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A further simplification of Tarski's axioms of geometry

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Abstract. A slight modification to one of Tarski's axioms of plane absolute geometry is proposed. This modification allows another of the axioms to be omitted from the set of axioms and proven as a theorem. This change to the system of axioms simplifies the system as a whole, without sacrificing the useful modularity of some of its axioms. The new system is shown to possess all of the known independence properties of the system on which it was based; in addition, another of the axioms is shown to be independent in the new system.

Keywords: foundations of geometry, absolute geometry, Tarski's axioms, independence

MSC 2010 classification: primary 03B30, secondary 51F05, 51F10

1 Background

Alfred Tarski's axioms of geometry were first described in a course he gave at the University of Warsaw in 1926–1927. Since then, they have undergone numerous improvements, with some axioms modified, and other superfluous axioms removed; for a history of the changes, see [11] (especially Section 2), or for a summary, see Figure 2 in [4].

The axioms are expressed in a one-sorted language, with individual variables to be interpreted as *points*, and two primitive relations: betweenness and congruence. Congruence is denoted $ab \equiv cd$, and can be interpreted as asserting that the line segment from a to b is congruent to the line segment from c to d. Betweenness is denoted $\mathrm{B}\,abc$, and can be interpreted as asserting that b lies on the segment from a to c (and may be equal to a or c).

The version of the axioms used in [7] (see pages 10–14) consists of ten first-order axioms, together with either a first-order axiom schema, or a single higher-order axiom. This version has been adopted in later publications, such as [3] (see Sections 2.3 and 2.4) and [4] (see Figure 3), and Victor Pambuccian has called it the "most polished form" of Tarski's axioms (see [6], page 122).

This semi-canonical version of the system is the result of many simplifications to the original system of twenty axioms plus one axiom schema. At least one of these simplifications appears to have taken the form of a slight alteration to one axiom in order to allow another axiom to be dropped and subsequently proven as a theorem.

Specifically, in [9] (see note 18), axiom (ix) is a version of what is called the *axiom of Pasch*, which states (modulo notational differences):[1]

$$\mathrm{B}\,apc \wedge \mathrm{B}\,qcb \longrightarrow \exists x.\ \mathrm{B}\,axq \wedge \mathrm{B}\,xpb \tag{OP}$$

In [10], this has been replaced by axiom A9, which states (again, modulo notational changes):

$$\exists x.\ \mathrm{B}\,apc \wedge \mathrm{B}\,qcb \longrightarrow \mathrm{B}\,axq \wedge \mathrm{B}\,bpx \tag{OP$'$}$$

The significant change between the two is the reversal of the order of points in the final betweenness relation. They are easily shown to be equivalent using the symmetry of betweenness, which states:

$$\mathrm{B}\,abc \longrightarrow \mathrm{B}\,cba \tag{SB}$$

The interesting thing is that in [9], (SB) is an axiom (axiom (iii)), but in [10], it has been removed from the list of axioms. Haragauri Gupta's proof of (SB) (see [2], Theorem 2.18) relies on the precise ordering of points in the final betweenness relation in (OP′).

Also, Wolfram Schwabhäuser, Wanda Szmielew, and Alfred Tarski, on page 12 of [7], draw attention to the ordering of points in their version of the axiom of Pasch (labelled (IP) in this paper), noting that it is important until after the proof of (SB) (which they call Satz 3.2). Again, their proof of (SB) (which is essentially the same as this paper's proof of Lemma 4) relies on the precise ordering of points in (IP).

Therefore, although it does not appear to be explicitly acknowledged in the published literature, it seems likely that the change from (OP) to (OP′) was necessary to allow the removal of (SB) from the set of axioms. It may be the case that (SB) is a theorem even with (OP), and that this was not known when it was replaced by (OP′), or that it was known, but (OP′) allows a simpler proof.

In any case, it appears that Tarski was willing to reorder points in his axioms to allow the simplification of the axiom system as a whole, either by removing axioms or merely by simplifying proofs of theorems.

In the tradition of such simplifications, this paper presents one further simplification of the axiom system; one of the axioms is slightly modified, allowing another of the traditional axioms to be proven as a theorem, rather than assumed as an axiom.

[1] Throughout this article, universal quantifiers scoped over a whole formula are omitted in order to improve readability.

2 The axioms

Tarski's axioms, as stated in [7], pages 10–14, are as follows. The names are adopted from [3] (Section 2.4), which provides some diagrams and intuitive explanations for the axioms.

- Reflexivity axiom for equidistance

$$ab \equiv ba \tag{RE}$$

- Transitivity axiom for equidistance

$$ab \equiv pq \wedge ab \equiv rs \longrightarrow pq \equiv rs \tag{TE}$$

- Identity axiom for equidistance

$$ab \equiv cc \longrightarrow a = b \tag{IE}$$

- Axiom of segment construction

$$\exists x.\ \mathrm{B}\, qax \wedge ax \equiv bc \tag{SC}$$

- Five-segments axiom

$$\begin{aligned} &a \neq b \wedge \mathrm{B}\, abc \wedge \mathrm{B}\, a'b'c' \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd' \wedge bd \equiv b'd' \\ &\qquad \longrightarrow cd \equiv c'd' \end{aligned} \tag{FS}$$

- Identity axiom for betweenness

$$\mathrm{B}\, aba \longrightarrow a = b \tag{IB}$$

- Axiom of Pasch

$$\mathrm{B}\, apc \wedge \mathrm{B}\, bqc \longrightarrow \exists x.\ \mathrm{B}\, pxb \wedge \mathrm{B}\, qxa \tag{IP}$$

- Lower 2-dimensional axiom

$$\exists a, b, c.\ \neg\, \mathrm{B}\, abc \wedge \neg\, \mathrm{B}\, bca \wedge \neg\, \mathrm{B}\, cab \tag{Lo_2}$$

- Upper 2-dimensional axiom

$$p \neq q \wedge ap \equiv aq \wedge bp \equiv bq \wedge cp \equiv cq \longrightarrow (\mathrm{B}\, abc \vee \mathrm{B}\, bca \vee \mathrm{B}\, cab) \tag{Up_2}$$

- Euclidean axiom

$$\mathrm{B}\,adt \wedge \mathrm{B}\,bdc \wedge a \neq d \longrightarrow \exists x, y.\ \mathrm{B}\,abx \wedge \mathrm{B}\,acy \wedge \mathrm{B}\,xty \tag{Eu}$$

- Axiom of continuity

$$\begin{aligned}&(\exists a.\ \forall x, y.\ x \in X \wedge y \in Y \longrightarrow \mathrm{B}\,axy)\\ &\qquad \longrightarrow (\exists b.\ \forall x, y.\ x \in X \wedge y \in Y \longrightarrow \mathrm{B}\,xby)\end{aligned} \tag{Co}$$

This collection of eleven axioms, Tarski's axioms of the continuous Euclidean plane, will be denoted CE_2. A similar collection of axioms, denoted CE_2', is obtained by removing (RE) from CE_2 and replacing (FS) with

$$\begin{aligned}&a \neq b \wedge \mathrm{B}\,abc \wedge \mathrm{B}\,a'b'c' \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge ad \equiv a'd' \wedge bd \equiv b'd'\\ &\qquad \longrightarrow dc \equiv c'd'\end{aligned} \tag{FS$'$}$$

The only difference between (FS) and (FS′) is the reversal of the first two points in the last congruence relation.

This paper will show that CE_2' is equivalent to CE_2, and will, in fact, show a stronger result — Theorem 1 — about a smaller set of axioms.

One of the features of Tarski's axiom system is its modularity: some texts omit or delay the introduction of (Co) (see, for example, [1], page 61); (Eu) can be replaced by another axiom in order to investigate hyperbolic geometry, or omitted entirely, for absolute geometry (see [5], pages 331–333); (Lo_2) and (Up_2) can be replaced by other axioms that characterize other dimensions (see [10], footnote 5). For this reason, let A denote the collection of axioms (RE), (TE), (IE), (SC), (FS), (IB), and (IP). These are Tarski's axioms of absolute dimension-free geometry without the axiom of continuity. Let A' denote the collection of axioms (TE), (IE), (SC), (FS′), (IB), and (IP).

The stronger result that this paper shows is that A' is equivalent to A. Thus, the modularity of axioms (Lo_2), (Up_2), (Eu), and (Co) is unaffected by the proposed change to Tarski's axiom system.

3 Proof of equivalence

Lemma 1. *If* (TE) *and* (SC) *hold, then given any points a and b, we have $ab \equiv ab$.*

Proof. Given a and b, (SC) lets us obtain a point x such that $ax \equiv ab$. Using this twice in (TE) gives us $ab \equiv ab$. $\boxed{QED}$

Lemma 2. *If* (TE) *and* (SC) *hold and a, b, c, and d are points such that $ab \equiv cd$, then $cd \equiv ab$.*

Proof. By Lemma 1, we have $ab \equiv ab$. Using $ab \equiv cd$ and $ab \equiv ab$, (TE) tells us that $cd \equiv ab$. $\boxed{QED}$

Lemma 3. *If* (IE) *and* (SC) *hold, then given any points a and b, we have* $\mathrm{B}\,abb$.

Proof. Given a and b, (SC) lets us obtain a point x such that $\mathrm{B}\,abx$ and $bx \equiv bb$. Then (IE) tells us that $b = x$, so $\mathrm{B}\,abb$. $\boxed{QED}$

Lemma 4. (IE)*,* (SC)*,* (IB)*, and* (IP) *together imply* (SB)*.*

Proof. Suppose we are given points a, b, and c such that $\mathrm{B}\,abc$. We also have $\mathrm{B}\,bcc$, by Lemma 3. Then (IP) lets us obtain a point x such that $\mathrm{B}\,bxb$ and $\mathrm{B}\,cxa$. According to (IB), the former implies $b = x$, so the latter tells us that $\mathrm{B}\,cba$. $\boxed{QED}$

Lemma 5. A' *implies* (RE)*.*

Proof. Given arbitrary points a and b, (SC) lets us obtain a point x such that $\mathrm{B}\,bax$ and $ax \equiv ba$. We consider two cases: $x = a$ and $x \neq a$.

If $x = a$, then $aa \equiv ba$. By Lemma 2, we have $ba \equiv aa$, so by (IE), we have $b = a$. Substituting this back into the congruence as necessary gives us $ab \equiv ba$, as desired.

Suppose, on the other hand, that $x \neq a$. Lemma 4 and $\mathrm{B}\,bax$ tell us that $\mathrm{B}\,xab$. Lemma 1 tells us that $xa \equiv xa$, $ab \equiv ab$, and $aa \equiv aa$. We make the following substitutions in (FS$'$): $a, a' \mapsto x$; $b, b', d, d' \mapsto a$; and $c, c' \mapsto b$. Then all of the hypotheses of (FS$'$) are satisfied, and its conclusion is that $ab \equiv ba$. $\boxed{QED}$

Lemma 6. *If* (RE) *and* (TE) *hold, then* (FS$'$) *is equivalent to* (FS)*.*

Proof. Because the hypotheses of (FS) and (FS$'$) are identical, we need only show that their conclusions are equivalent.

(RE) tells us that $cd \equiv dc$ and $dc \equiv cd$.

If $cd \equiv c'd'$, then $cd \equiv dc$ together with this fact and (TE) let us conclude that $dc \equiv c'd'$.

Similarly, if $dc \equiv c'd'$, then $dc \equiv cd$ together with this fact and (TE) let us conclude that $cd \equiv c'd'$.

Therefore (FS$'$) is equivalent to (FS). $\boxed{QED}$

Theorem 1. A' *is equivalent to* A.

Proof. By Lemmas 5 and 6, A' implies (RE) and (FS). A' contains all of the other axioms of A, so A' implies A.

By Lemma 6, A implies (FS′), and it contains all of the other axioms of A', so A implies A'.

Therefore A' is equivalent to A. $\boxed{QED}$

As an immediate corollary, we have the following:

Corollary 1. CE_2' *is equivalent to* CE_2. $\boxed{QED}$

4 Independence results

The first part of Section 5 of [11] (see pages 199 and 200) concerns the independence of Tarski's axioms. One problem seen there is that the various historical changes to Tarski's axioms often force a reconsideration of previously established independence results. This paper's suggested simplification of Tarski's axioms is no exception, so this section aims to establish which of the known independence results apply to the specific set of axioms CE_2'.

Because CE_2 and CE_2' differ only in their subsets A and A', Theorem 1 tells us that the axioms in CE_2' but not in A' are independent if and only if they are independent in CE_2. In fact, we can go further than this.

Theorem 2. *Suppose that* (Ax) *is an axiom of* CE_2' *other than* (TE) *or* (FS′). *If* (Ax) *is independent in* CE_2 *then* (Ax) *is also independent in* CE_2'.

Proof. Note that (Ax) is not (TE), because this was explicitly excluded; nor is it (RE) or (FS), because these are not axioms of CE_2', from which (Ax) was chosen. Therefore $\mathsf{CE}_2 \setminus \{ (\mathrm{Ax}) \}$ contains (RE), (TE), and (FS), so by Lemma 6, $\mathsf{CE}_2 \setminus \{ (\mathrm{Ax}) \} \vdash (\mathrm{FS}')$.

All of the axioms of CE_2' other than (FS′) are also axioms of CE_2, so $\mathsf{CE}_2 \setminus \{ (\mathrm{Ax}) \} \vdash \mathsf{CE}_2' \setminus \{ (\mathrm{Ax}) \}$. Therefore, if $\mathsf{CE}_2' \setminus \{ (\mathrm{Ax}) \} \vdash (\mathrm{Ax})$, then $\mathsf{CE}_2 \setminus \{ (\mathrm{Ax}) \} \vdash (\mathrm{Ax})$. Taking the contrapositive of this statement, we see that if (Ax) is independent in CE_2 then it is independent in CE_2'. $\boxed{QED}$

This allows us to adopt almost all of the independence results known for CE_2.

Corollary 2. (SC), (IB), (Lo_2), (Up_2), (Eu), *and* (Co) *are each individually independent in* CE_2'.

Proof. The independence of each of these axioms in CE_2 is noted in [7], page 26. $\boxed{QED}$

The remaining independence result noted in [7] can also be adapted to CE_2':

Theorem 3. (FS′) *is independent in* CE_2'.

Proof. The existence of a model $\mathcal{M}$ demonstrating the independence of (FS) in CE_2 is noted in [7], page 26. Because $\mathcal{M}$ satisfies (RE) and (TE), but violates (FS), we can conclude, using Lemma 6, that $\mathcal{M}$ also violates (FS′). $\mathcal{M}$ satisfies every other axiom of CE_2', because all such axioms are also axioms of CE_2 (and are not equal to (FS)); therefore, $\mathcal{M}$ demonstrates the independence of (FS′) in CE_2'. $\boxed{QED}$

Because of the absence of (RE) from CE_2', we can easily show the independence of (TE) in that axiom system:

Theorem 4. (TE) *is independent in* CE_2'.

Proof. We proceed as is usual in independence proofs, by defining a model $\mathcal{M}$ of every axiom of CE_2' except (TE). As in the real Cartesian plane (the standard model of CE_2, and hence of CE_2'), we take $\mathbb{R}^2$ to be the set of points, and define betweenness so that $\mathrm{B}\,abc$ if and only if $b = ta + (1-t)\,c$, for some t with $0 \le t \le 1$. Departing from the standard model, we define congruence so that $ab \equiv cd$ if and only if $a = b$.

Because the real Cartesian plane is a model of CE_2', and $\mathcal{M}$ differs from the real Cartesian plane only in its definition of congruence, we can conclude that $\mathcal{M}$ is a model of all of the axioms of CE_2' that make no mention of congruence. Those axioms are (IB), (IP), (Lo_2), (Eu), and (Co).

The definition of congruence ensures that $\mathcal{M}$ trivially satisfies (IE).

Choosing $x = a$ ensures that (SC) is satisfied.

The hypotheses of (FS′) include $a \neq b$ and $ab \equiv a'b'$, which implies $a = b$; this contradiction in the hypotheses means that (FS′) is vacuously true.

The hypotheses of (Up_2) imply that $a = b = c = p$; it is always the case that $\mathrm{B}\,ppp$, so (Up_2) is satisfied.

Finally, in $\mathcal{M}$, it is the case that $(0,0)\,(0,0) \equiv (0,0)\,(0,1)$, but not the case that $(0,0)\,(0,1) \equiv (0,0)\,(0,1)$, so $\mathcal{M}$ does not satisfy (TE).

Because $\mathcal{M}$ satisfies every axiom of CE_2' except (TE), we can conclude that (TE) is independent in CE_2'. $\boxed{QED}$

Notice that $\mathcal{M}$ in the proof of Theorem 4 also violates (RE) (because it is not the case that $(0,0)\,(0,1) \equiv (0,1)\,(0,0)$), so this model would not demonstrate the independence of (TE) in CE_2, which is, as far as the author is aware, still an open question.

We now have independence results for all of the axioms of CE_2' except (IE) and (IP). As far as the author knows, the independence of these in CE_2 is still an open question (see also [11], pages 199 and 200), although there are independence results relating to these axioms in other versions of Tarski's axiom system.

Gupta shows the independence of (IE) (his axiom A5; see [2], pages 41 and 41a), but only in the context of a particular variant of Tarski's axiom system. This variant uses a weaker but more complicated form of the upper 2-dimensional axiom, Gupta's A′12;[2] his independence model for (IE) also violates (Up_2), which is trivially equivalent to his original axiom A12.

Lesław Szczerba established the independence of a version of the axiom of Pasch within a certain variant of Tarski's axiom system (see [8]). This variant used, instead of (Eu), an axiom essentially asserting that any three non-collinear points have a circumcentre.[3]

5 Conclusions

Although this paper has not answered the question of whether (RE) is independent in the axiom system of [7], it has demonstrated that (RE) is superfluous to Tarski's axioms of geometry. Even in a proper subset of the axioms, a slight modification to (FS) allows (RE) to be proven as a theorem, and therefore removed from the set of axioms. This simplification of the axiom system does not diminish its deductive power or the important ways in which it exhibits modularity.

Besides removing one of the axioms not known to be independent in the unmodified axiom system, the modified system allows an easy proof of the independence of (TE), which was also not known to be independent in the unmodified system. The two remaining independence questions for the new system are, as far as the author knows, still open questions for the old one; furthermore, if either axiom is shown to be independent in the old system, then Theorem 2 would immediately establish its independence in the new one.

As well as trying to resolve these remaining independence questions, future work might seek other slight modifications to the axioms that may allow even known independent axioms to be dropped. That this may be possible can be

[2] It seems that A′12 in Gupta's thesis ought to include $u \neq v$ among the hypotheses, as A12 does. Taken exactly as it is printed, A′12 is violated by the real Cartesian plane whenever x, y, and z are any non-collinear points and $u = v$.

[3] There appears to be a typographical error in the statement of this axiom in [8] (page 492). His axiom A8′ states (in our notation):

$$\exists p.\, \neg\, (\mathrm{B}\, abc \vee \mathrm{B}\, bca \vee \mathrm{B}\, cab) \longrightarrow ap \equiv bp \wedge bp \equiv cb$$

It seems that the second congruence relation in the consequent should state $bp \equiv cp$; see also [11], pages 199 and 184.

seen by considering Gupta's fully independent version of Tarski's axioms for plane Euclidean geometry (see [2], pages 41–41c).

His version had eleven axioms,[4] some of which were deliberately made more complicated in order to allow easy proofs of the independence of other axioms (see, for example, his note on the independence of his axiom A7 on page 41b). The system adopted by [7] already has *simpler* axioms than Gupta's system (which he used only for the demonstration that a fully independent system is possible), but this article's further simplification now shows that a reduction in the *number* of axioms is also possible, without making any of them more complex, despite Gupta's system being fully independent.

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[4]Strictly speaking, as he presented it, it had infinitely many axioms, but if his axiom schema $A'13$ is replaced by a comparable second-order axiom, then there are eleven axioms in total, and the second-order axiom is shown to be independent by his example on page 41c.

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