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REMARKS ON COMPACTNESS OF OPERATORS DEFINED ON L_p

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Dedicated to the memory of Professor Gottfried Köthe

This note presents several observations on Banach spaces X such that, for fixed $1 \leq \leq p \leq \infty$, every operator from an L_p-space into X which is weakly compact is already compact. The interest in such objects is due to the fact that a Banach space X has the above property for $2 \leq p < \infty$ if and only if, for some and then all $2 \leq q < \infty$, every strictly q-integral operator with values in X is already q-integral. Recall that a Banach space X has the Radon-Nikodym property iff every strictly 1-integral X-valued operator is nuclear. We shall, however, not discuss any Radon-Nikodym aspects here; these can be found in C. Cardassi's thesis [3].

We shall use standard terminology and notation from Banach space theory. Given $1 \leq \leq p \leq \infty$, let us denote by

$$\mathcal{K}_p$$

the class of all Banach spaces X such that for every $L_p(\mu)$-space Y every weakly compact operator from Y into X is compact. We shall use $\mathscr{W}$ and $\mathscr{K}$ to denote the ideals of all weakly compact operators, resp. compact operators between Banach spaces.

The use of $\mathscr{W}$ in our context is of course only of significance when $p = 1$ or $p = \infty$. In the latter case, we could work with $C(K)$-spaces rather than with $L_\infty(\mu)$-spaces. Also, rather than to work with arbitrary $L_p(\mu)$-spaces, it suffices to test our condition with operators defined on $L_p[0,1]$ (Lebesgue measure): consideration of sequences is all what is involved and this brings us down to separable subspaces anyway. Let us agree to write

$$L_p$$

instead of $L_p[0,1]$.

It is easy to verify that a Banach space X belongs to $\mathcal{K}_p$ if and only if every (separable) subspace of X belongs to $\mathcal{K}_p$. Simple examples show, however, that the property «$X \in \mathcal{K}_p$» is, in general, not preserved under the formation of quotients.

Our goal is to investigate relations between the classes $\mathcal{K}_p$ for various values of p. Our main result can be represented by a diagram.

If $1 < r < p < 2 < q < \infty$, then

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$$\begin{array}{ccccc} & & \mathcal{K}_r \subsetneq \mathcal{K}_p & & \\ & \subsetneq & & \subsetneq & \\ \mathcal{K}_1 & & & & \mathcal{K}_2 \quad = \mathcal{K}_q. \\ & \subsetneq & & \subset & \\ & & \mathcal{K}_\infty & & \end{array}$$

It remains open if $\mathcal{K}_\infty \subset \mathcal{K}_2$ is a proper inclusion.

We are going to discuss the various inclusions in the sequel. But first of all let us show that at least the class $\mathcal{K}_1$ is well-understood. In fact, we have:

(1). *A Banach space X belongs to $\mathcal{K}_1$ if and only if it has the Schur property.*

Proof. Clearly, if X has the Schur property, then $X \in \mathcal{K}_1$. Suppose conversely that X belongs to $\mathcal{K}_1$. To say that X has the Schur property is equivalent to saying that $\mathscr{W}(Y,X) = \mathscr{K}(Y,X)$ holds for every separable Banach space Y. It suffices to take $Y = \ell_1$, and this is taken care of by the hypothesis. ∎

We start by investigating the interval $[1,2]$:

(2). *If $1 \leq r < p \leq 2$, then $\mathcal{K}_r \subset \mathcal{K}_p$, and this is a proper inclusion.*

Proof. It follows from (1) that $\mathcal{K}_1$ is contained in all $\mathcal{K}_p$. Let us show that $\mathcal{K}_r$ is contained in $\mathcal{K}_p$ whenever $1 < r < p \leq 2$.

We argue contrapositively and suppose that some $X \in \mathcal{K}_r$ does not belong to $\mathcal{K}_p$. Accordingly, there exists a non-compact operator $u : L_p \to X$, and so some normalized weak null sequence (f_n) in L_p must satisfy $\| uf_n \| > \varepsilon$ for all integers n and some $\varepsilon > 0$. A well known result of C. Bessaga and A. Pełczyński [1] asserts that (f_n) admits a subsequence which is equivalent to a block basis sequence of the Haar system. So we may assume from the beginning that (f_n) itself is an unconditional basic sequence.

But now $e_n \mapsto f_n$ defines a bounded linear operator $v : \ell_p \to L_p$, where (e_n) denotes the standard basis of ℓ_p; compare e.g. with H.P. Rosenthal [9]. If $i : \ell_r \to \ell_p$ is the formal identity, then $uvi : \ell_r \to X$ fails to be compact: contradiction.

It remains to show that $\mathcal{K}_p \setminus \mathcal{K}_r$ is non-empty when $1 \leq r < p \leq 2$. By H.P. Rosenthal's generalization of Pitti's Theorem (cf. [9]), we have $\ell_s \in \mathcal{K}_p$ whenever $r < s < p$: but $\ell_s \notin \mathcal{K}_r$ since, for example, the formal identity $\ell_r \to \ell_s$ fails to be compact. ∎

The interval $[2, \infty[$ is settled by similar methods:

(3). *If $2 < q < \infty$, then $\mathcal{K}_q = \mathcal{K}_2$.*

Proof. That $\mathcal{K}_q$ is contained in $\mathcal{K}_2$ is an easy consequence of the well known fact that ℓ_2 is isomorphic to a complemented subspace of L_q.

As for the converse, suppose again that there exists a Banach space X which belongs to $\mathcal{K}_2$ but which does not belong to $\mathcal{K}_q$. Let again $u : L_q \to X$ be a non-compact operator, and let (f_n) be a normalized weak null sequence in L_q such that $\| uf_n \| > \varepsilon$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$. A theorem of M.I. Kadec and A. Pełczyński [7] tells us that (f_n) admits of a subsequence which is equivalent to the standard basis of ℓ_s, where $s = q$ or $s = 2$. Let $v : \ell_s \to L_q$ be the corresponding isomorphic embedding. By construction, uv cannot be compact, and so our assumption $X \in \mathcal{K}_2$ rules out the case $s = 2$. But $s = q$ is impossible as well, for otherwise we would run into an analogous situation by composing uv with the formal identity $\ell_2 \to \ell_q$. ∎

Note the following immediate consequence of (3):

(4). *No Banach space in $\mathcal{K}_2$ can contain a copy of c_0, or of any ℓ_q, for $2 \leq q < \infty$.*

We complete our program by showing:

(5). *$\mathcal{K}_1$ is properly contained in $\mathcal{K}_\infty$, and $\mathcal{K}_\infty$ is contained in $\mathcal{K}_2$.*

Proof. $\mathcal{K}_1 \subset \mathcal{K}_\infty$ follows easily from (1), and $\mathcal{K}_\infty \subset \mathcal{K}_2$ follows immediately from the fact that ℓ_2 is isomorphic to a quotient of L_∞. Simple examples of Banach spaces in $\mathcal{K}_\infty \setminus \mathcal{K}_1$ are given by the sequence spaces ℓ_p, $1 \leq p < 2$; for further examples see (7) below. ∎

We do not have, however, a complete answer to the following question:

Problem 1. Does there exist a Banach space belonging to $\mathcal{K}_2$ but not to $\mathcal{K}_\infty$?

Another problem concerns the relations between classes $\mathcal{K}_\infty$ and $\mathcal{K}_p$ for $1 < p < 2$. It is clear that $\mathcal{K}_\infty$ cannot be contained in any of the classes $\mathcal{K}_p$, $1 < p < 2$: just consider once more the sequence spaces ℓ_p. On the other hand, the following is open:

Problem 2. Is $\mathcal{K}_p$ contained in $\mathcal{K}_\infty$ when $1 < p < 2$?

Because of the following observation, counter examples to both problems can only exist among Banach spaces having «worst possible cotype» (see G. Pisier [8] for this notion).

(6). *Suppose that X does not contain the ℓ_∞^n's uniformly. If X belongs to $\mathcal{K}_p$ for some $p \leq 2$, then it belongs to $\mathcal{K}_\infty$.*

Proof. Our hypothesis means that X has finite cotype, hence every operator $u : L_\infty \to X$ factors through $L_q(\mu)$ for some (probability) measure μ and some $2 \leq q < \infty$ which depends only on X. Because of $\mathcal{K}_p \subseteq \mathcal{K}_q$, u must be compact. ∎

As we have already seen there is, for each $1 < p \leq \infty$, a Banach space in $\mathcal{K}_p$ which does not belong to $\mathcal{K}_1$. Our next result improves this observation:

(7). *There are reflexive as well as non-reflexive Banach spaces in* $\bigcap_{1<p\leq\infty} \mathcal{K}_p$ *which do not belong to* $\mathcal{K}_1$.

Proof. Let T be the (dual of) Tsirelson's space; we refer to P.G. Casazza and T.J. Shura [4] for the construction and an in-depth analysis of this space and its relatives. It was shown by E.W. Straeuli [10] that every Banach-Saks operator with range T is compact; see J. Diestel and C.J. Seifert [5] for details on Banach-Saks operators. In particular, if $1 < p < \infty$, then every operator from L_p into T is compact, i.e. $T \in \mathcal{K}_p$. But every weakly compact operator from L_∞ into T is also Banach-Saks (cf. [5]), so that $T \in \mathcal{K}_\infty$, too. Reflexivity prevents T from being a member of $\mathcal{K}_1$.

As was shown by J. Bourgain and G. Pisier [2], we may consider T as a subspace of a separable $\mathscr{L}_\infty$-space, say T_∞, such that T_∞/T has the Schur property (the Radon-Nikodym property is available as well). By a straightforward three space argument we find that $T_\infty-$ valued Banach-Saks operators are again compact. Reasoning as before, we see that T_∞ belongs to all classes $\mathcal{K}_p$, $1 < p \leq \infty$, whereas $T_\infty \notin \mathcal{K}_1$ follows from $T \subset T_\infty$. ∎

Weakly compact operators on L_∞-spaces can even be approximated uniformly by L_2- factorable operators, provided one is willing to accept an appropriate enlargement of the given operator's range space; this implies of course that these operators in Banach-Saks, by the results proved in [5]. The above result remains true if the L_∞-spaces are replaced by C^*-algebras, cf. [6]. Thus T and T_∞ actually have the property that all (weakly compact) operators defined on a C^*-algebra and taking values in either of these spaces, must be compact.

We conclude by posing a related question:

Problem 2. Does there exist a Banach space having the Banach-Saks property which belongs to all $\mathcal{K}_p$, for $1 < p \leq \infty$?

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