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LOCAL SPACES OF DISTRIBUTIONS

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Dedicated to the memory of Professor Gottfried Köthe

INTRODUCTION

A space of distributions E is local if, roughly, a distribution T belongs to E whenever T belongs to E in the neighborhood of every point. A space E, in whose definition growth conditions enter, is not local but one can associate with E a local space E_{loc}. This is classical for the spaces L^p [6], and was done for the Sobolev spaces $\mathscr{H}^s$ by Laurent Schwartz in his 1956 Bogotà lectures [8], where he presented an expository account of B. Malgrange's doctoral dissertation. In the present paper I establish some simple properties of the space E_{loc} attached to a space of distributions E.

To a distribution space E we can also attach the space E_c consisting of those elements of E which have compact support. At the end of the paper I make some remarks concerning the duality between local spaces and spaces of distributions with compact support.

1. LOCAL SPACES

Let Ω be an open subset of $\mathbf{R}^n$. Let us recall that a pair (E, j) consisting of a locally convex Hausdorff space E and a continuous injective linear map j from E into the space $\mathscr{D}'(\Omega)$ of Schwartz distributions on Ω is called an *injective pair* [5, Definition 4.2.2, p. 319]. The image $j(E)$ is said to be a *space of distributions* on Ω. In what follows we shall identify E with $j(E)$ but, of course, E keeps its own topology which is finer than the one induced by $\mathscr{D}'(\Omega)$.

If E is a space of distributions on Ω, we denote by E_{loc} the vector space of all distributions $T \in \mathscr{D}'(\Omega)$ such that, for all test functions φ belonging to the space $\mathscr{D}(\Omega)$ of infinitely differentiable functions with compact support, the distribution φT belongs to E [4, section 10.1, p. 13]. We equip E_{loc} with the coarsest topology for which the maps $T \mapsto \varphi T$ from E_{loc} into E are continuous for all $\varphi \in \mathscr{D}(\Omega)$.

If E coincides with E_{loc} as a vector space, then E is said to be a *local space.* If furthermore the topology of E is the same as that of E_{loc}, then we say that E is *topologically local.*

We shall see below that for $0 \leqq m \leqq \infty$ the space $\mathscr{E}^m(\Omega)$ of the m times continuously differentiable functions on Ω, equipped with the topology of uniform convergence on each compact subset of Ω of the functions and their derivatives, is topologically local.

Proposition 1. *E_{loc} is a space of distributions.*

Proof. We have to prove that the canonical injection $E_{\text{loc}} \hookrightarrow \mathscr{D}'(\Omega)$ is continuous. Let W be a neighborhood of 0 in $\mathscr{D}'(\Omega)$. We may assume that W is the polar B^o of a bounded subset B of $\mathscr{D}(\Omega)$. There exists a compact subset K of Ω such that Supp $\varphi \subset K$ for all $\varphi \in B$. Let $\psi \in \mathscr{D}(\Omega)$ be such that $\psi(x) = 1$ in a neighborhood of K. Since $E \hookrightarrow \mathscr{D}'(\Omega)$ is continuous, there exists a neighborhood V of 0 in E such that $T \in V$ implies $T \in W$. The inverse image U of V with respect to the map $T \mapsto \psi T$ from E_{loc} into E is a neighborhood of 0 in E_{loc}. If $T \in U$, then $|\langle T, \varphi\rangle| = |\langle \psi T, \varphi\rangle| \leq 1$ for all $\varphi \in B$, i.e., $T \in B^o = W$. ∎

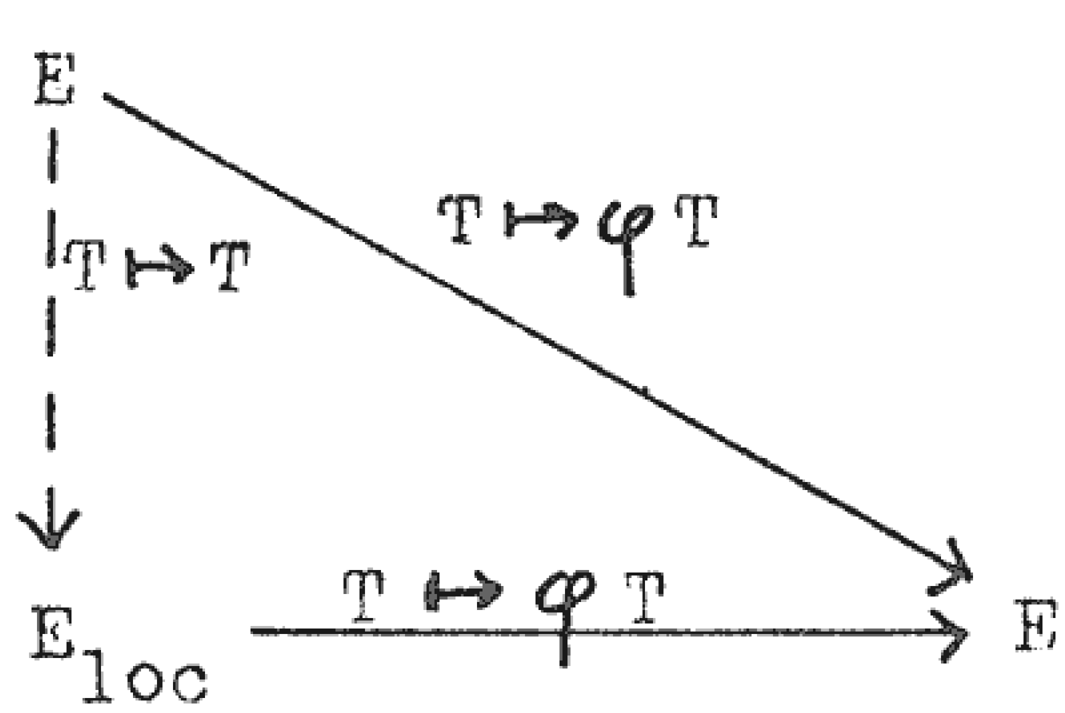
It can happen that $E_{\text{loc}} = \{0\}$. This is the case for instance if $\Omega = \mathbb{R}$ and E consists of restrictions to $\mathbb{R}$ of functions which are holomorphic in $\mathbb{C}$. To obviate this possibility, Hörmander introduced the following definition [4, Definition 10.1.18, p. 13]: E is *semi-local* if $E \subset E_{\text{loc}}$, i.e., if $T \in E$ and $\varphi \in \mathscr{D}(\Omega)$ imply $\varphi T \in E$. Thus E is local if and only if it is semi-local and contains every distribution T such that $\varphi T \in E$ for every $\varphi \in \mathscr{D}(\Omega)$.

A semi-local distribution space will be said to be *topologically semi-local* if for each $\varphi \in \mathscr{D}(\Omega)$ the map $T \mapsto \varphi T$ from E into E is continuous.

For each $\varphi \in \mathscr{D}(\Omega)$ the map $T \mapsto \varphi T$ is continuous from $\mathscr{D}'(\Omega)$ into $\mathscr{D}'(\Omega)$ [5, p. 348], hence it is a fortiori continuous from E into $\mathscr{D}'(\Omega)$. If E is semi-local, then it maps E into E. Therefore, if the closed graph theorem holds for linear maps from E into E (e.g., if E is a barrelled infra-Pták space or an ultrabornological space with a web of type $\mathscr{C}$), then E is also topologically semi-local.

Proposition 2. *If E is topologically semi-local, then the canonical injection $E \hookrightarrow E_{\text{loc}}$ is continuous.*

Proof. For each $\varphi \in \mathscr{D}(\Omega)$ the map $T \mapsto \varphi T$ from E into E is continuous, hence by the universal property of the initial topology [1, chap. I, § 2, n. 3, Proposition 4] the map $T \mapsto T$ from E into E_{loc} is continuous (see diagram). ∎



Let us recall that a triple (i, E, j) consisting of a locally convex Hausdorff space E, a continuous injective linear map $i : \mathscr{D}(\Omega) \to E$, and a continuous injective linear map $j : E \to \mathscr{D}'(\Omega)$ is called a *normal triple* if $i(\mathscr{D}(\Omega))$ is dense in E, and $j \circ i : \mathscr{D}(\Omega) \to \to \mathscr{D}'(\Omega)$ is the map which associates with each $\varphi \in \mathscr{D}(\Omega)$ the distribution $T_\varphi : \psi \mapsto \mapsto \int_\Omega \varphi(x)\psi(x)\,dx$ [5, Definition 4.2.3, p. 319]. The image $j(E)$ is said to be a *normal space of distributions*. In this situation $(E', {}^t i)$ is an injective pair if we equip E' with the strong topology $\beta(E', E)$, hence ${}^t i(E')$ is a space of distributions on Ω. We shall always identify $\mathscr{D}(\Omega)$ with $i(\mathscr{D}(\Omega))$ and E with $j(E)$.

Proposition 3. *Let E be a normal, topologically semi-local space of distributions. Then E_{loc} is a normal space of distributions.*

Proof. Since $\mathscr{D}(\Omega) \subset E$ and $\mathscr{D}(\Omega) \hookrightarrow E$ is continuous, it follows that $\mathscr{D}(\Omega) \subset E_{\text{loc}}$, and by Proposition 2 the map $\mathscr{D}(\Omega) \hookrightarrow E_{\text{loc}}$ is continuous.

Since $\mathscr{D}(\Omega)$ is dense in E, it is sufficient to prove that E is dense in E_{loc}. Let W be a neighborhood of 0 in E_{loc}. We may assume that $W = \bigcap_{j=1}^k f_j^{-1}(V_j)$, where V_j is a neighborhood of 0 in E, f_j is the map $T \mapsto \varphi_j T$ from E_{loc} into E and $\varphi_j \in \mathscr{D}(\Omega)$, $1 \leq j \leq k$. Let $\psi \in \mathscr{D}(\Omega)$ be such that $\psi(x) = 1$ for all x belonging to the compact set $\bigcup_{j=1}^k \operatorname{Supp} \varphi_j$. It $T \in E_{\text{loc}}$, then $\psi T \in E$ and $\varphi_j(T - \psi T) = \varphi_j T - \varphi_j \psi T = \varphi_j T - \varphi_j T = = 0 \in V_j$ for $1 \leq j \leq k$, hence $T - \psi T \in W$. Thus E is indeed dense in E_{loc}. ∎

Proposition 4. *For any space of distributions E on Ω the space E_{loc} is topologically local.*

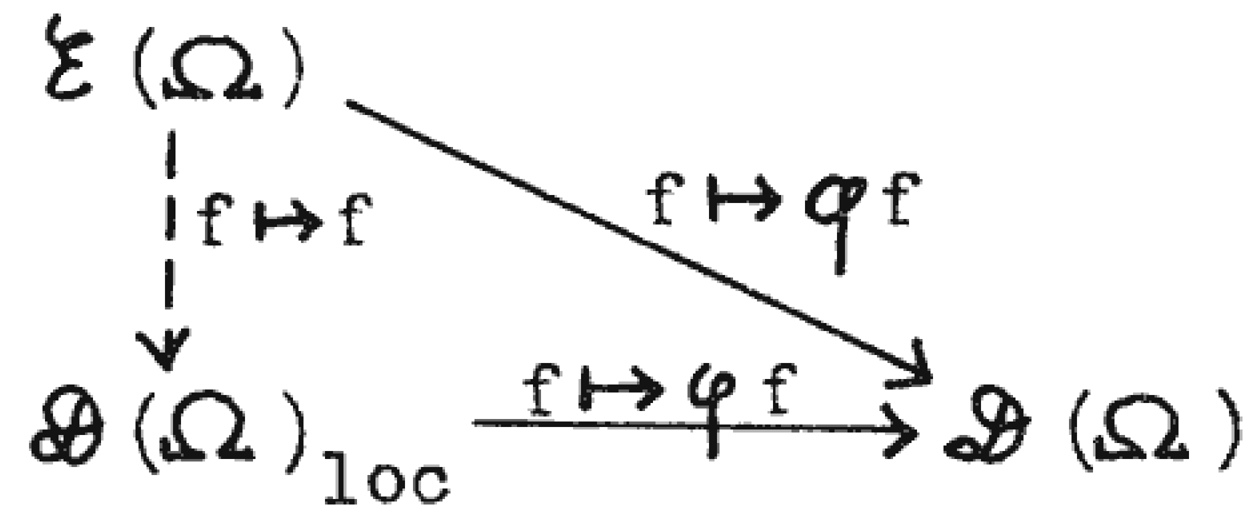
Proof. First I prove that $T \in \mathscr{D}'(\Omega)$ belongs to E_{loc} if and only if $\varphi T \in E_{\text{loc}}$ for every $\varphi \in \mathscr{D}(\Omega)$, i.e. that E_{loc} is a local space. If $T \in E_{\text{loc}}$ and φ, ψ are arbitrary elements of $\mathscr{D}(\Omega)$, then $\psi(\varphi T) = (\psi\varphi)T \in E$, hence $\varphi T \in E_{\text{loc}}$. Conversely, suppose that $\varphi T \in E_{\text{loc}}$ for every $\varphi \in \mathscr{D}(\Omega)$. Given any $\varphi \in \mathscr{D}(\Omega)$, choose $\psi \in \mathscr{D}(\Omega)$ so that $\psi(x) = 1$ for $x \in \operatorname{Supp} \varphi$. Since $\varphi T \in E_{\text{loc}}$, we have $\psi(\varphi T) \in E$. But $\psi\varphi = \varphi$, so $\varphi T \in E$, and therefore T belongs to E_{loc}.

It remains to prove that the two initial topologies on the vector space $(E_{\text{loc}})_{\text{loc}} = E_{\text{loc}}$ coincide. The composition of the maps $T \mapsto \varphi T$ from $(E_{\text{loc}})_{\text{loc}}$ into E_{loc} and $T \mapsto \mapsto \psi T$ from E_{loc} into E is the map $T \mapsto \psi\varphi T$ from $(E_{\text{loc}})_{\text{loc}}$ into E. Conversely, every map $T \mapsto \varphi T$ from $(E_{\text{loc}})_{\text{loc}}$ into E is of this form with $\psi(x) = 1$ for $x \in \operatorname{Supp} \varphi$. Therefore the topology on $(E_{\text{loc}})_{\text{loc}}$ is the initial topology for the maps $T \mapsto \varphi T$ from $(E_{\text{loc}})_{\text{loc}}$ into E by the transitivity of the initial topology [1, chap. I, § 2, n. 3, Proposition 5]. ∎

Example 1. $\mathscr{D}(\Omega)$ is a normal space of distributions, and it is topologically semi-local [7, chap. V, § 2, Théorème III, p. 119; 5, Proposition 4.6.1, p. 348]. I want to show that

$\mathscr{D}(\Omega)_{\mathrm{loc}} = \mathscr{E}(\Omega)$. Clearly if $f \in \mathscr{E}(\Omega)$ and $\varphi \in \mathscr{D}(\Omega)$, then $\varphi f \in \mathscr{D}(\Omega)$. Conversely, if $T \in \mathscr{D}'(\Omega)$ is such that $\varphi T \in \mathscr{D}(\Omega)$ for every $\varphi \in \mathscr{D}(\Omega)$, then choosing $\varphi(x) = 1$ for x near an arbitrary point of Ω we see that $T = f \in \mathscr{E}(\Omega)$. Thus $\mathscr{E}(\Omega)$ and $\mathscr{D}(\Omega)_{\mathrm{loc}}$ coincide as vector spaces.

The canonical bijection $\mathscr{E}(\Omega) \hookrightarrow \mathscr{D}(\Omega)_{\mathrm{loc}}$ is continuous, since for each $\varphi \in \mathscr{D}(\Omega)$ the map $f \mapsto \varphi f$ from $\mathscr{E}(\Omega)$ into $\mathscr{D}(\Omega)$ is continuous [5, Proposition 4.7.4, p. 360].



Finally, I show that the map $\mathscr{E}(\Omega) \hookrightarrow \mathscr{D}(\Omega)_{\mathrm{loc}}$ is open. Let U be a neighborhood of 0 in $\mathscr{E}(\Omega)$. We may assume that

$$U = \{f \in \mathscr{E}(\Omega);\ |\partial^\alpha f(x)| \leq \varepsilon,\ |\alpha| \leq m,\ x \in K\},$$

where $\varepsilon > 0$, $m \in \mathbb{N}$, and K is a compact subset of Ω. Let L be a compact neighborhood of K contained in Ω, and W_o the neighborhood of 0 in $\mathscr{D}(L)$ given by $W_o = \{\varphi \in \mathscr{D}(L); |\partial^\alpha \varphi(x)| \leq \varepsilon, |\alpha| \leq m\}$. Since $\mathscr{D}(L)$ is a subspace of $\mathscr{D}(\Omega)$ [2, chap. II, § 4, n. 6, Proposition 9], by the Lemme 2 of loc. cit. there exists a neighborhood W of 0 in $\mathscr{D}(\Omega)$ such that $W \cap \mathscr{D}(L) = W_o$. Let $\psi \in \mathscr{D}(L)$ be such that $\psi(x) = 1$ for x in neighborhood of K, and denote by V the inverse image of W with respect to the map $f \mapsto \psi f$ from $\mathscr{D}(\Omega)_{\mathrm{loc}}$ into $\mathscr{D}(\Omega)$. If $f \in V$, then $\psi f \in W$ and in particular $|\partial^\alpha f(x)| = |\partial^\alpha(\psi f)(x)| \leq \varepsilon$ for $x \in K$ and $|\alpha| \leq m$, i.e., $f \in U$. Thus $U \supset V$, and the map is indeed open.

Taking into account Proposition 4, we proved

Proposition 5. *$\mathscr{E}(\Omega)$ is a topologically local space of distributions. Its usual topology of uniform convergence on compact subsets of Ω of $f \in \mathscr{E}(\Omega)$ and all its derivatives is the coarsest topology for which the maps $f \mapsto \varphi f$ from $\mathscr{E}(\Omega)$ into $\mathscr{D}(\Omega)$ are continuous for all $\varphi \in \mathscr{D}(\Omega)$.*

Example 2. The space $\mathscr{K}(\Omega)$ of continuous functions with compact support, equipped with the finest locally convex topology for which the canonical injections $\mathscr{K}(K) \hookrightarrow \mathscr{K}(\Omega)$ are continuous, is a normal space of distributions [5, Proposition 4.4.1, p. 338] and it is topologically semi-local [5, Proposition 4.6.1, p. 348]. Here $\mathscr{K}(K)$ denotes the space of continuous functions with support in the compact subset K of Ω, equipped with the topology of uniform convergence.

The space $\mathcal{K}(\Omega)_{\text{loc}}$ is topologically isomorphic to the space $\mathcal{E}(\Omega)$ of continuous functions on Ω equipped with the topology of uniform convergence on compact subsets of Ω. The proof runs along the same lines as in Example 1, and it is even simpler since m is equal to zero.

Let X be a locally compact topological space. Similarly as in Example 2, one introduces for every compact set K in X space $\mathcal{K}(K)$ of continuous functions on X, having support in K, equipped with the topology of uniform convergence. The space $\mathcal{K}(X) = \cup\mathcal{K}(K)$ consists of all continuous functions with compact support, and is equipped with the finest locally convex topology for which the canonical injections $\mathcal{K}(K) \hookrightarrow \mathcal{K}(X)$ are continuous [3, chap. III, § 1, n. 1].

Let $\mathcal{E}(X)$ be the vector space of all continuous functions on X equipped with the topology of uniform convergence on compact subsets of X. For every $\varphi \in \mathcal{K}(X)$ the linear map $f \mapsto \varphi f$ from $\mathcal{E}(X)$ into $\mathcal{K}(X)$ is continuous. Indeed, if $K = \text{Supp}\ \varphi$ and $\| \varphi \|_\infty = \max |\varphi(x)|$, then $|f(x)| \le \varepsilon / \| \varphi \|_\infty$ for $x \in K$ implies $|\varphi(x) f(x)| \le \varepsilon$. The topology of $\mathcal{E}(X)$ is furthermore the coarsest for which the maps $f \mapsto \varphi f$ are continuous; this can be seen exactly as in the simplification alluded to in Example 2 of the proof in Example 1.

Returning to the case of an open subset Ω of $\mathbf{R}^n$, a distribution $T \in \mathcal{D}'(\Omega)$ belongs obviously to $\mathcal{E}(\Omega)$ if and only if $\varphi T \in \mathcal{K}(\Omega)$ for every $\varphi \in \mathcal{K}(\Omega)$. Combining with Example 2, we have therefore:

Theorem 1. *$\mathcal{E}(\Omega)$ is the space of all distributions $T \in \mathcal{D}'(\Omega)$ such that $\varphi T \in \mathcal{K}(\Omega)$ for all $\varphi \in \mathcal{D}(\Omega)$ or for all $\varphi \in \mathcal{K}(\Omega)$. The topology on $\mathcal{E}(\Omega)$ of uniform convergence on compact subsets of Ω coincides with the coarsest topology for which the maps $f \mapsto \varphi f$ from $\mathcal{E}(\Omega)$ into $\mathcal{K}(\Omega)$ are continuous for all $\varphi \in \mathcal{D}(\Omega)$ or for all $\varphi \in \mathcal{K}(\Omega)$.*

Statements analogous to Examples 1 and 2 can be made for $1 \le m < \infty$. In particular $\mathcal{D}^m(\Omega)_{\text{loc}} = \mathcal{E}^m(\Omega)$.

Example 3. Let $\mathcal{E}_b(\Omega)$ be the space of all bounded continuous functions on Ω, equipped with the topology of uniform convergence on Ω defined by the norm $\| f \|_\infty = \sup_{x\in\Omega} |f(x)|$. Denote by $\mathcal{E}_o(\Omega)$ the subspace consisting of the functions «tending to zero at infinity» (i.e., such that given $\varepsilon > 0$ there exists a compact subset K of Ω such that $|f(x)| \le \varepsilon$ for $x \notin K$). Then $\mathcal{E}_b(\Omega)_{\text{loc}} = \mathcal{E}_o(\Omega)_{\text{loc}} = \mathcal{E}(\Omega)$. The proof is again similar to the argument in Example 1, this time, however, not only can one take $m = 0$ but also the topology of $\mathcal{E}_b(\Omega)$ and $\mathcal{E}_o(\Omega)$ is simpler. Just as in the case of $\mathcal{K}(\Omega)$, we could also consider the maps $f \mapsto \varphi f$ for φ in $\mathcal{K}(\Omega)$.

Example 4. Let $\mathscr{B}(\Omega)$ be the space of all bounded continuous functions on Ω which have bounded continuous derivatives of all orders. One equips $\mathscr{B}(\Omega)$ with the topology of uniform convergence on Ω of $f \in \mathscr{B}(\Omega)$ and all its derivatives, defined by the family of semi-norms $f \mapsto \| \partial^\alpha f \|_\infty$, $\alpha \in \mathbb{N}^n$. We denote by $\mathscr{B}_o(\Omega)$ the subspace of $\mathscr{B}(\Omega)$ consisting of those functions which tend to 0 at infinity, together with all their derivatives [7, chap. VI, § 8, p. 199; 5, Examples 2.4.17 and 2.4.19, pp. 91-92]. One has $\mathscr{B}(\Omega)_{\text{loc}} = \mathscr{B}_o(\Omega)_{\text{loc}} = \mathscr{E}(\Omega)$.

Example 5. For $1 \leq p \leq \infty$ the space $L^p_{\text{loc}}(\Omega)$ can be defined as the space of (equivalence classes of) Lebesgue-measurable functions f such that for every compact subset K of Ω the function $\chi_K f$ belongs to $L^p(\Omega)$, equipped with the topology defined by the semi-norms $f \mapsto \| \chi_K f \|_p$ [6]. Here χ_K is the characteristic function of K,

$$\| f \|_p = \left(\int_\Omega |f(x)|^p \mathrm{d}\, x \right)^{1/p} \quad \text{for} \quad 1 \leq p \leq \infty,$$

and $\| f \|_\infty$ is the essential maximum of $|f|$.

Let us prove that $f \in L^p_{\text{loc}}(\Omega)$ if and only if $\varphi f \in L^p(\Omega)$ for every $\varphi \in \mathscr{D}(\Omega)$. Assume that $f \in L^p_{\text{loc}}(\Omega)$. Then φf is measurable. Setting $K = \operatorname{Supp} \varphi$ we have $|\varphi(x)| \leq$ $\leq \| \varphi \|_\infty \chi_K(x)$, hence $\int |\varphi f|^p < \infty$, i.e., $\varphi f \in L^p(\Omega)$. Conversely, assume that $\varphi f \in$ $\in L^p(\Omega)$ for all $\varphi \in \mathscr{D}(\Omega)$. Given any compact set $K \subset \Omega$, there exists a positive function $\varphi \in \mathscr{D}(\Omega)$ such that $\varphi(x) = 1$ for $x \in K$ [5, Proposition 2.12.5, p. 169]. Then $\chi_K(x) \leq \varphi(x)$ and $\chi_K \varphi = \chi_K$, hence $\chi_K f = \chi_K \varphi f$ is measurable and $\int \chi_K |f|^p < \infty$, i.e., $f \in L^p_{\text{loc}}(\Omega)$.

The same reasoning shows also that $L^p_{\text{loc}}(\Omega)$ is equipped with the coarsest topology for which the maps $f \mapsto \varphi f$ from $L^p_{\text{loc}}(\Omega)$ into $L^p(\Omega)$ are continuous for all $\varphi \in \mathscr{D}(\Omega)$.

Example 6. The Sobolev space $\mathscr{H}^s$ is defined as the space of those tempered distributions $T \in \mathscr{S}'$ on $\mathbb{R}^n$ whose Fourier transform $\widehat{T}$ satisfies $(1 + |\xi|^2)^{s/2} \widehat{T}(\xi) \in L^2(\mathbb{R}^n)$. It is equipped with the norm

$$\| T \|_s = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{T}(\xi)|^2 \mathrm{d}\, \xi \right)^{1/2}$$

[8, p. 4]. Laurent Schwartz introduced the corresponding space $\mathscr{H}^s_{\text{loc}}$ and denoted it $\mathscr{L}^s$ [8, p. 16].

2. DISTRIBUTIONS WITH COMPACT SUPPORT AND DUALITY

If E is a space of distributions on the open subset Ω of $\mathbf{R}^n$, denote by E_c the subspace of E consisting of those $T \in E$ whose support is compact. For any compact subset K of Ω we denote by $E(K)$ the space of those $T \in E$ whose support is in K; then $E_c = \cup E(K)$. Equip each $E(K)$ with the topology induced by E, and E_c with the finest locally convex topology for which the canonical injections $E(K) \hookrightarrow E_c$ are continuous. The canonical injection $E_c \hookrightarrow E$ is continuous and therefore E_c is a space of distributions.

Let (K_j) be a sequence of compact subsets of Ω such that $K_j \subset \overset{\circ}{K}_{j+1}$ and $\bigcup_j K_j = \Omega$. Then the topology of E_c is also the locally convex inductive limit of the sequence $(E(K_j))_j$. If the compact set $K \subset \Omega$ is contained in the compact set $L \subset \Omega$, then $E(K)$ is closed in $E(L)$. Indeed, if $T_o \in E(L)$ is adherent to $E(K)$ for the topology induced by E, then it is a fortiori adherent to $E(K)$ for the weak topology $\sigma(\mathscr{D}'(\Omega), \mathscr{D}(\Omega))$. Let $\varphi \in \mathscr{D}(\Omega)$ be such that Supp $\varphi \cap K = \emptyset$. For any $\varepsilon > 0$ there exists a $T \in E(K)$ such that $|\langle T - T_o, \varphi\rangle| = |\langle T, \varphi\rangle - \langle T_o, \varphi\rangle| \leq \varepsilon$. But $\langle T, \varphi\rangle = 0$, so $\langle T_o, \varphi\rangle = 0$, i.e., $T \in E(K)$. It follows [2, chap. II, § 4, n. 6, Proposition 9, p. II.35] that E_c induces on each $E(K)$ the same topology as E, and each $E(K)$ is closed in E_c.

If E is a normal space of distributions, then in all examples so is E_c but this does not seem to be true in general.

Let E be a normal, topologically semi-local space of distributions on Ω, and E' its dual equipped with $\beta(E', E)$. Given $T \in (E')_c$, let $\psi \in \mathscr{D}(\Omega)$ be such that $\psi(x) = 1$ in a neighborhood of Supp T. For every $S \in E_{\text{loc}}$ the expression $\langle T, \psi S\rangle$ is well defined and independent of the particular choice of ψ. The linear form $L_T : S \mapsto \langle T, \psi S\rangle$ on E_{loc} is continuous, since the map $S \mapsto \psi S$ from E_{loc} into E is continuous by definition, and $T \in (E')_c \subset E'$. In all known examples the linear map $T \mapsto L_T$ from $(E')_c$ into $(E_{\text{loc}})'$ is a topological isomorphism. In the general case I was only able to establish

Theorem 2. *Let E be a normal, topologically semi-local space of distributions. The map which associates with $T \in (E')_c$ the linear form*

$$L_T : S \mapsto \langle T, \psi S\rangle$$

on E_{loc} is a continuous bijection from $(E')_c$ onto $(E_{\text{loc}})'$ equipped with the strong topology $\beta((E_{\text{loc}})', E_{\text{loc}})$.

Proof. 1) First I prove that the map $T \mapsto L_T$ is injective. Assume that $L_T = 0$. Let $x \in \Omega$ and K be a compact neighborhood of x. Choose $\psi \in \mathscr{D}(\Omega)$ so that $\psi(x) = 1$ for $x \in K \cup$ $\cup$ Supp T. Then for all $\varphi \in \mathscr{D}(K)$ we have $\langle T, \varphi\rangle = \langle T, \psi\varphi\rangle = L_T(\varphi) = 0$. Thus $T = 0$ in the neighborhood of any point of Ω, hence by the localization principle [7, chap. I, § 3, Théorème IV, p. 27; 5, Proposition 4.2.1, p. 318] $T = 0$.

2) Next, I want to prove that the map $T \mapsto L_T$ is continuous. Let W be a strong neighborhood of 0 in $(E_{\text{loc}})'$. There exists a bounded subset B of E_{loc} whose polar B^o is contained in W. Let (K_j) be a sequence of compact subsets of Ω such that $K_j \subset \overset{\circ}{K}_{j+1}$ and $\bigcup K_j = \Omega$. Let (ψ_j) be a sequence of functions in $\mathscr{D}(\Omega)$ such that $\psi_j(x) = 1$ for $x \in K_j$ and $\psi_j(x) = 0$ outside K_{j+1}. Since the maps $S \mapsto \psi_j S$ from E_{loc} into E are continuous, for each j the set $\psi_j B$ is bounded in E. If $(\psi_j B)^o$ denotes the polar in E', then $V_j = E'(K_j) \cap (\psi_j B)^o$ is a neighborhood of 0 for the topology induced by $\beta(E', E)$ on $E'(K_j)$. The balanced, convex hull V of $\bigcup V_j$ is a neighborhood of 0 in $(E')_c$.

Let $T \in V$. There exist distributions $T_j \in V_j$ and scalars λ_j with $\sum |\lambda_j| \leq 1$ and $\lambda_j = 0$ except for indices j belonging to a finite set J such that $T = \sum \lambda_j T_j$. Choose $\psi \in \mathscr{D}(\Omega)$ such that $\psi(x) = 1$ in a neighborhood of Supp $T \cup \bigcup_{j \in J} K_{j+1}$. Then $\psi \psi_j = \psi_j$ for $j \in J$. If S belongs to B, then

$$L_T(S) = \langle T, \psi S \rangle = \sum_{j \in J} \langle \lambda_j T_j, \psi_j S \rangle$$

and so

$$|L_T(S)| \leq \sum |\lambda_j| \, |\langle T_j, \psi_j S \rangle| \leq \sum |\lambda_j| \leq 1,$$

i.e. $L_T \in B^o \subset W$.

3) Finally I show that the map $T \mapsto L_T$ is surjective. First I prove that given $L \in (E_{\text{loc}})'$ there exists a compact subset K of Ω such that $L(S) = 0$ for all $S \in E_{\text{loc}}$ with Supp $S \cap K = \emptyset$. There exists a neighborhood U of 0 in E_{loc} such that $|L(S)| \leq 1$ for all $S \in U$. We may assume that $U = \bigcap_{j=1}^{k} f_j^{-1}(U_j)$, where U_j is a neighborhood of 0 in E and $f_j : E_{\text{loc}} \to E$ is the map $S \mapsto \varphi_j S$ for some $\varphi_j \in \mathscr{D}(\Omega)$. Let $K = \bigcup_{j=1}^{k}$ Supp φ_j and assume that Supp $S \cap K = \emptyset$. Then for any $m \in \mathbb{N}$ and $1 \leq j \leq k$ we have $\varphi_j \cdot mS = 0 \in U_j$ and therefore $mS \in U$. Consequently $|L(S)| \leq \frac{1}{m}$ for all $m > 0$ and so $L(S) = 0$.

By Proposition 2 the canonical injection $E \hookrightarrow E_{\text{loc}}$ is continuous, hence the restriction of L to E is a continuous linear form on E and there exists $T \in E'$ such that $L(S) = \langle T, S \rangle$ for all $S \in E$. Choosing $\varphi \in \mathscr{D}(\Omega) \subset E$ with Supp $\varphi \cap K = \emptyset$ we have $\langle T, \varphi \rangle = L(\varphi) = 0$, hence Supp $T \subset K$ and in particular $T \in (E')_c$. Let $\psi \in \mathscr{D}(\Omega)$ be equal to 1 in a neighborhood of Supp T. If $S \in E_{\text{loc}}$, then $(1 - \psi) S \in E_{\text{loc}}$ and Supp $(1 - \psi) S \cap K = \emptyset$, hence

$$L(S) = L(\psi S) + L((1 - \psi) S) = \langle T, \psi S \rangle,$$

i.e., $L = L_T$. ∎

Let now T be an element of $(E')_{\text{loc}}$. For any $S \in E_c$ let $\psi \in \mathscr{D}(\Omega)$ be such that $\psi(x) = 1$ for x in a neighborhood of Supp S. Then $\langle \psi T, S \rangle$ is well defined and independent of the choice of ψ. The linear form $L_T : S \mapsto \langle \psi T, S \rangle$ is continuous on E_c, and it is easy to prove that the linear map $T \mapsto L_T$ from $(E')_{\text{loc}}$ into $(E_c)'$ is injective and continuous. It might be of some interest to know whether it is surjective and open.

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