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# REMARKS ON JOHN'S THEOREM ON THE ELLIPSOID OF MAXIMAL VOLUME INSCRIBED INTO A CONVEX SYMMETRIC BODY IN $\mathbf{R}^n$

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*Dedicated to the memory of Professor Gottfried Köthe*

**Abstract.** *Relationships between some geometric properties of finite dimensional Banach spaces and distribution of contact points of the boundary of the unit ball of the space with the ellipsoid of maximal volume inscribed into the space are discussed. It is shown that every $n$-dimensional Banach space is isometrically isomorphic to a norm one complemented $k$-dimensional Banach space whose unit ball is affinely equivalent to a convex body which* $(\star)$ *lies between the Euclidean unit ball and the standard cube circumscribed about the Euclidean unit ball, where* $k \leq \frac{1}{2}n(n+1)$. *Every $n$-dimensional space whose unit ball satisfies* $(\star)$ *is within Banach-Mazur distance* $\sqrt{2}$ *from some Banach space with the ball $D$ whose boundary has* $\frac{1}{2}n(n+1)$ *«equiweighted» contact points with the ellipsoid of maximal volume inscribed into $D$.*

## INTRODUCTION

Fritz John [J] had discovered in 1948 that contact points of a boundary of a convex body in $\mathbf{R}^n$ withe the ellipsoid of maximal volume inscribed into the body satisfy with certain weights a Parseval type identity. Dan Lewis [L] gave later for symmetric convex bodies an alternative proof of John's result in a more general setting. Lewis' approach bases on trace duality and Banach ideals technique. The minimal number $k$ of contact points involved in John's Parseval type identity for $n$-dimensional convex symmetric body satisfies $n \leq k \leq \frac{1}{2}n(n+1)$. The case $k = n$ holds iff $(\star)$ the body is affinely equivalent to a body which lies between the Euclidean unit ball and the standard cube circumscribed about the ball.

A recent deep result of Szarek [S] asserts that there is a convex symmetric body in $\mathbf{R}^n$ such that it generates a Banach space which is «far» (i.e., within a Banach-Mazur distance at least $C \log n$) from any Banach space whose unit ball satisfies $(\star)$. However we show that every $n$-dimensional Banach space is isometric to a norm one complemented subspace of a $k$-dimensional Banach space whose unit ball satisfies $(\star)$ where $k$ is the number of contact points in a John's Parseval type identify for the unit ball of the original $n$-dimensional space. We also show that if the unit ball of an $n$-dimensional real Banach space satisfies $(\star)$ then the space is within Banach Mazur distance at most $\sqrt{2}$ from a Banach space whose unit ball satisfies John's Parseval type identity with $\frac{1}{2}n(n+1)$ contact points with equal weights. The paper also contains a simple elementary proof of a well known converse to John's Theorem as well as a construction for a given weights (satisfying the necessary condition: the sum of the weights equals the dimension of the space) a system of versors which satisfy with these weights John's Parseval type identity. We are indebted to Stanislaw Kwapien for supplying the main ingredient in the construction.

The results are stated for real spaces. Extending to the complex case requires usual changes, e.g. replacing $\frac{1}{2}n(n+1)$ by $n^2$.

## PRELIMINARIES

We consider finite dimensional Banach spaces over the field $\mathbf{R}$ of reals. An $n$-dimensional Banach space we usually identify with a pair $(\mathbb{R}^n, C)$ for appropriate $C \in \mathcal{C}_n$ where $\mathcal{C}_n$ stands for the family of all compact convex bodies in $\mathbf{R}^n$ which are symmetric with respect to the origin. The Euclidean ball $B_n = \left\{x = (x(j)) \in \mathbb{R}^n : |x|_2 = \left(\sum_{j=1}^n [x(j)]^2\right)^{1/2} \leq 1\right\}$ and the standard cube $Q^n = \left\{x = (x(j)) \in \mathbf{R}^n : \max_{1\leq i \leq n} |x(j)| \leq 1\right\}$ are members of $\mathcal{C}_n$. We write $l_n^2 = (\mathbb{R}^n, B_n); \ell_n^\infty = (\mathbb{R}^n, Q^n)$. By $i_{2,\infty}^n : \ell_n^2 \to \ell_n^\infty$ we denote the formal identity map. For fixed $m = 1, 2, \ldots$ we put $\langle x, y\rangle = \sum_{j=1}^m x(i)y(i)$ for $x = (x(j)) \in \mathbb{R}^m$ and $y = (y(j)) \in \mathbf{R}^m$. Given integers $k \geq n \geq 1$ the natural embedding $i_{n,k} : \mathbf{R}^n \to \mathbb{R}^k$, and the natural projection $P_{k,n} : \mathbb{R}^k \to \mathbf{R}^n$ are defined by $(i_{n,k}x)(j) = x(j)$ for $1 \leq j \leq n$ and $(i_{n,k}x)(j) = 0$ for $n < j \leq k$; $P_{k,n}(y)(j) = y(j)$ for $1 \leq j \leq n$ $(y = (y(j)) \in \mathbf{R}^k)$. By $|\cdot|_C$ we denote the gauge functional of a $C \in \mathcal{C}_n$. By $d(X,Y) = \inf\{\|T : X \to Y\|\ \|T^{-1} : Y \to X\| : T \text{ isomorphism }\}$ we denote the Banach-Mazur distance between Banach space $X$ and $Y$. If $E \in \mathcal{C}_n$ is an ellipsoid (= the image of $B_n$ under a non singular affine transformation, say $T : \mathbb{R}^n \to \mathbb{R}^n$), then we put

$$\langle x, y\rangle_E = \langle T^{-1}x, T^{-1}y\rangle \text{ for } x, y \in \mathbb{R}^n.$$

Given $C \in \mathcal{C}_n$ by $E_C$ we denote (the unique) ellipsoid of maximal volume inscribed into $C$.

## 1. JOHN'S PARSEVAL TYPE IDENTITY

Recall [J], [L], [Pi], [T].

**Theorem.** *(F. John) For every $C \in \mathcal{C}_n$ the ellipsoid $E_C$ satisfies: For some $k$ with $n \leq k \leq n(n+1)/2$ there are $x_1, x_2, \ldots, x_k \in \mathbf{R}^n$ and positive numbers $d_1, d_2, \ldots, d_k$ such that* *(i)* $|x_j|_C + |x_j|_{E_C} = 1 \ (j = 1, 2, \dots, k)$

*(ii)* $x = \sum_{j=1}^{k} d_j \langle x, x_j \rangle_{E_C^{x_j}}$ *for* $x \in \mathbf{R}^n$.

Note that (i) and (ii) formally imply (iii), and (i) together with the inclusion $E_C \subset C$ imply (iv) where

*(iii)* $\sum_{j=1}^{k} d_j = n, \quad d_j \leq 1$ *for* $1 \leq j \leq k$, (iv) $\sup\{|\langle x_j, y\rangle_{E_C}| = 1 : y \in C\}$.

If (i) and (ii) are satisfied for some $k$, not necessarily $k \leq \frac{1}{2}n(n+1)$, then we say that the vectors $x_1, x_2, \dots, x_k$ satisfy *John's Parseval type identity for* $E_C$ *with the weights* $d_1, d_2, \dots, d_k$.

A converse to John's Theorem also holds (cf. e.g. [P-T], p. 132). The proof can be easily deduced from [T], Proposition 14.3 or from the material presented in [Pi], Chapt. 3. Since the argument in both books involves trace duality and Banach ideals technique, we begin with an elementary proof of this result.

**Theorem 1.1.** *Let* $C \in C_n$, *and let* $E \in C_n$ *be an ellipsoid. Assume that* $E \subset C$ *and that a John's Parseval type identity is satisfied with* $E_C$ *replaced by* $E$. *Then* $E$ *is the ellipsoid of maximal volume inscribed into* $C$.

*Proof.* Without loss of generality one may assume that $E = B_n$. Let $D$ be another ellipsoid with $D \subset C$. Let $e_1, e_2, \dots, e_n$ be the orthonormal basis in $\ell_n^2$ consisting of versors of the principal axes of $D$. Then

$$D = \left\{ x \in \mathbf{R}^n : \sum_{m=1}^{n} c_m \langle x, e_m \rangle^2 = |x|_D^2 \leq 1 \right\}$$

where $c_m^{1/2}$ is the length of the half axes of $D$ parallel to $e_m (m = 1, 2, \dots, n)$. The polar ellipsoid

$$D^* = \{x \in \mathbb{R}^n : |\langle x, y\rangle| \leq 1 \text{ for } y \in D\}$$

satisfies

$$D^* = \left\{ x \in \mathbf{R}^n : \sum_{m=1}^{n} c_m^{-1} \langle x, e_m \rangle^2 = |x|_{D^*}^2 \leq 1 \right\}.$$

It follows from (iv) and the inclusion $D \subset C$ that

$$|x_j|_{D^*} = \sup_{|y|_D \leq 1} \left| \langle x_j, y \rangle \right| \leq \sup_{|y|_C \leq 1} \left| \langle x_j, y \rangle \right| = 1.$$

Thus

$$n=\sum_{j=1}^{k} d_j \geq \sum_{j=1}^{k} d_j |x_j|_D^2.$$

$$=\sum_{j=1}^{k} d_j \sum_{m=1}^{n} c_m^{-1} \langle x_j, e_m\rangle^2$$

$$=\sum_{m=1}^{n} c_m^{-1} \sum_{j=1}^{k} d_j \langle x_j, e_m\rangle^2$$

$$=\sum_{m=1}^{n} c_n^{-1} \langle e_m, e_m\rangle^2 \quad (\text{by}(ii))$$

$$=\sum_{m=1}^{n} c_m^{-1}.$$

Hence using the inequality between arithmetic and geometric means one gets $\prod_{m=1}^{n} c_m^{-1} \leq 1$.

Thus $\text{vol } D = \left(\prod_{m=1}^{n} c_m^{-1/2}\right) \text{vol } B_n \leq \text{vol } B_n$.

**Remarks.** 1) The equality in the latter inequality is possible only when $c_m = 1$ for $m = 1, 2, \ldots, n$, i.e., when $D = B_n$. This shows uniqueness of $E_C$. 2) Ball [B] generalized the proof of Theorem 1.1 to non centrally symmetric convex bodies.

Our aim is to construct various sequences of versors satisfying (ii). First we recall

**Proposition 1.1.** *Given integers $2 \leq n \leq k$ and reals $0 < d_j \leq 1$ for $j = 1, 2, \ldots, k$ with $\sum_{j=1}^{k} d_j = n$. Then the following are equivalent*

*(1) there exist $x_1, x_2, \ldots, x_k \in \mathbb{R}^n$ with $|x_j|_2 = 1$ for $j = 1, 2, \ldots, k$ such that*

$$x = \sum_{j=1}^{k} d_j \langle x, x_j\rangle x_j \quad \text{for } x \in \mathbb{R}^n;$$

*(2) there exists an $n \times k$ real matrix $(a_{i,j})_{\substack{1\leq i\leq n\\ 1\leq j\leq k}}$ such that the column vectors of the matrix are orthonormal, i.e.*

$$\sum_{j=1}^{k} a_{ij}^2 = 1; \ \sum_{j=1}^{k} a_{i,j} a_{i',j} = 0 \ \text{for } i \neq i' \ (i, i' = 1, 2, \ldots, n)$$

*and the square of the length of the j-th row vector equals $d_j$, i.e.*

$$\sum_{i=1}^{n} a_{ij}^2 = d_j \quad \textit{for } j = 1, 2, \ldots, k;$$

*(3) there is an orthonormal basis $(g_j)_{1 \le j \le k}$ in $\mathbf{R}^k$ such that*

$$|P_{k,n} g_j|_2 = \sqrt{d_j} \textit{ for } j = 1, 2, \ldots, k.$$

For the proof see [P-T], Proposition 2.6, although the result goes back to [K] and [M]. Note that the connection between the conditions (1), (2), (3) are

$$\sqrt{d_j} x_j = (a_{ij})_{1 \le i \le n} = P_{k,n}(g_j) \quad (j = 1, 2, \ldots, k).$$

Let us observe at this point that if $d_{j_0} = 1$ and if the versors $x_1, x_2, \ldots, x_k$ together with positive numbers $d_1, d_2, \ldots, d_k$ satisfy (1) then $x_{j_0}$ is orthogonal to all $x_j$ with $j \neq j_0$. Indeed

$$\langle x_{j_0}, x_{j_0} \rangle^2 = \langle x_{j_0}, x_{j_0} \rangle^2 + \sum_{j \neq j_0} d_j \langle x_{j_0}, x_{j_0} \rangle^2.$$

Hence $\langle x_{j_0}, x_{j_0} \rangle = 0$ whenever $j \neq j_0$. This observation implies for instance that if $d_1 = d_2 = \ldots = d_{n-1} = 1$ then all the vectors $x_n, x_{n+1}, \ldots, x_k$ must be colinear. To simplify the formulation of the next result we shall assume that $d_j < 1$ for $j = 1, 2, \ldots, k$. Note that the assumption of non colinearity of the versors $x_j$'s satisfying (1) means in terms of John's Theorem that we deal with essentially distinct contact points. Now we are ready to state the main result of this section.

**Theorem 1.2.** *We are given positive integers $2 \le n < k$ and real numbers $1 > d_j > 0$ for $j = 1, 2, \ldots, k$ with $\sum_{j=1}^{k} d_j = n$. Then there exists an $n \times k$ real matrix whose column vectors are orthonormal, the square of the length of the j-th row vector equals $d_j$ for $j = 1, 2, \ldots, k$, and no two distinct row vectors are colinear.*

*Proof.* 1° $n = 2$. We are looking for $y_1, y_2, \ldots, y_k \in \mathbb{R}^2$ of the form $y_j = \left(\sqrt{d_j} \cos \varphi_j, \sqrt{d_j} \sin \varphi_j\right)$, with $0 \le \varphi_j < \pi$ for $j = 1, 2, \ldots, k$ such that

$$(+) \qquad \sum_{j=1}^{k} d_j \cos^2 \varphi_j = \sum_{j=1}^{k} d_j \sin^2 \varphi_j = 1; \sum_{j=1}^{k} d_j \cos \varphi_j \sin \varphi_j = 0.$$

The condition (+) is equivalent to

$$(++) \qquad \sum_{j=1}^{k} d_j \cos 2\varphi_j = 0, \quad \sum_{j=1}^{k} d_j \sin 2\varphi_j = 0,$$

which, in view of $\sum_{j=1}^{k} d_j = 2$, is equivalent to

(+++) there exists in the plane a (convex) $k$-gone with lengths of sides $d_1, d_2, \dots, d_k$ respectively.

The existence of the desired $k$-gone follows from

**Lemma 1.1.** *Let* $1 > d_1 \geq d_2 \geq \dots \geq d_k > 0$, $\sum_{j=1}^{k} d_j = 2$, $k \geq 3$. *Then there exists a convex k-gone in the plane, say* $A_1 A_2 \dots A_k$, *with lengths of consecutive sides equal* $d_1, d_2, \dots, d_k$ *respectively, i.e.* $|A_j A_{j+1}| = d_{j+1}$ *for* $j = 1, 2, \dots, k-1$ *and* $|A_k, A_1| = d_k$. *Moreover except the case*

$$(*) \qquad k = 4, \quad d_1 = d_2 = d_3 = d_4 = 1/2$$

*the k-gone has no pair of parallel sides.*

The proof of Lemma 1.1 is given at the end of this section.

Note that the «moreover part» of the Lemma guarantees the non colinarity of each pair of distinct vectors $y_1, y_2, \dots, y_k$ except the case $(*)$. In the case $(*)$ we fix $\varphi$ with $0 < \varphi < \frac{\pi}{2}$ and put

$$y_1 = \frac{1}{\sqrt{2}}(\cos\varphi, \sin\varphi), y_2 = \frac{1}{\sqrt{2}}(-\sin\varphi, \cos\varphi),$$

$$y_3 = \frac{1}{\sqrt{2}}(-\cos\varphi, \sin\varphi), y_4 = \frac{1}{\sqrt{2}}(\sin\varphi, \cos\varphi),$$

2° General case. We proceed by induction with respect to $n$. The case $n = 2$ has been already considered. Fix $n \geq 3$ and assume the validity of the assertion of Theorem 1.2 for all pairs $(n-1, k)$ with $2 \leq n-1 < k$. Let $1 > d_1 \geq d_2 \geq \dots \geq d_k > 0$ with $\sum_{j=1}^{k} d_j = n$ be given. Pick the index $m$ so that $1 > \sum_{j=1}^{m} d_j \geq 1 - d_{m+1}$. Put $a = 1 - \sum_{j=1}^{m} d_j$. Clearly $0 < a \leq d_{m+1}$. Pick $a_1, a_2, \dots, a_m$ so that $a = \sum_{s=1}^{m} a_s, d_{m+s} - a_s > 0$

for $s = 1, 2, \ldots, m$. (For instance put $a_s = \min\left(\frac{1}{2}d_{m+s}, \frac{a}{m+1}\right.$ for $s = 2, 3, \ldots, m$ and $\left. a_1 = a - \sum_{s=2}^{m} a_s\right)$. Note that $3m < k$, in particular $2m < k$. Indeed $md_m \le \sum_{j=1}^{m} d_j < 1$ and $(k-m)d_m \ge \sum_{j=m+1}^{k} d_j \ge n-1 \ge 2$. Hence $md_m < \frac{k-m}{2} d_m$ which yields $3m < k$.

Put $d'_{m-s+1} = d_{m-s+1} + a_s$ and $d'_{m+s} = d_{m+s} - a_s$ for $s = 1, 2, \ldots, m$; $d'_j = d_j$ for $j = 2m+1, 2m+2, \ldots, k$. Observe that $1 > d'_j > 0$ for $j = m+1, m+2, \ldots, k$ and $\sum_{j=m+1}^{k} d'_j = n-1$, consequently $k - m > n - 1$. Hence, by the inductive hypothesis, there exists an $(n-k) \times (k-m)$ real matrix, say $(y'_j(i))_{\substack{1 \le i \le n-1 \\ m+1 \le j \le k}}$ such that its column vectors are orthonormal; the row vectors satisfy $\sum_{i=1}^{n} \left[y'_j(i)\right]^2 = d'_j$ for $j = m+1, m+2, \ldots, k$; distinct row vectors are not colinear. Now we consider the $n \times k$ matrix

$$\begin{bmatrix} 0, & 0, & \ldots, & 0, & \sqrt{d'_1} \\ 0, & 0, & \ldots, & 0, & \sqrt{d'_2} \\ \cdot & \cdot & \ldots, & \cdot & \cdot \\ 0, & 0, & \ldots, & 0, & \sqrt{d'_m} \\ y'_{m+1}(1), & y'_{m+1}(2), & \ldots, & y'_{m+1}(n-1)), & 0 \\ y'_{m+2}(1), & y'_{m+2}(2), & \ldots, & y'_{m+2}(n-1)), & 0 \\ \cdot & \cdot & \ldots, & \cdot & \cdot \\ y'_k(1), & y'_k(2), & \ldots, & y'_k(n-1), & 0 \end{bmatrix}$$

Next we construct a new $n \times k$ matrix. Its row vectors $y_1, y_2, \ldots, y_k$ are defined as follows: $y_j = y'_j = (y'_j(1), y'_2(2), \ldots, y'_j(n-1), 0)$ for $j = 2m+1, 2m+2, \ldots, k$; each pair of row vectors $(y_{m-s+1}, y_{m+s})$ is obtained from the pair of row vectors $(y'_{m-s+1}, y'_{m+s})$ of the «old» matrix by applying the rotation by an angle $\varphi_s$ with $0 < \varphi_s < \pi$, i.e., according to the rule

$$\begin{bmatrix} \cos\varphi_s, \sin\varphi_s \\ -\sin\varphi_s, \cos\varphi_s \end{bmatrix} \begin{bmatrix} y'_{m-s+1} \\ y'_{m+s} \end{bmatrix} = \begin{bmatrix} y_{m-s+1} \\ y_{m+s} \end{bmatrix}.$$

Thus

$$y_{m-s+1} = \left(y'_{m+s}(1)\sin\varphi_s, y'_{m+s}(2)\sin\varphi_s, \ldots, y'_{m+s}(n-1)\sin\varphi_s, \sqrt{d'_{m-s+1}}\cos\varphi_s\right),$$

$$y_{m+s} = \left(y'_{m+s}(1)\cos\varphi_s, y'_{m+s}(2)\cos\varphi_s, \dots, y'_{m+s}(n-1)\cos\varphi_s, -\sqrt{d'_{m-s+1}}\sin\varphi_s\right)$$

for $s = 1, 2 \dots, m$.

It can be easily verified that for arbitrary angles $\varphi_1, \varphi_2, \dots, \varphi_s$ we get in that way from our «old» matrix an $n \times k$ matrix with orthonormal column vectors. Next we choose $\varphi_s$ so that

$$|y_{m-s+1}|_2^2 = d_{m-s+1} = d'_{m+s}\sin^2\varphi_s + d'_{m-s+1}\cos^2\varphi_s$$
$$|y_{m+s}|_2^2 = d_{m+s} = d'_{m+s}\cos^2\varphi_s + d'_{m-s+1}\sin^2\varphi_s$$

To this end $\varphi_s$ should satisfy

$$\sin^2\varphi_s = a_s(d_{m-s+1} - d_{m+s} + 2a_s)^{-1} \text{ for } s = 1, 2, \dots, m.$$

The latter identify suffices because $d_{m-s+1} + d_{m+s} = d'_{m-s+1} + d'_{m+s}$ for $s = 1, 2, \dots, m$.

Finally we verify that if $1 \le r < t \le k$ then the row vectors $y_r$ and $y_s$ are not colinear. Indeed if the pair $(r,t)$ is different from a pair $(m-s+1, m+s)$ for all $s = 1, 2, \dots, m$ then for $1 \le i' < 1 \le n-1$ we have

$$\det\begin{bmatrix} y_r(i'), y_r(i) \\ y_t(i'), y_t(i) \end{bmatrix} = C(r,t)\det\begin{bmatrix} y'_p(i'), y'_p(i) \\ y'_q(i'), y'_q(i) \end{bmatrix}$$

where $p = p(r,t) \neq q = q(r,t) \in \{m+1, m, +2, \dots, k\}$ and $C(r,t) \neq 0$. Thus the condition that the row vectors $y'_p$ and $y'_q$ are not colinear implies the non colinearity of the row vectors $y_r$ and $y_t$. For a pair $(m-s+1, m+s)$ we have

$$\sum_{i=1}^{n-1}\left(\det\begin{bmatrix} y_{m-s+1}(i), y_{m-s+1}(n) \\ y_{m+s}(i), y_{m+s}(n) \end{bmatrix}\right)^2 = d'_{m-s+1}d'_{m+s} \neq 0$$

which implies the non colinearity of the pair $(y_{m-s+1}, y_{m+s})$.

*Proof of lemma 1.2.* If

(**) $$0 < \min(a,b,c) \le \max(a,b,c) < 1; a+b+c = 2$$

then $b + c > 1 > a > |b-c|$, hence

(***) there is a triangle $\triangle ABC$ with $|AB| = c, |BC| = a, |CB| = b$.

This proves the Lemma for $k = 3$.

If $(*)$ holds then there is square with sides of length half.

Now assume that $k \geq 4$ and $(*)$ does not hold. Then $d_k + d_{k-1} < 1/2 + 1/2 = 1$. Hence there exists an index $m$ with $k - 1 \geq m \geq 3$ such that $1 - d_{m-1} \leq \sum_{j=m}^{k} d_j < 1$. Let us put

$$a = \sum_{j=m}^{k} d_j, \quad b = \sum_{j=2}^{m-1} d_j, \quad c = d_1 .$$

Then $a, b, c$ satisfy $(**)$. Let $\triangle ABC$ satisfies $(***)$. Let $\alpha, \beta, \gamma$ denote measures of the angles of $\triangle ABC$ at the vertices $A, B, C$ respectively. Then either $\max(\alpha, \gamma) < \frac{\pi}{2}$ or $\max(\beta, \gamma) < \frac{\pi}{2}$. We consider only the case $\max(\alpha, \gamma) < \frac{\pi}{2}$; the argument in the remaining cases is similar.

Fix $\varepsilon > 0$ sufficiently small. Since $\gamma < \frac{\pi}{2}$, there exists a point $C^\varepsilon$ on the circle centered at $A$ of radius $|AC| = b$ such that $|BC^\varepsilon| < |BC| = a$ and $\max(\alpha_\varepsilon, \gamma_\varepsilon) < \frac{\pi}{2}$ where $\alpha_\varepsilon, \gamma_\varepsilon$ are measures of the angles $\measuredangle BAC^\varepsilon, \measuredangle AC^\varepsilon B$ respectively. Since $|BC^\varepsilon| < a = d_k + d_{k+1} + \ldots + d_m$ for $\varepsilon > 0$ sufficiently small, there exists a convex polygone $A_k A_{k-1} \ldots A_m C^\varepsilon$ where $B = A_k$ such that: $|A_j A_{j+1}| = d_{j+1}$ for $j = m, m+1, \ldots, k-1$; $|C^\varepsilon A_m| = d_m$; the points $A_{k-1}, A_{k-2} \ldots, A_m$ are separated from $A$ by the straight line passing through $B$ and $C^\varepsilon$; the measure of the angles $\measuredangle AC^\varepsilon A_m$ and $\measuredangle A_{k-1}BA$ are less than $\frac{\pi}{2}$ and $\pi$ respectively. Now if $m = 3$ then the polygone $AA_k \ldots A_m C^\varepsilon$ has the desired properties because then $b = d_2 = |AC| = |AC^\varepsilon|$ and $c = d_1 = |AA_k| = |AB|$. If $m > 3$ then because the measure of the angle $\measuredangle AC^\varepsilon A_m$ is less than $\frac{\pi}{2}$ for $\varepsilon > 0$ sufficiently small there exists a point $A_{m-1}$ on the circle centered at $A_m$ of radius $|A_m C^\varepsilon|$ such that $|AA_m| < |AC^\varepsilon| = b = \sum_{j=2}^{m-1} d_j$. Hence there exists a convex polygone $A_1, A_2 \ldots A_{m-1}$ where $A_1 = A$ such that $|A_j A_{j+1}| = d_{j+1}$ for $j = 1, 2, \ldots, m-2$, the points $A_2, A_3, \ldots, A_{m-2}$ are separated from the points $A_m, A_{m+1}, \ldots, A_k$ by the straight line passing through $A = A_1$ and $A_{m-1}$; the measure of the angles $\measuredangle A_m A_{m-1} A_{m-2}$ and $\measuredangle A_k A_1 A_2$ are less than $\frac{\pi}{2}$. The polygone $A_1, A_2, \ldots, A_k$ has the desired properties. Note that if $\varepsilon > 0$ is sufficiently small then each side of the gone is «almost parallel» to one side of the triangle $\triangle ABC$. Therefore two sides of the gone which are «almost parallel» to different sides of the triangle cannot be parallel each to other. Two sides which are «almost parallel» to the same side of the triangle are not parallel between themselves for convexity reason.

§ 2. Given a Banach space $X$ with $\dim X = n$, a *Dvoretzky-Rogers factorization through* $X$ is a pair of linear operators $u : \ell^2_n \to X$ and $v : X \to \ell^\infty_n$ such that $vu = i^n_{\infty,2}$.

We put

$$dr(X) = \inf ||u||\ ||v||$$

where the infinum extends over all Dvrotezky-Rogers factorizations through $X$.

The following is routine

**Proposition 2.1.** *Given* $X$ *with* $\dim X = n$. *Then*

*(a)* $dr(X) \geq 1$;

*(b) there exists a Dvoretzky-Rogers factorization through* $X$, *say* $(u, v)$ *such that* $dr(X) = ||u||\ ||v||$;

*(c) if* $S : X \to Y$ *is an isomorphism then* $dr(X) \leq ||S||\ ||S^{-1}||dr(Y)$;

*(d) there exists an* $n$*-dimensional Banach space* $Y$ *such that*

$$dr(Y) = 1 \text{ and } d(X,Y) = dr(X).$$

*Proof.* (a) Note that if $vu = i^n_{\infty,2}$ then $||u||\ ||v|| \geq ||i^n_{\infty,2}|| = 1$. (b) Use a standard compactness argument for finite dimensional spaces.

(c) Note that if $(u, v)$ is a Dvoretzky-Rogers factorization through $Y$ then $(S^{-1}u, vS)$ so does through $X$.

(d) Let $(u, v)$ be as in (b). One can additionally assume that $||u|| = 1$, and one can identified $X$ with its isometrically isometric copy $(\mathbb{R}^n, C)$ where $C = u^{-1}\{x \in X : ||x|| \leq 1\}$. Then $u$ and $v$ become formal identities. Let $a = ||v||^{-1}$. Put $C_1 = conv(B_n \cup aC)$ and $Y = (\mathbb{R}^n, C_1)$. Clearly $B_n \subset C_1$ and $C_1 \subset Q^n$ because $B_n \subset Q^n$ and $||v|| = a^{-1}$ yields $aC \subset Q^n$. Thus $||u : \ell^2_n \to Y|| = ||v : Y \to \ell^\infty_n|| = 1$. Hence $dr(Y) = 1$. Furthermore $aC \subset C_1 \subset C$ (because $||u : \ell^2_n \to X|| = 1$ yields $B_n \subset C$ while $a \leq 1$ yields $aC \subset C$). Thus $d(X,Y) \leq a^{-1} = dr(X)$. On the other hand, by (c), $dr(X) \leq d(X,Y)dr(Y) = d(X,Y)$.

We left to the reader the routine proof of the next

**Proposition 2.2.** *For a Banach space* $X$ *with* $\dim X = n$ *the following are equivalent*

*(j)* $dr(X) = 1$

*(jj)* $X$ *is isometrically isomorphic to* $(\mathbb{R}^n, C)$ *with* $C$ *satisfying* $B_n \subset C \subset Q^n$.

*(jjj)* $X$ *is isometrically isomorphic to* $(\mathbb{R}^n, C)$ *with* $C$ *satisfying* 1°. $B_n \subset C$, 2° *there exists an orthonormal basis, say* $(g_j)_{1\leq j\leq n}$, *such that* $|g_j|_2 = |g_j|_C = 1$ *for* $j = 1, 2, \ldots, n$.

It was proved by Szarek [S] that there exists a sequence $(X_n)$ of Banach spaces such that $\dim X_n = n$ for $n = 1, 2, \ldots,$ and $\lim_n dr(X_n) = +\infty$. On the other hand from the classical Dvoretzky-Rogers Lemma (cf [D.R], [T], Proposition 15.6) it follows that each $X$ with dim $X = n$ contains a subspace $Z$ with $dr(Z) \leq 2$ and dim $Z \geq \sqrt{n}$. The latter result has been recently improved to dim $Z \geq cn$ (cf [B-S]). It is interesting to compare these facts with the following

**Theorem 2.1.** *If $X(\mathbb{R}^n, C)$ is a Banach space such that a John's Parseval type identity is satisfied for $E_C$ with $k$ contact points then there exists a $k$-dimensional Banach space $Y$ with $dr(Y) = 1$ such that $X$ is isometrically isomorphic to a norm one complemented subspace of $Y$.*

*Proof.* Without loss of generality assume that $E_C = B_n$. In particular $B_n \subset C$. Put $D = conv(C \cup B_k)$ and $Y = (\mathbb{R}^k, D)$ (we identify here $C$ with $i_{n,k}(C)$). Observe that $i_{n,k}$ regarded as an operator from $(\mathbb{R}^n, C)$ into $(\mathbb{R}^k, D)$ is an isometric embedding because $D \cap \mathbb{R}^n = C$. Also $\|P_{k,n} : (\mathbb{R}^k, D) \to (\mathbb{R}^n, C)\| \leq 1$. Indeed if $y \in D$ then $y = tb + (1-t)c$ for some $b \in B_k$, $c \in C$, $0 \leq t \leq 1$. Thus $P_{k,n}(y) = tP_{k,n}(b) + (1-t)c \in C$ because for every $b \in B_k$, $P_{k,n}(b) \in B_n \subset C$.

It remains to show that $dr(Y) = 1$. Since $B_k \subset D$, in view of Proposition 2.2 it is enough to show that there is an orthonormal basis, say $(g_j)_{1 \leq j \leq k}$ in $\mathbb{R}^k$ with $|g_j|_2 = |g_j|_D = 1$ for $j = 1, 2, \ldots, k$. Let $(g_j)_{1 \leq j \leq k}$ be the orthonormal basis appearing in Proposition 1.1 (3) which satisfies $P_{k,n}(g_j) = \sqrt{d_j} x_j$. We are going to show that $|g_j|_D = 1$ for $j = 1, 2, \ldots, k$. Fix an integer $j$ with $1 \leq j \leq k$. Since $g_j \in B_k \subset D$, we have $1 = |g_j|_2 \geq |g_j|_D$. Since $g_j \in D$, there exist $g \in B_k$ and $c \in C$ such that $|g_j|_D = |g|_2 + |c|_C$ and $g_j = g + c$.
Hence

$$\begin{aligned}
|g_j|_D &\leq 1 \\
&= \langle g_j, g_j \rangle \\
&= \langle g_j, g \rangle + \langle g_j, c \rangle \\
&\leq |g_j|_2 |g|_2 + \langle g_j, c \rangle \\
&= |g|_2 + \langle g_j, P_{k,n}(c) \rangle \\
&= |g_j|_2 + \langle P_{k,n}(g_j), c \rangle \\
&= |g_j|_2 + \sqrt{d_j} \langle x_j, c \rangle \\
&\leq |g_j|_2 + \sqrt{d_j} |c|_C \quad (\text{by(iv)}) \\
&\leq |g_j|_2 + |c|_C \qquad (\text{because} 0 < d_j \leq 1)
\end{aligned}$$

$$= |g_j|_D$$

Thus $|g_j|_D = 1$.

**Remark.** In [P-T] for every fixed $k$ with $n \leq k \leq \frac{1}{2}n(n+1)$ a body $C \in \mathcal{C}_n$ was constructed with the property that there exists only one John's Parseval type identity for $E_C$ and it involves all $k$ contact points of the boundary of $C$ with $E_C$. Therefore by Proposition 1.1 and Proposition 2.2 the space $X = (\mathbb{R}^n, C)$ is not isometrically isomorphic to a norm one complemented subspace of any Banach space $Y$ with $dr(Y) = 1$ and $\dim Y < k$.

§ **3.** The convex body $D_n \in \mathcal{C}_n$ examined in [P-S], section 6 satisfies: $d((\mathbb{R}^n, D_n), \ell_n^\infty) < 2$ and there exists an equiweighted John's Parseval type identity for $E_{D_n}$ with $\frac{1}{2}n(n+1)$ contact points. In this section we establish a more general fact.

**Theorem 3.1.** *If $X$ is an $n$-dimensional Banach space with $dr(X) = 1$ then for every $k \geq n$ of the form $k = \frac{1}{2}pn$ where $p > 1$ is an integer, in particular for $k = \frac{1}{2}n(n+1)$, there exists a Banach space $Y = (\mathbb{R}^n, D)$ such that $d(X,Y) \leq \sqrt{2}$ and a John's Parseval type identity is satisfied for $E_D$ by some $k$ contact points with equal weights.*

*Proof.* Theorem 3.1 is an immediate consequence of the next two lemmas Proposition 1.1 and Theorem 1.1.

**Lemma 3.1.** *Let $C \in \mathcal{C}_n$ satisfies $B_n \subset C \subset Q^n$. Put $D = conv(B_n \cup p^{-1/2}C)$ where $p$ is a fixed integer with $1 \leq p \leq n$. Then $|x|_D = |x|_2$ for every $x \in \mathbb{R}^n$ which has at most $p$ non-zero coordinates.*

*Proof.* Fix $x = (x(i))$ in question. Put $\sigma = \{i : x(i) \neq 0\}$ and define $P_\sigma$ by $(P_\sigma(y))(i) = y(i)$ for $i \in \sigma$ and $(P_\sigma(y))(i) = 0$ for $i \neq \sigma$. Clearly $P_\sigma$ is an orthogonal projection which satisfies

$$(+) \qquad |P_\sigma(y)|_2 \leq p^{1/2}|y|_{Q^n} \text{ for } y \in \mathbb{R}^n$$

because $\sigma$ has at most $p$ elements. Next pick $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$ so that $|x|_D = |b|_2 + p^{1/2}|c|_C$ and $x = b + c$. In particular $P_\sigma(x) = P_\sigma(b) + P_\sigma(c)$. We have

$$\begin{aligned}
|x|_D &\leq |x|_2 && \text{(because } B_n \subset D\text{)}\\
&= |P_\sigma(x)|_2 && \text{(because } P_\sigma(x) = x\text{)}\\
&\leq |P_\sigma(b)|_2 + |P_\sigma(c)|_2 && \text{(by the triangle inequality)}\\
&\leq |b|_2 + |P_\sigma(c)|_2 && \text{(because the projection } P_\sigma \text{ is orthogonal)}\\
&\leq |b|_2 + p^{1/2}|c|_{Q^n} && \text{(by (+))}\\
&= |x|_D
\end{aligned}$$

Thus $|x|_D = |x|_2$.

**Lemma 3.2.** *Let* $k = mn$ *where either (I)* $m = 1, 2, \ldots$; $n = 2, 3 \ldots$, *or (II)* $m = (2r + 1)/2$, $n = 2q$ $(r, q = 1, 2, \ldots)$. *Then there exists an* $n \times k$ *real matrix such that the column vectors of the matrix are orthonormal, the row vectors have the same length, no two row vectors are collinar, each row vector has at most two non-zero coordinates.*

*Proof.* Case (I). Let $s = m/2$ for $m$ even and $s = (m-1)/2$ for $m$ odd. Let $0 < \varepsilon < 1$ and let reals $a_1, a_2, \ldots a_s$ satisfy $1 > a_1 > a_2 > \ldots > a_s > 1 - \varepsilon^2/2$. Put $b_t = \sqrt{1 - a_t^2}$ for $t = 1, 2, \ldots, s$. Define auxiliary column vectors $A$ and $B$ of length $m$ as follows: for even $m$

$$A = m^{1/2}(a_1, a_1, a_2, a_2, \ldots, a_s, a_s),$$

$$B = m^{-1/2}(b_1, -b_1, b_2, -b_2, \ldots, b_s, -b_s);$$

for odd $m$

$$A = m^{1/2}(1, a_1, a_1, a_2, a_2, \ldots, a_s, a_s),$$

$$B = m^{-1/2}(0, b_1, -b_1, b_2, -b_2, \ldots, b_s, -b_s);$$

Let us put

$$M = \begin{bmatrix} A & B & 0 & , & \cdot & \cdot & \cdot & 0 \\ 0 & A & B & , & \cdot & \cdot & \cdot & 0 \\ & \cdot & \cdot & & \cdot & & & \\ 0 & 0 & \cdot & \cdot & \cdot & , & A & B \\ B & 0 & \cdot & \cdot & \cdot & , & 0 & A \end{bmatrix}$$

Precisely if $y_j(i)$ is the $j$-th coordinate of the $i$-th column of the matrix $M(1 \le i \le n;\ 1 \le j \le k)$, then

$y_j(i) =$ the $[j - (i-1)m]$-th coordinate of $A$ for $(i-1)m < j \le im$

$y_j(1) =$ the $[j - (n-1)m]$-th coordinate of $B$ for $(n-1)m < j \le nm$

$y_j(i) =$ the $[j - (i-2)m]$-th coordinate of $B$ for $(i-2)m < j \le (i-1)m$ for $i = 2, 3, \ldots, n$

$y_j(i) = 0$ elsewhere

We omit a routine verification that $M$ satisfies the assertion of the Lemma.

Case (II). Define the auxiliary $2 \times (2r+1)$ matrix $N_r = (n_{p,q})_{p=1,2;q=1,2,\dots,2r+1}$ as follows

$$N_r = \sqrt{\frac{2}{2r+1}} \begin{bmatrix} 1 & 0 \\ \cos\frac{\pi}{2r+1} & \sin\frac{\pi}{2r+1} \\ \cos 2\frac{\pi}{2r+1} & \sin 2\frac{\pi}{2r+1} \\ \dots & \\ \cos 2r\frac{\pi}{2r+1} & \sin 2r\frac{\pi}{2r+1} \end{bmatrix}$$

(The matrix $N_r$ corresponds to the equiweighted $(2r+1)$ – contact point representation for the regular $2(2r+1)$ – gone in the plane).

The desired $n \times k$ matrix is

$$M = \begin{bmatrix} N_r & 0 & \dots & 0 \\ 0 & N_r & \dots & 0 \\ \cdot & \cdot & \cdot & \\ 0 & 0 & \dots & N_r \end{bmatrix}$$

Precisely if $M = (y_j(i))_{\substack{1\le i\le n \\ 1\le j\le k}}$ then

$y_j(i) = n_{j-(2r+1)(t-1),1}$ for $(2r+1)(t-1) < j \le (2r+1)t;\, i = 2t-1$
$y_j(i) = n_{j-(2r+1)(t-1),2}$ for $(2r+1)(t-1) < j \le (2r+1)t;\, i = 2t \ (t = 1,2,\dots,s)$
$y_j(i) = 0$ elsewhere.

We omit a routine verification that the matrix $M$ has the desired properties.

**Remark 1.** In general the constant $\sqrt{2}$ can not be improved to $1+\eta$ for arbitrary $\eta > 0$. Indeed if $Y = (\mathbb{R}^2, D)$ is a Banach space with $E_D = B_2$ such that an equiweighted John's Parseval type identity is satisfied by 3 contact points, say $x_1, x_2, x_3$, then $B_2 \subset D \subset S$ where $S$ is a regular hexagon circumscribed about $B_2$. For, putting in (ii) $x_1, x_2, x_3$ instead of $x$ and using (i) we get the system of equations

$$1 = \frac{2}{3}\sum_{j=1}^{3}\langle x_i, x_j\rangle^2 = \frac{2}{3} + \frac{2}{3}\sum_{j\neq i}\langle x_i, x_j\rangle^2 \text{ for } i = 1,2,3.$$

Solving this system of equations we get $\langle x_i, x_j\rangle^2 = \frac{1}{4}$ for $i \neq j$. Hence the measures of the angles between two different versors $x_1, x_2, x_3$ is either $\frac{\pi}{3}$ or $\frac{2\pi}{3}$. Thus $D$ is contained in the regular hexagon $S$ determined by the tangent lines to $B_2$ passing through the points $x_1, x_2, x_3$. The inclusion $B_2 \subset D \subset S$ easily implies that $d(Y, (\mathbf{R}^2, S)) \leq \frac{2}{\sqrt{3}}$. By a result of Asplund [A], $d(\ell_2^\infty, (\mathbf{R}^2, S)) = \frac{3}{2}$. Thus

$$d(\ell_2^\infty, Y) \geq d(\ell_2^\infty, (\mathbb{R}^2, S)) \cdot d(Y, (\mathbb{R}^2, S))^{-1} \geq \frac{3\sqrt{3}}{4} > 1.$$

**Remark 2.** In case (I), $k = m \cdot n\,(m = 1, 2, \ldots;\ n = 2, 3, \ldots)$ for every $\eta > 0$ there exists $Y$ satisfying the assertion of Theorem 3.1 such that $d(X, Y) < 1 + \eta$ because in case (I) for given $\varepsilon$ with $0 < \varepsilon < 1$ the matrix $M$ is constructed so that its row vectors can be divided into $n$ groups, each group having $m$ elements and all elements of the $i$-th group are within Euclidean distance $\varepsilon$ from the $i$-th unit vector of $\mathbf{R}^n (i = 1, 2, \ldots, n)$.

**REFERENCES**


[A] E. ASPLUND, *Comparison between plane convex bodies and parallelograms*, Math. Scand. **8** (1960), 171-180.

[B] K. BALL, *Ellipsoids of maximal volume in convex bodies*, Geometria Dedicata **41** (1992), 241-250.

[B-S] J. BOURGAIN, S.I. SZAREK, *The Banach-Mazur distance to the cube and the Dvoretzky-Rogers factorization*, Israel J. Math. **62** (1988), 169-180.

[D-R] A. DVORETZKY, C.A. ROGERS, *Absolute and unconditional convergence in normed linear space*, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 192-197.

[J] F. JOHN, *Extremum problems with inequalities as subsidiary conditions*, Courant Anniversary Volume, pp. 187-204, Interscience, New York 1948.

[K] H.KÖNIG, *Spaces with large projection constant*, Israel J. Math **50** (1985), 181-188.

[L] D. R. LEWIS, *Ellipsoid defined by Banach ideal norms*, Mathematika **26** (1979), 18-29.

[M] A.A. MELKMAN, n-*widths of octahedra*, Proc. Int. Sympos. Quantitat. Approximation Theory Bonn 1979, pp. 209-216, Academic Press 1980.

[Pi] G. PISIER, *The volume of convex bodies and Banach space geometry*, Cambridge Tracts in Math. **94** Cambridge University Press, London-New York (1989).

[P-S] A. PELCZYNSKI, S.J. SZAREK, *On parallelepipeds of maximal volume containing a convex symmetric body in* $\mathbb{R}^n$, Math. Proc. Camb. Phil. Soc. **109** (1991), 125-148.

[P-T] A. PELCZYNSKI, N. TOMCZAK-JAEGERMANN, *On the length of faithful representations of finite rank operators*, Mathematika **35** (1988), 126-143.

[S] S.J. SZAREK, *Spaces with large distance to* $\ell_\infty^n$ *and random matrices*, American J. Math., **112** (1990), 899-942.

[T] N. TOMCZAK-JAEGERMANN, *Banach-Mazur distances and finite-dimensional operator ideals*, Pitman Monographs and Surveys in Pure and Applied Mathematics **38**, Longman House 1989.


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