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A REMARK ON BASES IN QUOTIENTS OF ℓ_p WHEN $0 < p < 1$

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Dedicated to the memory of Professor Gottfried Köthe

In [9] Stiles showed that if $0 < p < 1$, ℓ_p has an infinite-dimensional closed subspace which contains no complemented copy of ℓ_p; this contrasts with the well-known result of Pelczynski [5] for $1 \leq p < \infty$. The following curious theorem is the main result of this note:

Theorem 1. *Let M be an infinite-dimensional closed subspace of ℓ_p where $0 < p < 1$. Suppose ℓ_p/M has a basis. Then M contains a subspace isomorphic to ℓ_p and complemented in ℓ_p.*

Before proving the result we make some remarks. First note that every separable p-Banach space is a quotient of ℓ_p ([6], [8]). The condition that ℓ_p/M has a basis cannot be weakened substantially. Indeed, take any normalized block basic sequence (f_n) in ℓ_p so that $\| f_n \|_1 \to 0$; let $M = [f_n]$. Then ℓ_p/M is isomorphic to $\ell_p(Y_n)$ where each Y_n is the quotient of some ℓ_p^m by a subspace of dimension one. However M contains no complemented copy of ℓ_p (see [9]). Thus the space $\ell_p(Y_n)$, which has a finite-dimensional Schauder decomposition (see [1]), has no basis (cf. [10] for a counter-example in the much more difficult Banach space setting).

Proof of Theorem 1. We shall suppose that dim $M = \infty$, ℓ_p/M has a basis and that M contains no complemented copy of ℓ_p; we then derive a contradiction. It will be convenient to regard ℓ_p as a subspace of its Banach envelope ℓ_1. If T is a bounded linear operator on ℓ_1, i.e. $T \in \mathcal{L}(\ell_1)$ then

$$\| T \|_1 = \sup_{\|x\|_1 \leq 1} \| Tx \|_1$$

and

$$\| T \|_p = \sup_{\|x\|_p \leq 1} \| Tx \|_p .$$

Then $\mathcal{L}(\ell_p)$ can be identified as the subalgebra of $\mathcal{L}(\ell_1)$ of all T such that $\| T \|_p < \infty$. Note that $\| T \|_1 \leq \| T \|_p$ for all T. We also let $\mathcal{K}(\ell_1)$ be the subalgebra of compact operators on ℓ_1 and let $\mathcal{L}(\ell_p; M)$ be the subalgebra of $\mathcal{L}(\ell_p)$ of all T such that $T(\ell_p) \subset M$.

By a theorem of Stiles [9], the set $M \cap \{x : \| x \|_p \leq 1\}$ is relatively compact in ℓ_1 so that $\mathcal{L}(\ell_p; M) \subset \mathcal{K}(\ell_1)$. We further observe two properties of $\mathcal{L}(\ell_p; M)$:

(A1) Suppose (A_n) is a bounded sequence in $\mathcal{L}(\ell_p; M)$ with $\| A_n x \|_1 \to 0$ $\forall x \in \ell_1$. Then $\lim \| A_n^2 \|_1 = 0$.
(A2) For each $\varepsilon > 0$ there is a finite-dimensional subspace F_ε of M so that if $A \in$ $\in \mathcal{L}(\ell_p; M)$

$$d_1 (Ax, F_\varepsilon) = \inf_{f \in F_\varepsilon} \| x - f \|_1 \leq \varepsilon \| A \|_p \| x \|_1$$

for $x \in \ell_1$.

To prove (A1) simply note that the set $K = \{A_n x : \| x \|_1 \leq 1, n \in N\}$ is relatively compact in ℓ_1 and hence $\| A_n x \|_1 \to 0$ uniformly on K.

For (A2) pick F_ε so that if $x \in M$ then $d_1(x, F_\varepsilon) \leq \varepsilon \| x \|_p$. If $A \in \mathcal{L}(\ell_p; M)$, we have $d_1(Ax, F_\varepsilon) \leq \varepsilon \| A \|_p \| x \|_p$ for $x \in \ell_p$. As $d_1(Ax, F_\varepsilon)$ is a seminorm on ℓ_p we conclude that $d_1(Ax, F_\varepsilon) \leq \varepsilon \| A \|_p \| x \|_1$ for $x \in \ell_1$.

Since ℓ_p / M has a basis, M is weakly closed in ℓ_p. If M_1 is the closure of M in ℓ_1 then $M = M_1 \cap \ell_p$. Let $\pi : \ell_1 \to \ell_1 / M_1$ be the quotient map. Then ℓ_1 / M_1 can be identified with the Banach envelope of ℓ_p / M and then $\pi | \ell_p$ is the quotient map onto ℓ_p / M.

Let (b_n) be a normalized basis for ℓ_p / M with biorthogonal functionals (b_n^*). Let S_n be the partial sum operators i.e. $S_0 = 0$ and

$$S_n x = \sum_{k=1}^{n} b_k^*(x) b_k.$$

Let C be a constant such that $C > 2^{1/p}$ and $\| S_m - S_n \| \leq C$ for all $m > n$. Then fix $\varepsilon = \frac{1}{48} C^{-4}$ and choose $F = F_\varepsilon$ as in (A2).

We now lift S_n to ℓ_p. Pick $a_k \in \ell_p$ so that $\| a_k \| < 2$ and $\pi a_k = b_k$. Define $D_n \in \mathcal{L}(\ell_p)$ by $D_n(x) = b_n^*(\pi x) a_n$. Then $\| D_n \|_p \leq 2 \| b_n^* \| \leq 2C$ $\forall n$.

Now $\pi(D_1 + \ldots + D_n) = S_n \pi$ but the sequence $D_1 + \ldots + D_n$ need not be bounded.

Next we obtain another lifting of S_n. Let e_n be the standard basis of ℓ_p. For each n, k select $u_{n,k} \in \ell_p$ so that $\pi u_{n,k} = \pi e_k - S_n \pi e_k$ and $\| u_{n,k} \|_p \leq 2 \| \pi e_k - S_n \pi e_k \|$. Then set

$$R_n x = \sum_{k=1}^{\infty} x_k u_{n,k}.$$

We have

$$\| R_n \|_p \leq \sup_k \| u_{n,k} \|_p \leq 2C.$$

We also have that $\| R_n x \|_p \to 0$ for $x \in \ell_p$.

Let Q be any projection onto the finite-dimensional space F with $\| Q \|_p < \infty$; such a projection exists since ℓ_p has a separating dual. Then $\| R_n Q \|_p \to 0$ and so there exists N_1 so that if $n \geq N_1$, $\| r_n - R_n Q \|_p \leq 4C$. Set $T_n = I - R_n + R_n Q$. Then:

$$(T1) \qquad \| T_n \|_p \leq 4C^2 \qquad n \geq N_1$$

$$(T2) \qquad \lim_{n\to\infty} \| T_n x - x \|_p = 0 \qquad x \in \ell_p$$

$$(T3) \qquad T_n x = x \qquad x \in F$$

$$(T4) \qquad \pi T_n = S_n \pi \qquad n \in \mathbb{N}.$$

Note that $T_n - (D_1 + \ldots + D_n) \in \mathcal{L}(\ell_p, M) \subset \mathcal{K}(\ell_1)$ so that $T_n \in \mathcal{K}(\ell_1)$ for each $n \in N$. Unfortunately, the operators T_n are not projections in general. However $T_n - T_n^2 \in \mathcal{L}(\ell_p, M)$ for $n \in N$ and so by (A1) we obtain:

$$(T5) \qquad \lim_{n\to\infty} \| (T_n - T_n^2)^2 \|_1 = 0.$$

Now let $V_n = 3T_n^2 - 2T_n^3$. It is easy to verify:

$$(V1) \qquad \| V_n \|_p \leq 12C^3 \qquad n \geq N_1$$

$$(V2) \qquad \lim_{n\to\infty} \| V_n x - x \|_p = 0 \qquad x \in \ell_p$$

$$(V3) \qquad V_n x = x \qquad x \in F$$

$$(V4) \qquad \pi V_n = S_n \pi \qquad n \in \mathbb{N}.$$

Fix $N_2 \geq N_1$ so that for $n \geq N_2$,

$$\| (T_n - T_n^2)^2 \|_1 < \frac{1}{16}$$

using (T5). For $n \geq N_2$ we can define $P_n \in \mathcal{L}(\ell_1)$ by

$$P_n = \frac{1}{2}\left(I - (I - 2T_n)\sum_{m=0}^{\infty}\binom{2m}{m}(T_n - T_n^2)^m\right)$$

where $\binom{0}{0} = 1$. We do not claim $P_n \in \mathcal{L}(\ell_p)$. However we shall prove:

$(P1)$
$$\lim_{n\to\infty} \| P_n - V_n \|_1 = 0$$

$(P2)$
$$\lim_{n\to\infty} \| P_n x - x \|_1 = 0 \qquad x \in \ell_p$$

$(P3)$
$$P_n^2 = P_n \qquad n \geq N_2$$

$(P4)$
$$P_n x = x \quad n \geq N_2, \qquad x \in F$$

$(P5)$
$$\dim P_n(\ell_1) < \infty \qquad n \geq N_2$$

$(P6)$
$$\pi P_n = S_n \pi \qquad n \geq N_2.$$

In (P6) S_n denotes the natural extension of S_n to the Banach envelope ℓ_1/M_1 of ℓ_p/M. To prove (P1) simply note that

$$P_n - V_n = \frac{1}{2}(2T_n - I)\sum_{m=2}^{\infty}(T_n - T_n^2)^m$$

and use (T5). Then (P2) follows from (P1) and (V2).

For (P3) we need only show that

$$(I - 2T_n)\left(\sum_{m=0}^{\infty}\binom{2m}{m}(T_n - T_n^2)^m\right)^2 = I.$$

But this follows from the formal manipulation of power series since

$$(1-4z)^{1/2} = \sum_{m=0}^{\infty} \binom{2m}{m} z^m$$

for $|z| < \frac{1}{4}$.

(P4) is trivial from (T3). Since T_n and V_n are compact on ℓ_1, so is P_n and hence by (P3) it is finite-rank. Thus (P5) is proven. Finally, for (P6) note that $\pi(T_n - T_n^2) = 0$.

From (P2) we deduce that $\lim_{n\to\infty} \| P_n x - x \|_1 = 0$ for every $x \in \ell_1$. Also from (P6) M_1 is invariant for each P_n. Thus if $G_n = P_n(M_1)$ then dim $G_n \to \infty$. We derive our contradiction by showing, on the contrary, that the sequence dim G_n is bounded.

Pick $N_3 \geq N_2$ so that if $n \geq N_3$, we have $\| P_n - V_n \| \leq \frac{1}{4}$. For $n \geq N_3$ we have $V_n - V_{n-1} - D_n \in \mathcal{L}(\ell_p, M)$ and

$$\| V_n - V_{n-1} - D_n \|_p \leq 12C^4 .$$

Thus, for $x \in \ell_1$,

$$\mathrm{d}_1 \left(V_n x - V_{n-1} x - D_n x, F\right) \leq 12C^4 \varepsilon \| x \|_1 = \frac{1}{4} \| x \|_1 .$$

If $x \in M_1$ then $D_n x = 0$ and so

$$\mathrm{d}_1 \left(P_n x - P_{n-1} x, F\right) \leq \frac{3}{4} \| x \|_1 .$$

Now (P4) implies $F \subset G_{n-1}$ and so

$$\mathrm{d}_1 \left(P_n x, G_{n-1}\right) \leq \frac{3}{4} \| x \|_1$$

for all $x \in M_1$. In particular if $x \in G_n$ and $\| x \|_1 = 1$ then $\mathrm{d}_1(x, G_{n-1}) \leq \frac{3}{4}$. This implies by a classical result of [3] (see [7], p. 269) that dim $G_n \leq$ dim G_{n-1} (for $n \geq N_3$). This gives us the desired contradiction.

We now give a simple application. Lindenstrauss and Rosenthal [4] showed that if M_1 and M_2 are two infinite-dimensional closed subspaces of ℓ_1 such that the quotients ℓ_1/M_1 and ℓ_1/M_2 are isomorphic then there is an automorphism $\tau : \ell_1 \to \ell_1$ so that $\tau(M_1) = M_2$. The corresponding result for $p < 1$ is false, but is true under the additional hypotheses that M_1 and M_2 contain copies of ℓ_p complemented in the whole space (see [2]). Thus we have:

Theorem 2. *Let M_1 and M_2 be two closed subspaces of ℓ_p, where $0 < p < 1$. Suppose the quotients ℓ_p/M_1 and ℓ_p/M_2 are isomorphic and have a basis. Then there is an automorphism τ of ℓ_p with $\tau(M_1) = M_2$.*

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