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THE STRUCTURE THEOREM FOR LINEAR TRANSFER SYSTEMS

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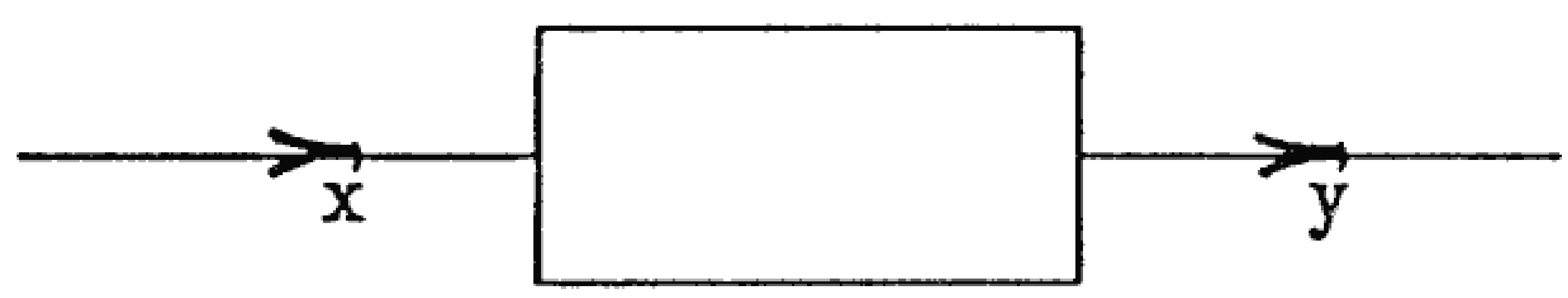
Dedicated to the memory of Professor Gottfried Köthe

Abstract. *The aim of this article is to show that a few reasonable assumptions lead to a complete theory describing linear transfer systems.*

1. INTRODUCTION

At first sight a linear transfer system is the famous black box



which transforms a linear space of input signals $x : \mathbb{R} \to \mathbb{R}$ into output signals $y : \mathbb{R} \to \mathbb{R}$, $y = A(x)$, belonging to another linear space. Linearity of the transfer system means linearity of the mapping A.

Simple examples of transfer systems are technical devices as
a thermometer or
a speedometer,
where x is the quantity to be measured and y is the quantity shown by the instrument, but it could also describe the dependence of the income y of an economic community on the interst rate x.

Of course, the real world is non-linear, but as a first approximation or for stability investigations we would be content with a linear model.

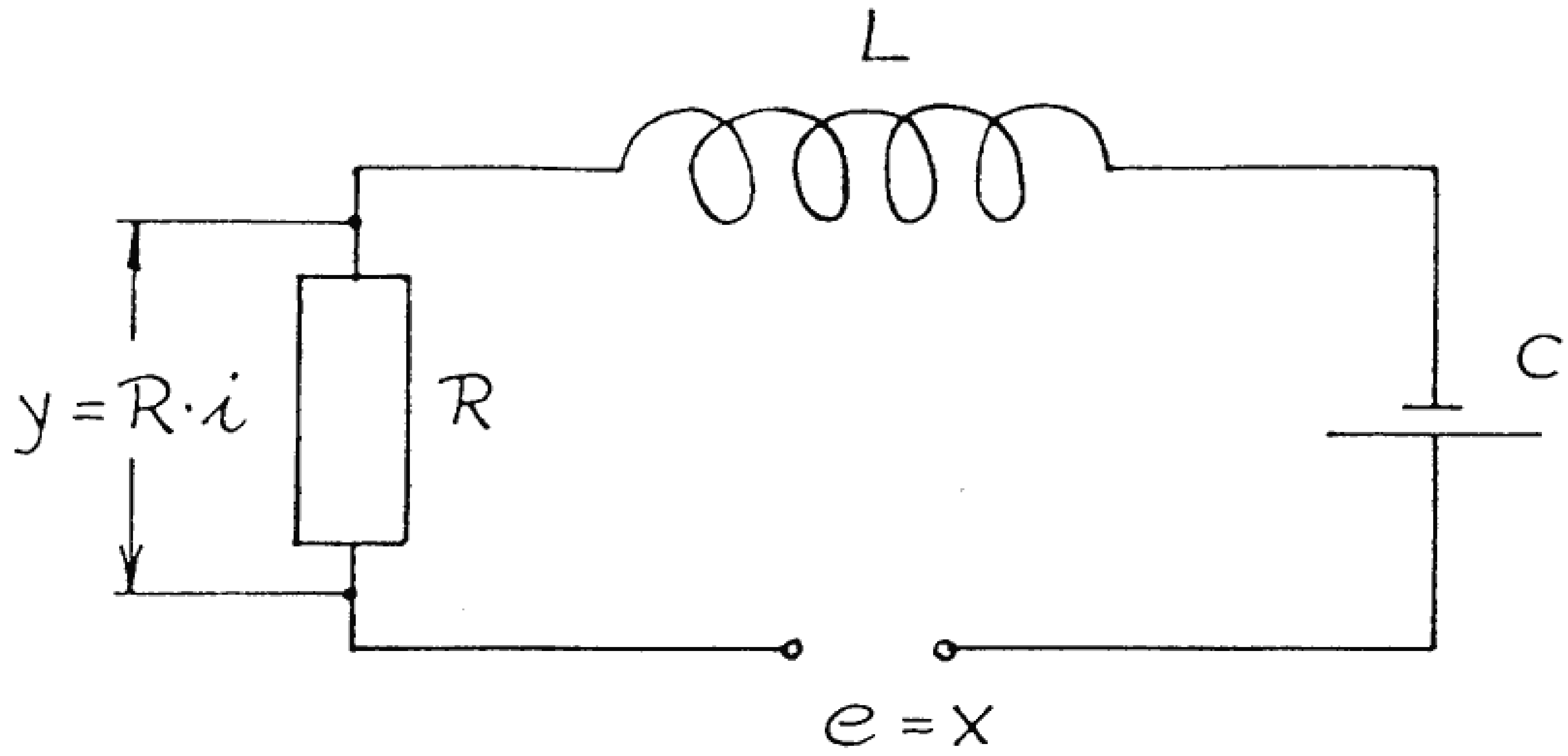
The problem of *system identification* is: Given the black box, find the mapping A. The above ideas are of course too vague to make up a mathematical theory. So let us look at some examples of technical transfer systems. There are essentially three types of technical devices

- eletrical,
- hydraulical, and
- mechanical

systems.

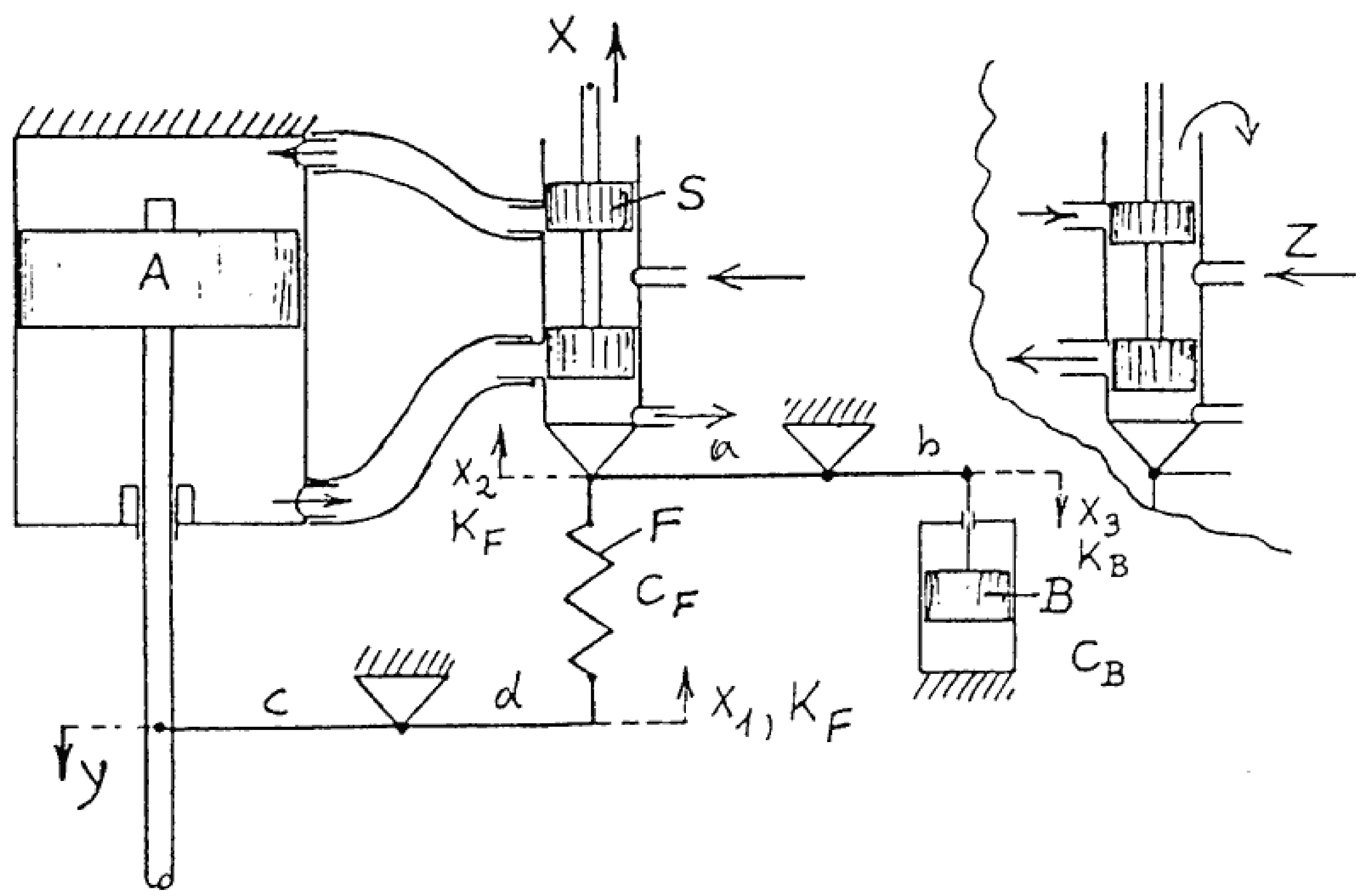
2. EXAMPLES

We first consider a simple electrical circuit, where x is the electromotive force and y is the voltage drop in the resistor R.



$$C L \cdot \ddot{y} + C R \cdot \dot{y} + y = C R \cdot \dot{x}$$

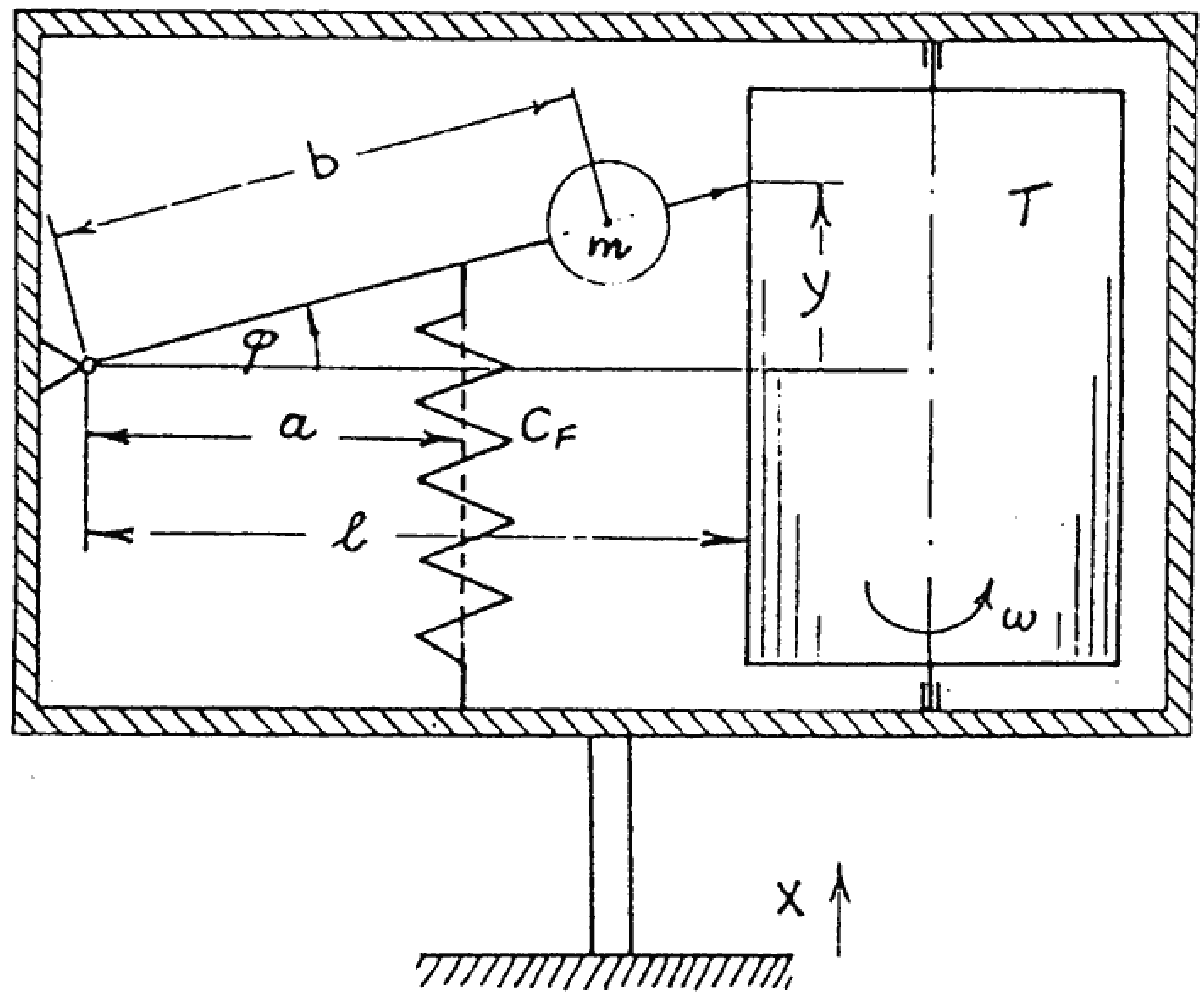
Our second example is a hydraulic motor.



$$a_2 \cdot \ddot{y} + a_1 \cdot \dot{y} + y = r_0 \cdot x + r_1 \cdot \dot{x}$$

The control piston S is connected to an infinite reservoir. If we move up the control piston it opens a valve and the working piston A moves down. With a delay caused by the spring F and the shock absorber B, the cylinder of the control piston follows the movement of the piston and closes the valve.

As our last example, we shall consider a seismograph.



Vibrations x of the earth surface move the box. By inertia the mass m tries not to move, and so we get a movement of the mass relative to the box that is written onto the cylinder T which turns with a constant speed ω.

Using the Lagrange formalism we obtain the following non-linear equation describing the seismograph.

$$\frac{b}{\cos\varphi}\ddot{\varphi} + \frac{a^2 \cdot c_F}{m \cdot b \cdot \cos\varphi}\left[\tan\varphi - \tan\varphi^*\right] = -\ddot{x} - g.$$

Here φ^* is the angle where the spring exerts no force. If $\tan\varphi^* = \frac{m \cdot g \cdot b}{a^2 \cdot c_F}$ then $\varphi = 0$ is a stationary solution and we get

$$\frac{b}{\cos\varphi}\ddot{\varphi} + \frac{c_F}{m} \cdot \frac{a^2}{b} \cdot \frac{\tan\varphi}{\cos\varphi} + g\left(1 - \frac{1}{\cos\varphi}\right) = -\ddot{x}.$$

We linearize this equation by setting

$$y = \ell \cdot \tan\varphi \approx \ell \cdot \varphi$$

to obtain

$$\frac{b}{\ell} \cdot \ddot{y} + \frac{c_F}{m} \cdot \frac{a^2}{b \cdot \ell} y = -\ddot{x}.$$

We note that the invariance principle of Galilei implies that neither x nor $\dot{x}$ can enter the above equation.

3. SYSTEM IDENTIFICATION

In our examples the linearized equation describing the transfer system is of the type

$$(3.1) \qquad p(D)y = q(D)x$$

with polynomials $p, q \in \mathbf{R}[X]$ satisfying $\deg(p) \geq \deg(q)$, and D denoting differentiation with respect to time.

The books on technical control theory (e.g. Leonhard, 1962, Schwarz, 1967) offer two methods to identify a linear transfer system of the above type.

One method uses the *Fourier transformation.* By a formal Fourier transformation of equation (3.1) we obtain

$$p(2\pi i\omega) \cdot \mathscr{F}(y)(\omega) = q(2\pi i\omega) \cdot \mathscr{F}(x)(\omega)$$

and thus

$$(3.2) \qquad \frac{\mathscr{F}(y)(\omega)}{\mathscr{F}(x)(\omega)} = \frac{q(2\pi i\omega)}{p(2\pi i\omega)}.$$

Up to a common polynomial factor, p and q are determined by the quotient of the Fourier transformation of the output signal y by the Fourier transformation of the corresponding input signal x.

In practice one uses a whole family of periodic input signals $x_\omega(t) = \exp(2\pi i\omega t)$ ($\omega \in I$, I some interval in $\mathbf{R}$) and measures the corresponding output signals. Especially for stability calculations this method is rather popular.

The other method uses the *transfer function.* This function is defined to be the output signal corresponding to the unit step function (or Heaviside function)

$$x(t) := Y(t) := \begin{cases} 1 & (t \geq 0) \\ 0 & (t < 0) \end{cases}$$

as an input signal. Most books on control theory hint that a linear transfer system is completely determined by its transfer function. But only a few books tell you why this is true. As a byproduct the developments of the next section will provide a proof of this.

4. THE SYSTEM DISTRIBUTION

Let us assume that an equation of the type (3.1) describes our transfer system. We want to calculate the corresponding transfer function. If $\deg(q) \geq 1$ the derivatives of the unit step function occuring in the right hand side of (3.1) are no longer functions. Thus we have to solve a linear differential equation with constant coefficients and a *distributional right hand side.* This suggests that we probably should use distributions from the very beginning. So let us return to the problem to calculate the mapping A from the equation (3.1). To do this, let E be the unique fundamental solution to the differential operator $p(D)$ which has its support in $[0,\infty)$ (cf. Treves, 1975, p. 26, Zemanian, 1965, p. 157). If all the occuring convolution products exist in the $\mathscr{D}(\mathbf{R})'$-sense (cf. Schwartz, 1954, exposé n. 22, Wladimirov, 1972, p. 100, Dierolf, Vogt, 1978), we obtain

$$E*(p(D)y) = (p(D)E)*y = \delta * y = y = E*(q(D)x) = (q(D)E)*x, \text{ i.e. } y = (q(D)E)*x. \tag{4.1}$$

Because of $\operatorname{supp}(E) \subset [0,\infty)$ the convolution product $T*E$ exists for all $T \in \mathscr{D}(\mathbf{R})'$ whose support is bounded to the left, i.e. which satisfy $\inf \operatorname{supp}(T) > -\infty$. This condition does not really restrict the class of input signals x.

The relation $y = (q(D)E)*x$ defines a mapping

$$A : \mathscr{D}(\mathbf{R}) \to \mathscr{D}(\mathbf{R})',\ A(x) = (q(D)E)*x \tag{4.2}$$

with the following properties:
(I) A is linear.
(II) A commutes with translations, $\tau_h \cdot A = A \cdot \tau_h$ for all $h \in \mathbf{R}$, where $\tau_h(x)(t) = x(t-h)$ $(t \in \mathbf{R})$.
(III) A is continuous with respect to the standard topologies on $\mathscr{D}(\mathbf{R})$ and $\mathscr{D}(\mathbf{R})'$, respectively.

The following theorem of Schwartz, 1966, p.197-198, characterizes these mappings.

Theorem 4.3. *Let $A : \mathscr{D}(\mathbf{R}) \to \mathscr{D}(\mathbf{R})'$ satisfy (I), (II) and (III). Then there exists a unique distribution $S \in \mathscr{D}(\mathbf{R})'$ such that $A(\varphi) = S * \varphi$ for all $\varphi \in \mathscr{D}(\mathbf{R})$.*

Are the properties (I), (II) and (III) natural assumptions to *define* a linear transfer system?

Linearity of A is our basic assumption. That A commutes with translations reflects the homogeneity of time of classical mechanics. The continuity of A, however, is an artificial assumption. Linear transfer systems are much older than the theory of distributions.

This was the state of the art up to 1979, when E. Albrecht and M. Neumann proved a result on automatic continuity which cleared the situation. From $\operatorname{supp}(E) \subset [0,\infty)$ we obtain the following additional property of the mapping $A(x) = (q(D)E)*x$:
(IV) $\forall T \in \mathbf{R} : x = 0$ on $(-\infty, T)$ implies $A(x) = 0$ on $(-\infty, T)$.

In physical terms (IV) means *causality* of our linear system in the sense that there cannot be an output signal before there was an input signal. This is of course a very natural assumption. One of the important results of Albrecht, Neumann, 1979, is

Theorem 4.4. *Let* $A : \mathscr{D}(\mathbf{R}) \to \mathscr{D}(\mathbf{R})'$ *be a mapping which is linear, translation invariant and casual, i.e. satisfies (I), (II) and (IV), then* A *is continuous with respect to the standard topologies on* $\mathscr{D}(\mathbf{R})$ *and* $\mathscr{D}(\mathbf{R})'$, *respectively.*

We are now ready to *define* a linear transfer system. A linear transfer system is a «black box» which transforms a class of input signals containing at least $\mathscr{D}(\mathbf{R})$ into a class of output signals such that the mapping $A : \mathscr{D}(\mathbf{R}) \to \mathscr{D}(\mathbf{R})'$

- is linear,
- commutes with translations, and
- is causal.

The distribution S which according to (4.3) and (4.2) represents the mapping A is called the *system distribution.*

Returning to the problem of system identification, i.e. to determine the distribution S, we observe that for $x = \delta$ = Dirac distribution as an input signal we would obtain $A(\delta) = S*\delta = S$. But δ cannot be realized as an input signal. So we take $x = Y$ = Heaviside function, and obtain

$$D(A(Y)) = D(S*Y) = S*(DY) = S*\delta = S.$$

The differential of the transfer function is the system distribution S. The differentiation of the transfer function can be carried out numerically.

A systematic treatment of stability of linear transfer systems exists for systems of the type (3.1) where it reduces to the well known Hurwitz criterion. Stability of general linear systems can be formulated in terms of continuity properties of the convolution operator $x \mapsto S*x$, where S is the system distribution. These continuity properties are strongly related to the size of the subspaces of $\mathscr{D}(\mathbf{R})'$ between which the convolution operator acts (cf. e.g. Dierolf, 1984).

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