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THE MACKEY DUAL OF A BANACH SPACE (*)

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Dedicated to the memory of Professor Gottfried Köthe

1. INTRODUCTION

If X is a Banach space, then the Mackey topology on the dual space X^* is the topology $\tau(X^*, X)$ of uniform convergence on weakly compact subsets of X. The classical Mackey-Arens Theorem tells us that τ is the finest locally convex topology on X^* whose dual space is X.

The topology τ is finer than the weak* topology $\sigma(X^*, X)$ and coarser than the norm topology on X^*, with equality holding only in special cases. The weak topology $\sigma(X^*, X^{**})$ also lies between the weak* and norm topologies; τ and weak are usually not comparable, but their compact sets often are, as shown by Grothendieck [11] in his study of the Dunford-Pettis and reciprocal Dunford-Pettis properties.

In general, the Mackey topology on X^* has received much less attention from functional analysts and topologists than its more famous relatives: norm, weak, and weak*. Recently, however, the authors have introduced and studied a new class of Banach spaces in which τ arises in a natural way [27]. A Banach space X is strongly WCG(SWCG) if and only if there is a weakly compact subset K of X such that for every weakly compact subset L of X and $\epsilon > 0$, there is a positive integer n with $L \subset nK + \epsilon B (B =$ closed unit ball of $X)$. This is a properly stronger notion than the familiar WCG property, since every SWCG space is weakly sequentially complete.

The relevant fact is that X is SWCG if and only if (B^*, τ), the dual unit ball with the relative Mackey topology, is (completely) metrizable [27, Th. 2.1]. Taking this as our starting point, we pursue here the topological study of (X^*, τ) and (B^*, τ). Two topological properties emerge as central. One is Michael's notion of an $\aleph_0$-space [18]: remarkably, if any of (B, weak), (X, weak), (B^*, τ), or (X^*, τ) has the $\aleph_0$-property, then so do all the others. This allows us to employ the powerful machinery of duality theory in a novel way. For example, it becomes easy to show that if X is a separable SWCG space, then (X, weak) must be an $\aleph_0$-space. The Banach space results in [18] can then be placed in a general context.

The other key topological notion is that of a k-space: a space in which the topology is determined by the compact subsets [6]. It is folklore that (B, weak) is a k-space iff X contains no isomorphic copy of l_1; we prove more than this (Theorem 5.1). If (B^*, τ) is a

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k-space, then X is weakly sequentially complete. If (X^*, τ) is a k-space, then either X is reflexive or X is hereditarily l_1.

In general topology, the k-and $\aleph_0$-spaces are precisely the quotients of separable metric spaces [18]. In the Banach space setting, more is true: $(B$, weak) is a k-and $\aleph_0$-space iff it is separably metrizable (iff X^* is norm separable). Also, if X is isomorphic to its square, then (B^*, τ) is a k-and $\aleph_0$-space iff it is separably metrizable (iff X is separable and SWCG). The point is that the k-and $\aleph_0$-properties are often a «factorization» of the separable metric property in this context.

The Batt-Hiermeyer space [3; 27, 2.6] is a separable, weakly sequentially complete dual space for which (B^*, τ) is neither a k-space nor an $\aleph_0$-space. The space $l_2(l_1)$ is a separable, weakly sequentially complete dual space for which (B^*, τ) is an $\aleph_0$-spaces, but not a k-space. The permanence properties of k-spaces and $\aleph_0$-spaces in this setting are also studied.

2. SOME FACTS ABOUT $\mathbf{X}^*, \tau$

We use [5, 26, 32] as references for Banach spaces and locally convex spaces. Section 1 of [27] presents some special results about (X^*, τ). This space is considered explicitly in [11, 13, 15, 16, 31].

The term «operator» means continuous linear operator from one Banach space into another. An operator $T : X \to Y$ is said to be a Dunford-Pettis operator (or completely continuous) iff it maps weakly compact sets to norm compact sets. Then X is said to have the Dunford-Pettis property (DP) iff every weakly compact operator $T : X \to Y$ is a Dunford-Pettis operator. Similarly, X is said to have the reciprocal Dunford-Pettis property (RDP) iff every Dunford-Pettis operator $T : X \to Y$ is weakly compact. If K is a compact Hausdorff space, then $X = C(K)$ enjoys both DP and RDP. Basic references for this area are [4, 11].

Proposition 2.1. *[11, p. 135, p. 152]. (a) X has DP iff every $\sigma(X^*, X^{**})$ -compact subset of X^* is $\tau(X^*, X)$-compact; (b) X has RDP iff every $\tau(X^*, X)$-compact subset of X^* is $\sigma(X^*, X^{**})$-compact.*

If (a) holds, then the weak and Mackey topologies agree on any weakly compact subset of X^*, since both topologies are compact and finer than the Hausdorff topology $\sigma(X^*, X)$. A similar result holds for τ-compact sets in (b).

Proposition 2.2. *[11, p. 134; 15]. A subset H of X^* is relatively τ-compact iff every weakly convergent sequence in X converges uniformly on H.*

A proof of the next result can be found in [8].

Proposition 2.3. *The following conditions on X are equivalent: (a) every Dunford-Pettis operator $T : X \to Y$ is compact; (b) every $\tau(X^*, X)$-compact set is norm-compact; (c) X*

contains no isomorphic copy of l_1.

Corollary 2.4. X *has the hereditary RDP iff* X *contains no isomorphic copy of* l_1.

Proposition 2.5. *The following conditions on* X *are equivalent: (a) every weakly compact operator* $T : X \to Y$ *is compact; (b) every* $\sigma(X^*, X^{**})$*-compact set is norm-compact (i.e.,* X^* *is a Schur space); (c)* X *has* DP, *and* X *contains no isomorphic copy of* l_1.

Proof. The equivalence of (a) and (b) is left to the reader. For (b) <=> (c), see [23; 4, pp. 23-24]. ■

The final result of this section should be compared with [5, p. 223, Ex. 2]. Let $S(X^*)$ be the set of vectors in X^* with norm 1.

Proposition 2.6. *(a) If* X *is reflexive, then* $S(X^*)$ *is* τ*-closed in* X^**; (b) if* X *is not reflexive, then* $S(X^*)$ *is* τ*-dense in* $B(X^*)$*; (c) if* X *contains no isomorphic copy of* l_1, *then* $S(X^*)$ *is* τ*-sequentially closed in* X^*.

Proof. (a) Since τ is the norm topology. (b) Let $x^* \in X^*$, $\|x^*\| < 1$, and let K be a weakly compact subset of X. For each $\epsilon > 0$, there is a member y^* of $S(X^*)$ such that $\sup\{|y^*(x)| : x \in K\} < \epsilon$; otherwise, the closed absolutely convex hull of K would contain $\epsilon \cdot B(X)$, and X would be reflexive. A simple argument shows that either $\|x^* + y^*\| \geq 1$ or $\|x^* - y^*\| \geq 1$. Hence there is a scalar $t, |t| \leq 1$, such that $z^* = x^* + ty^* \in S(X^*)$, and $\sup\{|z^*(x) - x^*(x)| : x \in K\} < \epsilon$. (c) This follows from 2.3. ■

3. COMPACTNESS, METRIZABILITY, AND SEPARABILITY

It is well-known that X with its weak topology is an angelic space [24], so that the various notions of compactness coincide. This need not be true for (X^*, τ). If $H \subset X^*$ is τ-sequentially compact, then the τ-closure of H is τ-totally bounded and τ-complete [27, 1.1], hence τ-compact. If $X = l_2[0,1]$, then $H = B^*$ is τ-compact, but not τ-sequentially compact [13]. If X is a WCG space, then $(B^*$, weak*) is an Eberlein compact [17, Th. 3.3]. Thus $(X^*$, weak*) is an angelic space, and so is (X^*, τ) [24, 0.5].

Proposition 3.1. *The following are equivalent: (a)* (B^*, τ) *is compact; (b)* (B^*, τ) *is locally compact; (c)* (B^*, τ) *is* σ*-compact; (d)* X *is a Schur space.*

Proof. (a) => (b) and (a) => c) are obvious.

(c) => (a): B^* is a countable union of τ-compact sets, which are norm-closed. Thus one of these sets has a norm-interior point, and it follows by standard arguments that (B^*, τ) is compact.

(b) => (d): Choose a weakly compact absolutely convex subset K of X such that $K^0 \cap \cap B^*$ is τ-compact. Let $x_n \to 0$ in $(X$, weak). Then $x_n \to 0$ uniformly on $K^0 \cap B^*$, by

2.2. Since $(K^0 \cap B^*)^0 = (K \cup B)^{00}$, for any $\epsilon > 0$ there is a positive integer n_0 such that $x_n \in \epsilon(K \cup B)^{00}$ for all $n \geq n_0$. Since $K \subset mB$ for some $m \geq 1$, we have $x_n \in \epsilon mB$ for all $n \geq n_0$. Thus $\|x_n\| \to 0$, and so X is a Schur space.

(d) => (a): the Mackey and weak* topologies coincide on B^* [26, p. 85]. ∎

The space (X, weak) is metrizable iff X is finite-dimensional; the space (X^*, τ) is metrizable iff X is reflexive (so that τ = norm topology on X^*). It is well-known that (B, weak) is metrizable iff X^* is norm separable. The question of complete metrizability of (B, weak) is investigated in [7]. The Banach spaces for which (B^*, τ) is (completely) metrizable are precisely the strongly WCG space [27]. Examples include reflexive spaces, separable Schur spaces, and spaces $L_1(\mu)$ for μ a σ-finite measure [27, 2.3].

If X is separable, then (X^*, weak*) is a countable union of compact metric spaces, hence hereditarily separable. In this case, it follows from the Hahn-Banach Theorem that any absolutely convex subset of X^* is separable for the Mackey topology. However, (B^*, τ) need not be hereditarily separable. Kirk [16] studied (X^*, τ) for $X = C(K)$, K a compact Hausdorff space, and showed that when K is first countable the natural image of K in B^* is τ-closed and discrete. Thus $X = C[0, 1]$ is a separable space for which (B^*, τ) is not hereditarily separable, and not Lindelöf. Indeed, since (B^*, τ) is separable with an uncountable closed discrete subset, it is not even normal.

If $X = l_\infty$, then (X^*, weak*) contains l_1 as a dense subspace, and so (X^*, τ) is separable, although X is not separable.

4. $\aleph_0$-SPACES

Topologists have studied a number of properties which fall under the heading of «generalized metric spaces» [12]. Among the most interesting of these is the notion of an $\aleph_0$-space introduced by Michael [18]. A collection $\mathscr{P}$ of (not necessarily open) subsets of a topological space T is called a pseudobase for T if, whenever $C \subset U$ with C compact and U open, then $C \subset P \subset U$ for some $P \in \mathscr{P}$. An $\aleph_0$-space is then a regular space with a countable pseudobase. A collection $\mathscr{P}$ of subsets of T is called a k-network for T if, whenever $C \subset U$ with C compact and U open, then $C \subset P_1 \cup \ldots \cup P_n \subset U$ for some finite collection $\{P_i\}$ of members of $\mathscr{P}$. Clearly T is an $\aleph_0$-space if and only if it is regular and has a countable k-network.

For the convenience of the reader, we summarize key results from [18].

Theorem 4.1 (Michael). *(a) All separable metric spaces and their regular quotient spaces are $\aleph_0$-spaces; (b) first countable or locally compact $\aleph_0$-spaces are separably metrizable; (c) every $\aleph_0$-space is separable, has the Lindelöf property, and every closed subset is a G_δ set; (d) the class of $\aleph_0$-spaces is preserved by all subsets and by countable products; (e) if (T, t) is an $\aleph_0$-space, and t' is a regular topology on T having the same compact sets as t, then*

(T, t') is an $\aleph_0$-space; (f) if S and T are $\aleph_0$-spaces, so is the continuous function space $C(S,T)$ with the compact-open topology; (g) to show that T is an $\aleph_0$-space, it suffices (in the definition of pseudobase) to consider open sets U selected from a sub-base for the topology; (h) if a regular space T is covered by closed $\aleph_0$-subspaces $(A_n)_{n\in N}$, and if each compact subset of T is contained in some A_n, then T is an $\aleph_0$-space; (i) a regular space is both an $\aleph_0$-space and a k-space if and only if it is a quotient of a separable metric space.

Our goal here is to incorporate these results into a Banach space setting. Already in [18] it was shown that

1) if X is a Banach space with separable dual, or if $X = l_1$, then $(X$, weak) is an $\aleph_0$-space;

and

2) if $X = C(K)$, K a compact Hausdorff space, then $(X$, weak) is an $\aleph_0$-space if and only if K is countable.

According to 4.1 (c), we need only consider separable Banach spaces.

Theorem 4.2. *If any of the spaces $(X$, weak), $(B$, weak), (X^*, τ) and (B^*, τ) has the $\aleph_0$-property, so do all the others.*

Proof. The results 4.1 (d) and (h) show that X and B satisfy or fail the $\aleph_0$-condition together, with a similar outcome for X^* and B^*. If $S = (X$, weak) is an $\aleph_0$-space and $T = \mathbf{R}$, then (X^*, τ) is a subspace of $C(S,T)$, endowed with the compact-open topology. Hence (X^*, τ) is an $\aleph_0$-space, by 4.1 (d) and (f).

Finally, if $S = (X^*, \tau)$ is an $\aleph_0$-space, and $T = \mathbf{R}$, let γ denote the topology on X of uniform convergence on compact subsets of S. Then (X, γ) is a subspace of $C(S,T)$ with the compact-open topology and thus an $\aleph_0$-space. The topologies γ and $\sigma(X, X^*)$ have exactly the same compact sets. This follows from a standard result in duality theory [26, p. 85], with E the complete locally convex space (X^*, τ), $F = R$, and H a weakly compact subset of $L(E, F) = X$. Now 4.1 (e) shows that $(X$, weak) is an $\aleph_0$-space. ∎

This remarkable duality has no analogue for the property of metrizability. If $X = c_0$, then $(B$, weak) is metrizable, since X has separable dual; but (B^*, τ) is not metrizable, since c_0 is not $SWCG$. If $X = l_1$, then (B^*, τ) is metrizable, since X is $SWCG$, but $(B$, weak) is not, since X^* is not separable. We will also see later (5.2) that there is no analogue for the property of being a k-space.

Theorem 4.3. *If X is separable and $SWCG$, then $(X$, weak) is an $\aleph_0$-space.*

Dual Proof. (B^*, τ) is separable and metrizable, hence an $\aleph_0$-space, by 4.1(a). The result follows from 4.2.

Dual Proof. (We include this to show how a pseudobase can be constructed explicitly). Let (K_n) be a strongly generating sequence of weakly compact subsets of X, and let (f_m) be a countable dense subset of (B^*, τ). Consider the sub-base for the weak topology on X consisting of all sets $f^{-1}(-\infty, q)$, where $f \in X^*$, $\| f \| = 1$, and q is rational. With a view to applying 4.1(g), let L be a weakly compact subset of such a half-space $f^{-1}(-\infty, q)$. Choose a rational ϵ such that $\max f(L) < q - \epsilon$, choose n such that $L \subset K_n + (\epsilon/4)B$, and choose m such that $\max\{|f(t) - f_m(t)| : t \in L \cup K_n\} < \frac{\epsilon}{4}$.

We claim that $L \subset (f_m^{-1}(-\infty, q - \epsilon/2) \cap K_n) + (\epsilon/4)B \subset f^{-1}(-\infty, q)$. For the first inclusion, let $x \in L$, and choose $y \in K_n$, $z \in B$ with $x = y + (\epsilon/4)z$. Then $f_m(y) < q - \epsilon/2$; for if not, then $f(y) > f_m(y) - \epsilon/4 \geq q - 3\epsilon/4$, so that $f(x) = f(y) + (\epsilon/4)f(z) \geq f(y) - \epsilon/4 > q - \epsilon$, a contradiction. A short calculation now verifies the second inclusion. According to [18, p. 986], the family of all finite unions of finite intersections of sets $(f_m^{-1}(-\infty, q_1) \cap K_n) + q_2 B$, for q_1 and q_2 rational, is a countable pseudobase for $(X$, weak). ∎

The next result unifies (and goes a bit further than) the Banach space results in [18].

Proposition 4.4. *The class of Banach space X such that $(X$, weak) is an $\aleph_0$-space includes all spaces with separable dual, separable Schur spaces, and separable $L_1(\mu)$ spaces. It is preserved by closed subspaces, but not by quotients.*

Proof. If X has separable dual, then $(B$, weak) is separably metrizable, so 4.1(a) applies. The result for separable Schur spaces and separable $L_1(\mu)$ spaces follows from 4.3 and [27, 2.3].

The property passes to any subset, by 4.1(d). The space l_1 belongs to this class, but its quotient space $C[0,1]$ does not: (B^*, τ) is not Lindelöf, as noted in Section 3, so it cannot be an $\aleph_0$-space, by 4.1(c). The quotient map is also a quotient map for the weak topologies on l_1 and $C[0,1]$ [26, p. 135]. Now a quotient space of a quotient space is a quotient space (direct verification), yet this example does not violate 4.1(a). The reason (which will follow from 4.1(i) and 5.4) is that $(l_1$, weak) cannot be a quotient of a separable metric space. ∎

We remark that (**) can be easily established with the techniques used here. Let $X = C(K)$. If$(X$, weak) is an $\aleph_0$-space, then so is (X^*, τ), and so is $(X^*$, weak), using 2.1 and 4.1(e). Thus X^* is weakly (hence norm) separable, by 4.1(c). Since the point masses at points of K have distance 2 apart in X^*, this forces K to be countable. Conversely, if K is countable, then $X^* = l_1(K)$ is separable, and so $(X$, weak) is an $\aleph_0$-space, by 4.4.

Proposition 4.5. *The class of Banach spaces X such that $(X$, weak) (or (X^*, τ)) is an $\aleph_0$-space is preserved by countable l^p-sums, $1 \leq p < \infty$, and by countable c_0-sums.*

Proof. Let $\{X_n\}_{n=1}^{\infty}$ be Banach spaces such that (X_n, weak) is an $\aleph_0$-space for all n, and let $X = l_p(X_n) = \{(x_n) : x_n \in X_n, \sum_{n=1}^{\infty} \| x_n \|^p < \infty\}$. For $p = 1$, $(B(X^*), \tau(X^*, X))$ is the topological product of the $\aleph_0$-space $(B(X_n^*), \tau(X_n^*, X_n))$ [27, 1.2], so 4.1(d) can be applied.

Now let $1 < p < \infty$. For each n, let $\mathscr{P}_n$ be a countable pseudobase for $(B(X_n), \text{weak})$. Let $\mathscr{P}$ be the (countable) collection of all sets $(P_1 + P_2 + \ldots + P_m + Z_m) \cap B(X)$, where $P_i \in \mathscr{P}_i$, $1 \leq i \leq m$, and $Z_m = \{x \in X : x_i = 0, 1 \leq i \leq m\}$.

We apply 4.1(g): let K be weakly compact in $B(X)$, and consider $x^* \in X^*$ and $\alpha \in R$ such that $K \subset U = \{x \in B(X) : x^*(x) < \alpha\}$, a subbasic open set for the weak topology. Now $X^* = l_q(X_n^*)$, where p and q are conjugate exponents. Hence there exist m and a rational number β such that if $y^* = (x_1^*, x_2^*, \ldots, x_m^*, 0, 0 \ldots)$ is the m^{th} truncate of x^*, then $K \subset V = \{x \in B(X) : y^*(x) < \beta\} \subset U$.

Let $\mathscr{C}$ be the (countable) collection of all m-tuples of rational numbers whose sum is β. For each $i, 1 \leq i \leq m$, and rational q_i, let $S(i, q_i) = \{x_i \in B(X_i) : x_i^*(x_i) < q_i\}$ and $T(i, q_i) = \{x \in B(X) : x_i^*(x) < q_i\}$. Thus if π_i is the natural projection of $B(X)$ onto $B(X_i)$, $T(i, q_i) = \pi_i^{-1}(S(i, q_i))$. For each $C = (q_1^C, q_2^C, \ldots, q_m^C) \in \mathscr{C}$, let $W_C = \cap_{i=1}^m T(i, q_i^C) \subset B(X)$. Then $K \subset \cup\{W_C : C \in \mathscr{C}\} = V$, so $\exists C_1, \ldots, C_p \in \mathscr{C}$ with $K \subset \cup_{j=1}^k W_{C_j} \subset V \subset U$. It is a standard result that there exist weakly compact sets K_j with $\cup_{j=1}^k K_j = K$ and $K_j \subset W_{C_j}$ for all j.

For each j, $1 \leq j \leq k$, and i, $1 \leq i \leq m$, choose $P_{i,j} \in \mathscr{P}_i$ with $\pi_i(K_j) \subset \subset P_{i,j} \subset S(i, q_i^{C_j})$. Then $K_j \subset (P_{1,j} + P_{2,j} + \ldots + P_{m,j} + Z_m) \cap B(X) \subset W_{C_j}$, and so $K \subset \cup_{j=1}^k (P_{1,j} + P_{2,j} + \ldots + P_{m,j} + Z_m) \cap B(X) \subset V \subset U$. The result follows. ∎

The same argument works for countable c_0-sums, but not for countable l_∞-sums, since the truncates of x^* do not coverge to x^* in norm. Note that (l_∞, weak) is not an $\aleph_0$-space, since it is not separable.

The Batt-Hiermeyer space 4.6. *Let $X = BH$ be the separable, weakly sequentially complete dual space introduced in [3]. The space BH is a tree space which behaves like l_1 on totally ordered subsets of the binary tree, but like l_2 on subsets of the binary tree whose members are pairwise incomparable. In [27, 2.6] it is shown that X is not an SWCG space. Here we present the deeper result that (X, weak) (equivalently, (B^*, τ)) is not an $\aleph_0$-space (cf. 4.3).*

It suffices to find uncountable collections $(K_A)_{A\in\Gamma}$ and $(U_A)_{A\in\Gamma}$ of compact and open subsets of (X, weak) such that $K_A \subset U_A$ for all A, but $K_D \not\subset U_A$ for $A \neq D$. For then if (P_n) were a countable pseudobase for (X, weak), we would have $K_A \subset P_{n(A)} \subset U_A$, so that there would be a sequence (U_n) with each K_A contained in some U_n, a contradiction.

Following the notation of [27, 2.6], let Γ be the set of all infinite, convex, totally ordered subsets A of C, the binary tree, which begin at (0). Fix such an $A = \{t_n\}$, and let $x_n(A) = (1/2n)e_{t_n} + e_{(t_n,i_n)}$, where $i_n = 0$ or 1 is chosen so that $(t_n, i_n) \neq t_{n+1}$. Let $K_A = \{x_n(A)\} \cup \{0\}$, and let $U_A = \{x \in X : \exists m$ such that $|\langle x_m(A) - x, x_A^*\rangle| < \frac{1}{4}\}$. We show that $x_n(A) \to 0$ weakly. It will follow that K_A is weakly compact, U_A is weakly open, and $K_A \subset U_A$.

Let P_A denote the natural projection $x \to x|A$ on BH. Then the sequenec $(P_A(x_n(A)))$ converges to 0 in norm, so it suffices to show that $(x_n(A) - P_A(x_n(A)))$ is equivalent to the unit vector basis of l_2. This is an easy consequence of the fact that the (t_n, i_n) are pairwise incomparable: if (c_j) is a sequence of scalars, then

$$\|\sum_{j=1}^{m} c_j(x_j(A) - P_A(x_j(A)))\| = \left(\sum_{j=1}^{m} |c_j(x_j(A) - P_A(x_j(A)))(t_j, i_j)|^2\right)^{1/2}$$

$$= \left(\sum_{j=1}^{m} |c_j|^2\right)^{1/2}.$$

Fix $A, D \in \Gamma$, $A \neq D$, and let $s = \min\{t \in C : t \in A, t \notin D\}$. Let s_0 be the immediate predecessor of s, and suppose s_0 is the nth member of A (and D). Then $\langle x_A^*, x_n(D)\rangle = 1/2 + 1$, while $\langle x_A^*, x_m(A)\rangle \leq 1/2$ for all m. Thus $K_D \not\subset U_A$, to complete the argument.

5. k-SPACES AND TOPOLOGICAL FACTORIZATIONS

A Hausdorff space T is a k-space iff a subset which intersects each compact set in a closed set must be closed. Equivalently, the topology on T is the finest yielding the same collection of compact sets as itself. The class of k-spaces is extensive – it includes both locally compact and first countable spaces [6]. Its major defect is its failure to be preserved under finite products; this plays a role in 5.7.

The space $(X$, weak) cannot be a k-space unless X is finite-dimensional. Indeed if the k-space property holds, then the topology γ denied in the proof of 4.2 coincides with the weak topology. Thus every norm-convergent sequence in X^*, being τ-compact, must have finite-dimensional linear span. Hence X^* and X are finite-dimensional.

The situation for $(B$, weak) is far more interesting. A Hausdorff space T is said to be a Fréchet-Urysohn space iff whenever $p \in \overline{A} \subset T$, then some sequence in A converges to p. A space is Fréchet-Urysohn iff every subset is a k-space in the relative topology [2]. Also, T is said to be a sequential space iff every sequentially closed subset is closed.

The following result seems to be well-known, but we have not found it recorded in this form. An early version appears in [9].

Theorem 5.1. *The following conditions on a Banach space X are equivalent: (a) (B, weak) is a Fréchet-Urysohn space; (b) (B, weak) is a sequential space; (c) (B, weak) is a k-space; (d) X contains no isomorphic copy of l_1.*

Proof. (a) => (b) => (c) is easily seen to hold for any Hausdorff topological space.

(c) => (d): Suppose l_1 is isomorphic to a subspace of X. Then ($B(l_1)$, weak) is homeomorphic to a closed subset of ($nB(X)$, weak) for some n, hence is a k-space. But this is clearly not true, since the weak and norm topologies on $B(l_1)$ have the same compact sets.

(d) => (a): Let p belongs to the weak closure of A in $B(X)$. Every Banach space in its weak topology has the property of *countable tightness*: there is a countable subset C of A whose weak closure contains p [32, p. 229]. Then Y, the closed linear span of $\{p\}\cup C$ in X, is a separable space containing no isomorphic copy of l_1. By the Odell-Rosenthal Theorem [22], $(B(Y^{**}), w^*)$ is a pointwise compact set of Baire-1 functions on the compact metric space $(B(Y^*), w^*)$. Thus $(B(Y^{**}), w^*)$ is a Rosenthal compact in the sense of Godefroy [10], so it is a compact angelic space, and therefore has the Fréchet-Urysohn property. Since ($B(Y)$, weak) is a subspace of $(B(Y^{**}), w^*)$, some sequence in C must converge weakly to p. ■

In contrast to 4.2, the property of being a k-space is very far from being a «dual property».

Corollary 5.2. *(B, weak) is a k-space if and only if every τ-compact subset of X^* is norm-compact. Thus (B, weak) and (B^*, τ) are both k-spaces if and only if X is reflexive.*

*Proof.*The first assertion is an immediate consequence of 2.3 and 5.1. The condition «every τ-compact set is norm-compact» says that the k-space associated with (B^*, τ) is (B^*, norm). The second assertion now follows. ■

Corollary 5.3. *The class of Banach spaces such that (B, weak) is a k-space is preserved by closed subspaces, quotients, and finite products.*

*Proof.*This is immediate from 5.1 and the fact [4, p. 42] that the property «X contains no isomorphic copy of ℓ_1» is a three-space property. The families of k-spaces and Fréchet-Urysohn spaces are not preserved by two-fold products in general topology. ■

Factorization theorem for (B, weak). *The following are equivalent: (a) (B, weak) is (separably) metrizable; (b) (B, weak) is a k-and-$\aleph_0$-space; (c) X^* is separable.*

*Proof.*The equivalence of (c) with either version of (a) is well-known, and the separable version of (a) clearly implies (b).

(b) => (c): By 5.2, the Mackey and norm topologies on X^* admit the same compact sets. By 4.1(e) and (4.2), (X^*, norm) is an $\aleph_0$-space, so it is separable. ■

This is much stronger than the best available result in general topology (4.1(i)).

Example 5.5. *Let X be the James tree space [14], a separable Banach space which contains no isomorphic copy of l_1, yet has non-separable dual. Then since X is itself a dual space, (B, weak) is a k-space, and admits a coarser compact metric topology, but is not an $\aleph_0$-space.*

We turn now to the k-space question for (X^*, τ) and (B^*, τ).

Theorem 5.6. *If X is either reflexive or a Schur space, then (X^*, τ) is a k-space.*

Proof. If X is a Schur space, then τ is the topology of uniform convergence on norm-compact subsets of X. By the Banach-Dieudonné Theorem [26, p. 151], τ is the finest topology (locally convex or not) which agrees with the weak* topology on weak*-compact sets. Thus (X^*, τ) is a k-space. ∎

Towards a partial converse of this result, let A_n be a space homeomorphic to $\{\frac{1}{m}\}_{m=1}^{\infty} \cup \{0\}$, for each positive integer n, such that $A_n \cap A_p = \{0\}$ for $n \neq p$. The hedgehog space H is the quotient space of $\cup_{n=1}^{\infty} A_n$ obtained by identifying all the 0 points in the A_n. The space H is a k-space, but is not first countable at 0. A result of Michael [19] shows, in particular, that for a metrizable space T, $T \times H$ is a k-space iff T is locally compact. Hence if T is a non-locally compact metric space, and Y is a space containing a closed copy of H, then $T \times Y$ is not a k-space.

Now observe that for $X = l_1$, (X^*, τ) contains a closed copy of H. Indeed let $S = \{ne_m\}_{m,n=1}^{\infty} \cup \{0\} \subset l_\infty$, where e_m denotes the mth unit vector. For fixed n, the sequence $\{ne_m\}_{m=1}^{\infty}$ is τ-convergent to 0; hence there is a natural 1-1 correspondence between S and H. Moreover, the correspondence preserves compact sets (in both directions). Now H is a k-space, and so is S (it is a closed subset of (l_∞, τ), which is a k-space by 5.6). Hence the correspondence is a homeomorphism.

Theorem 5.7. *If (X^*, τ) is a k-space, then either X is reflexive or X is hereditarily l_1 (i.e., every infinite-dimensional closed subspace contains an isomorphic copy of l_1).*

Proof. First we show that if Z is an infinite-dimensional closed subspace of X, then either (a) Z is reflexive or (b) Z contains an isomorphic copy of l_1. Now (Z^*, τ) is a quotient space of (X^*, τ) [26, p. 135]. Since k-spaces are exactly the quotients of locally compact spaces [6, p. 248], and a quotient of a quotient is a quotient, we have that (Z^*, τ) is a k-space. If Z contains no isomorphic copy of l_1, then every τ-compact set in Z^* is norm-compact, by 2.3. Hence τ and norm must coincide on Z^*, so Z is reflexive.

Now suppose that both alternatives (a) and (b) occur. Thus X contains infinite-dimensional subspaces Z_1 and Z_2 which are reflexive and isomorphic to l_1, respectively. Then Z_1 and Z_2 are totally incomparable Banach spaces, so by [25, Th. 1], $Z_1 + Z_2$ is closed in X

and isomorphic to $Z_1 \times Z_2$. Now $T = (Z_1^*, norm)$ is a non-locally compact metric space, and $Y = (Z_2^*, \tau) = (l_\infty, \tau)$ contains a closed copy of H. Hence $T \times Y = ((Z_1 \times Z_2)^*, \tau)$ is not a k-space. This space is a quotient of (X^*, τ), so that space is not a k-space either, a contradiction. This completes the proof. ∎

Theorem 5.8. *If (B^*, τ) is a k-space, then X is weakly sequentially complete.*

Proof. Let γ again denote the topology on X of uniform convergence on τ-compact subsets of X^*. We show that (X, γ) is a complete locally convex space. Applying Grothendieck's Completeness Theorem, let f be a linear functional on the topological dual space $(X, \gamma)' = X^*$ such that $f|H$ is weak*-continuous for each τ-compact $H \subset X^*$. Then f is τ-continuous on each compact subset of the k-space (B^*, τ) and so $f|B^*$ is τ-continuous. Then $f^{-1}(0) \cap B^*$ is τ-closed and convex, hence weak*-closed, and so $f \in X$ [26, p. 149].

Now [31, 1.3, 1.4] shows that γ is the finest locally convex topology on X which has the same convergent (or Cauchy) sequences as the weak topology. The weak sequential completeness of X follows immediately. ∎

Since a metric space is a k-space, this is a stronger result than the fact that an $SWCG$ space is weakly sequentially complete [27, 2.5]. Note that for $X = L_1[0,1]$, (B^*, τ) is a k-space, but (X^*, τ) is not ([27, 2.3] and 5.7). Also it can be shown that the Banach space X_0 constructed in [1], a separable, hereditarily l_1 $SWCG$ space, fails to have (X_0^*, τ) a k-space. Thus the converse to 5.7 is false. We do not know if the converse to 5.6 is true.

If X is infinite-dimensional and enjoys both the Dunford-Pettis and reciprocal Dunford-Pettis properties, then (B^*, τ) cannot be a k-space. Indeed the topologies $\tau(X^*, X)$ and $\sigma(X^*, X^{**})$ have the same compact sets, by 2.1. Thus $\sigma(X^*, X^{**})$ would be coarser than $\tau(X^*, X)$ on B^*, so X would be reflexive. A reflexive space with DP is finite dimensional.

Example 5.9. *A weakly sequentially complete space X such that (B^*, τ) is not a k-space. Let $X = C[0,1]^*$, the space of bounded regular Borel measures on [0,1]. Then X can be expressed as an uncountable l_1-sum of separable $L_1(\mu)$ spaces. By [27, 1.2, 2.3], (B^*, τ) is homeomorphic to an uncountable product of complete separable metric spaces. Such a product is never a k-space unless all but countably many of the factors are compact [21], which is not the case here.*

The examples $l_2(l_1)$ and BH (4.5 and 4.6) are separable, weakly sequentially complete dual spaces for which (B^*, τ) is not a k-space (it is an $\aleph_0$-space for the first of these, but not the second). For $l_2(l_1)$ this will follow from the results presented below (5.10, 5.11). The rather lengthy proof for BH is omitted.

Theorem 5.10. *Let $\{X_n\}_{n=1}^{\infty}$ be $SWCG$ Banach spaces, and let $1 < p < \infty$. Then if $X = l_p(X_n)$, X is $SWCG$ iff all but finitely many of the X_n are reflexive.*

Proof. The sufficiency is straightforward. For the necessity, we may suppose that X_n is non-reflexive for all n. Let K_0 be a strongly generating, weakly compact subset of X, and let $K_n = \pi_n(K_0)$ for each n. Clearly K_n is strongly generating for X_n.

For each n, we can choose $x_n \in B(X_n)$ such that $x_n \notin nK_n + \frac{1}{2}B(X_n)$. Otherwise, we would have $B(X_n) \subset nK_n + \frac{1}{2}B(X_n)$, so $\frac{1}{2}B(X_n) \subset \frac{n}{2}K_n + \frac{1}{4}B(X_n)$, and thus $B(X_n) \subset (n+\frac{n}{2})K_n + \frac{1}{4}B(X_n)$, etc. An application of Grothendieck's Criterion [5, p. 227] shows that $B(X_n)$ is weakly compact, so X_n is reflexive, a contradiction.

Now let $L = \{(0,0,\ldots,x_n,0,\ldots)\}_{n=1}^{\infty} \cup \{0\}$, a weakly convergent sequence in X. Since K_0 is strongly generating, there is a positive integer m with $L \subset mK_0 + \frac{1}{2}B(X)$. But then $x_n \in mK_n + \frac{1}{2}B(X_n)$ for all n, a contradiction. ∎

This should be compared with [27, 2.9].

It is natural to inquire if an analogue of 5.4 holds for (B^*, τ). In other words, if (B^*, τ) is a k-and-$\aleph_0$-space, must it be metrizable, so that X is $SWCG$? Since (X, weak) will also be an $\aleph_0$-space under these conditions, the problem is restricted to separable X.

Although we have not solved this question completely, we are able to present some partial results, based on the penetrating topological work of Tanaka [28, 29, 30].

Theorem 5.11. *Let X be a separable Banach space such that X is isomorphic to $X \times X$. Then X is $SWCG$ if and only if (B^*, τ) is a k-and-$\aleph_0$-space.*

Proof. The necessity is clear, since (B^*, τ) is a separable metric space. If $D = (B^*, \tau)$ is a k-and-$\aleph_0$-space, and $T : X \to X \oplus_1 X$ is a linear homeomorphism, then T^* is a homeomorphism of $D \times D$ onto a closed subset of a suitable multiple of D, so $D \times D$ is a k-and-$\aleph_0$-space. According to [29, Th. 1.1], either D is a separable metric space or D is σ-compact. In the latter case, X is a separbale Schur space, by 3.1, so D is still metrizable. ∎

Since $X = l_2(l_1)$ is isomorphic to its square, it follows from 4.5, 5.10, and 5.11 that (B^*, τ) is not a k-space in this instance.

Theorem 5.12. *Let X be a separable Banach space, and let $Y = l_1(X)$, the l_1-sum of countably many copies of X. Then X is $SWCG$ if and only if $(B(Y^*), \tau)$ is a k-and-$\aleph_0$-space.*

Proof. The necessity follows from [27, 2.9]. Conversely, $(B(Y^*), \tau)$ is homeomorphic to a countable product of copies of $(B(Y^*), \tau)$, by [27, 1.2]. A k-and-$\aleph_0$-space is a sequential space [20, Th. 7.3]. A direct application of [28, Th. 1.3(i)] shows that $(B(Y^*), \tau)$ is metrizable, so X is $SWCG$. ∎

The remarkable result [30, Th. 4.4] reveals exactly how a k-and-$\aleph_0$-space (B^*, τ) could fail to be metrizable, if indeed this can occur at all. Let H again denote the hedgehog space. Let $S_2 = (N \times N) \cup (N \times \{0\}) \cup \{(0,0)\}$, where each point of $N \times N$ is isolated; a base of neighbourhoods of $(n, 0)$ consists of sets of the form $\{(n, 0)\} \cup \{(n, m) : m \geq m_0\}$; and U is a neighbourhood of $(0,0)$ if $(0,0) \in U$, and U is a neighbourhood of all but finitely many $(n, 0)$. Then (B^*, τ) is metrizable if and only if it contains no (closed) copy of H or S_2.

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