following

(i) $o(U)$

(ii) $o(V_1) \oplus o(V_2)$

(iii) $u(2)$, the unitary algebra of some complex structure on U if dim $U = 4$, where U is the orthogonal complement of the relative nullity space $N_1(x), V_1 \oplus V_2 = U$ and $o(U)$ is orthogonal algebra. Since we are considering product manifold the only case to be considered at a point x with index of relative nullity equal to zero will be ii) where V_1 and V_2 are orthogonal to each other.

Lemma 2.2. *Let $x = (x_1, x_2) \in M$ be a point with $x_1 \in M_1$ and $x_2 \in M_2$. If $r(x) = o(V_1) \oplus o(V_2)$ with $V_1 \subseteq T_{x_1}M_1$ and $V_2 \subseteq T_{x_2}M_2$ then for one $i = 1, 2$ we have $K(\sigma) > 0$ for all 2-plane $\sigma \subseteq V_i$.*

Proof. If $\dim V_1 > 1$ and $\dim V_2 > 1$ we have by [6] that the second fundamental form $\alpha(X,Y) = 0$ for $X \in V_1$ and $Y \in V_2$. Then, there is a choice of a tangent and a normal frames at x such that the matrices of the second fundamental operators have the form

$$\begin{pmatrix} A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where each A_i is is $(\dim V_i \times \dim V_i)$ non-singular matrix, $i = 1,2$. This means that both V_1 and V_2 satisfy the lemma.

Now, let us suppose that $\dim V_1 > 1$ and $\dim V_2 = 1$. Then, following the proof of Theorem 1 in [4], there is one normal vector ξ_1 such taht rank $A_{\xi_1} = 1$. If ξ_2 is a normal vector orthogonal to ξ_1 we have

$$(*) \qquad \rho = A_{\xi_2} \wedge A_{\xi_2}.$$

We will prove that $A_{\xi_2}|V_2 = 0$ and the range of A_{ξ_2} is contained in V_1. This together with (*) imply that A_{ξ_2} is non-singular on V_1 proving the lemma for V_1. Since $\dim(V_1 \oplus V_2) \geq 3$ there exist $X, Y \in T_x M$ such that $\rho(X \wedge Y) \neq 0$. Denoting by X' and X'' respectively the orthogonal projection of X on V_1 and V_2, we have:

$$\begin{aligned}\rho(X \wedge Y) = (A_{\xi_2}X)' \wedge (A_{\xi_2}Y)' + (A_{\xi_2}X)' \wedge (A_{\xi_2}Y)'' + \\ + (A_{\xi_2}X)'' \wedge (A_{\xi_2}Y)' + (A_{\xi_2}X)'' \wedge (A_{\xi_2}Y)''\end{aligned}$$

where $w_1 = (A_{\xi_2}X)'' \wedge (A_{\xi_2}Y)'' = 0$ and

$$w_2 = (A_{\xi_2}X)' \wedge (A_{\xi_2}Y'') + (A_{\xi_2}X)'' \wedge (A_{\xi_2}Y') = 0$$

since $\dim V_2 = 1$.

Taking interior product of w_2 with $(A_{\xi_2}X)'$ we get

$$0 = i(A_{\xi_2}X)'w_2 = \|(A_{\xi_2}X)'\|^2 (A_{\xi_2}Y)'' - \langle (A_{\xi_2}Y)', (A_{\xi_2}X)' \rangle (A_{\xi_2}X)''$$

and therefore

$$(A_{\xi_2}Y)'' = \langle (A_{\xi_2}Y)', (A_{\xi_2}X)' \rangle \|(A_{\xi_2}X)'\|^{-2} (A_{\xi_2}X)''.$$

Taking interior product with $(A_{\xi_2}Y)'$ we get

$$0 = i(A_{\xi_2}Y)'w_2 = \langle (A_{\xi_2}X)', (A_{\xi_2}Y)'\rangle (A_{\xi_2}Y)'' - \|(A_{\xi_2}Y)'\|^2 (A_{\xi_2}X)'' =$$
$$= \|(A_{\xi_2}X)'\|^2 \{ (\langle (A_{\xi_2}X)', (A_{\xi_2}Y')\rangle)^2 - \|(A_{\xi_2}X)'\|^2 \|(A_{\xi_2}Y)'\|^2 \} (A_{\xi_2}X)''.$$

If $(A_{\xi_2}X)'' \neq 0$ the above relation implies $(A_{\xi_2}Y)' = \lambda (A_{\xi_2}X)'$ and then $\rho(X \wedge Y) = (A_{\xi_2}X)' \wedge (A_{\xi_2}Y)' = 0$.

Hence,

(**) if $\rho(X \wedge Y) \neq 0$ we have $(A_{\xi_2}X)'' = (A_{\xi_2}Y)'' = 0$.

Consider now the orthonormal frame $\{Z_1, \ldots, Z_n\}$ which diagonalizes the operator A_{ξ_1} such that $A_{\xi_1}(Z_1) = \lambda Z_1$ and $A_{\xi_1}(Z_i) = 0, i \geq 2$. Since $\rho \neq 0$ at x, there exist Z_i and Z_j such that $\rho(Z_i \wedge Z_j) \neq 0$. By (**), if $Y \in V_2$ we have $\langle \alpha(Z_i, Y), \xi_2 \rangle = \langle \alpha(Z_j, Y), \xi_2 \rangle = 0$. This implies $\alpha(Z_i, Y) = 0$ for $i \geq 2$, as we have supposed $A_{\xi_1}(Z_i) = 0$. Since $\rho(Y \wedge Z_i) = \rho(Y \wedge Z_i') + \rho(Y \wedge Z_i'') = 0$ we will have in the Gauss equation $\langle \alpha(Z_i, Z_i), \alpha(Y, Y) \rangle = 0$. Because $\alpha(Z_i, Z_i)$ is orthogonal to ξ_1, we have $\alpha(Y, Y)$ orthogonal to ξ_2. Now, writing the Gauss equation for the sectional curvature of a plane spanned by $X \in V_1$ and $Y \in V_2$ we get:

$$0 = \langle A_{\xi_2}X, X \rangle \langle A_{\xi_2}Y, Y \rangle - (\langle A_{\xi_2}Y, X \rangle)^2 = -(\langle A_{\xi_2}Y, X \rangle)^2.$$

This together with (**) imply $A_{\xi_2}Y = 0$. Thus, range $A_{\xi_2} \subseteq V_1$ and $A_{\xi_2}Y = 0$, as we claimed.

Next, we state a result which fits the proof of Theorem.

Lemma 2.3. *Let $f : M^n \to R^{n+p}$ be an isometric immersion of M^n, a complete, non-compact manifold with $K \geq 0$, and* dim $N_1 \leq 2$. *If there exists $p \in M$ with $K_p > 0$ then M^n is diffeomorphic to R^n.*

Proof. As stated in the beginning of the section, M has non-negative curvature operator. Therefore the lemma follows from the result about the splitting of manifolds with non-negative curvature operator obtained by M.H. Noronha (cf. [8]), Corollary), namely, a complete, non-compact manifold with non-negative curvature operator is locally isometric to a product over its soul. In particular, if the curvature operator is positive at some point, then M^n is diffeomorphic to R^n.

3. PROOF OF THEOREM

By [6], either f is a product of hypersurface immersions or f takes a complete geodesic into a straight line. In the second case, this geodesic is a line in the sense that each segment of which realizes the distance between its end points, hence it must split off isometrically (cf. [9]). The projection of the line into either factor of $M_1 \times M_2$ must also be a line. Then it must lie in the M_2 factor since M_1 is irreducible.

Therefore we can write $M_2 = M_2' \times R$ and

$$f = f' \times id : (M_1 \times M_2') \times R \to R^{n+1} \times R.$$

We repeat the same argument to f'. After at most $(n_2 - 1)$ steps we conclude that either:

(i) f is n_2-cylindrical and M_2 is flat;

(ii) the non-cylindrical factor of f is a product of hypersurface immersions; or

(iii) f is $(n_2 - 1)$-cylindrical and M_2 is flat.

Suppose then we have a non-cylindrical f as a product

$$f = f_1 \times f_2, f_1 : M_1^{n_1} \to R^{n_1+1}, f_2 : M_2^{n_2} \to R^{n_2+1},$$

$n_i \geq 2, M_1$ non-flat and irreducible. In this case, there exist $x_1 \in M_1$ and $x_2 \in M_2$ such that $i_{f_1}(x_1) = 0 = i_{f_2}(x_2)$, where $i_f(x)$ denotes the index of the relative nullity of f at a point x. Thus the sectional curvatures of M_1 and M_2 are strictly positive at x_1 and x_2 respectively, implying that $M_i^{n_i}$ is either homeomorphic to S^{n_i} or R^{n_i}. This proves (a) of Theorem.

Suppose now that M_2 is flat and f is $(n_2 - 1)$-cylindrical. Consider a non-cylindrical immersion $f : M = M_1^{n_1} \times M_2^1 \to R^{n_1+3}$ and fix $x = (x_1, x_2) \in M$ where $i_f(x) = 0$. At this point we apply the lemma 2.2 in the case dim $V_2 = 1$, and we have $r(x) = o(T_{x_1} M_1)$ and $K(\sigma) > 0$ for all 2-planes $\sigma \subseteq T_{x_1} M_1$.

If M_1 is compact and $n_1 \geq 3$ then M_1 is homeomorphic to S^{n_1} by 2.1. When $n_1 = 2$, M_1 is homeomorphic to either S^2 or RP^2.

If M_1 is non-compact then M_1 is diffeomorphic to R^{n_1} by lemma 2.3.

This concludes the proof.

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Yuriko Y. Baldin
Departamento de Matemàtica
Univ. Federal de São Carlos
13560 - S. Carlos, SP
Brazil

Maria Helena Noronha
IMECC-UNICAMP
C.P. 6065
13081 - Campinas, SP
Brazil