Note di Matematica Vol. XII, 137-144 (1992)

RELATIVE DOMAINS OF INTEGRAL OPERATORS

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Dedicated to the memory of Professor Gottfried Köthe

SUMMARY. *The paper generalizes the known construction of the extended domain of an integral operator relative to an arbitrary range space. The aim of the generalization is to get rid of excessive solidity hypotheses imposed in the previous work on the subject.*

1. INTRODUCTION

Let $(S, ds), (T, dt)$ be σ-finite measure spaces, $L^0(S), L^0(T)$ be the corresponding spaces of almost everywhere finite scalar valued measurable functions on S and T, and let k – a measurable function on $S \times T$ – be the kernel of the integral operator K:

$$Ku(t) = \int_S k(t,s)u(s)\,ds.$$

The *proper domain* D_K of K is defined in the usual way:

$$D_K = \{u \in L^0(S) : |K||u| < \infty \text{ a.e.}\}, \quad |K||u|(t) = \int_S |k(t,s)||u(s)|\,ds.$$

The spaces $L^0(S)$ and $L^0(T)$ are furnished with the topologies of convergence in measure on all subsets of finite measure, defined by some fixed monotone F-norms ρ_S and ρ_T, respectively.

We recall, [AS, 1967], that D_K is furnished with the natural topology given by the F-norm $u \to \rho_S(u) + \rho_T(|K||u|)$. D_K is a solid F-space and $K : D_K \to L^0$ is continuous.

We assume that D_K is order dense in L^0. This nonsingularity assumption implies, by completeness of D_K, that D_K contains a function which is positive a.e. on S.

By a subsace of L^0 we mean a vector subspace; in the case when the subspace has its own topology, we always tacitly assume that its inclusion in L^0 is continuous.

Throughout this paper (L, ρ_L) will denote an F-subspace of $L^0(T)$; we stress that we do not assume that L is solid.

In [LS, 1988] the maximal extension of an integral operator K with values in L was studied under the hypothesis that the space

(1.1) $$D_{KL} = \{u \in D_K : Ku \in L\}$$

is solid. While realistic in some cases, the hypothesis need not be satisfied in some examples of interest.

Such an example is provided by the Fourier transform, i.e., $S = T = \mathbb{R}$, $k(t, s) = e^{-ist}$, and by the range space $L = L^q, q > 2$. In this case (as one would expect) the space $\{f \in L^1 : Kf \in L^q\} = D_{KL^q}$ is not solid. An elegant proof of this fact was communicated to us by C. Datry and G. Muraz.

The extended domain of the Fourier transform with values in L^q was studied in an *ad hoc* manner in [S, 1980].

With the possibility in mind that D_{KL} need not be solid, we take up again the project of constructing the maximal domain of K relative to L, assuming this time that the space D_{KL} contains a solid subspace which is order dense in L^0; in the preceding example, by the M. Riesz Theorem, D_{KL} contains L^p, with $\frac{1}{p} + \frac{1}{q} = 1$.

The construction in this paper supersedes that of [LS, 1988], which in turn was a generalization of the one in [AS, 1967] dealing with the case when $L = L^0$.

2. EXTENDED DOMAINS RELATIVE TO L

We recall the definition of the extended domain of K (relative to L^0), [AS, 1967]. Define

$$d_K(u) = \sup\{\rho_T(Kv) : v \in D_K \cap [u]\}, \qquad u \in L^0(S), \tag{2.1}$$

where $[u]$ is the order interval $[u] = \{v \in L^0 : |v| \leq |u|\}$.

Then d_K is a (not necessarily finite) group semi-norm on L^0 and equipped with the corresponding (not necessarily finite) group norm

$$\tilde{\rho}_K = \rho_S + d_K, \tag{2.2}$$

L^0 is a complete metric group. The extended domain $\tilde{D}_K$ of K is the closure of D_K in $(L^0, \tilde{\rho}_K)$; $\tilde{D}_K$ is a solid F-space and the operator K can be extended by continuity to $\tilde{D}_K$. We denote this extension by $\tilde{K}$. The following comments and the theorem explain the usefullness of $\tilde{K}$.

Let V be a topological vector subspace of $L^0(S)$. We say that K is $V - L$ semiregular if

(i) $D_K \cap V$ is dense in V;

(ii) $K(D_K \cap V) \subset L$ and the operator $K : D_K \cap V \to L$ is $V - L$ continuous.

If these conditions are satisfied, then K can be extended by continuity to V; this extension is denoted by K_{VL}. When $L = L^0$, we suppress the symbol L and use the term V-semiregular and the symbol K_V. Clearly, $V - L$ semiregular implies V-semiregular.

Theorem 2.1. *Let V be a solid (topological vector) subspace of $L^0(S)$ such that K is V semiregular. Then V is continuously contained in $\tilde{D}_K$ and $K_V = \tilde{K}_{|V}$ (where $_{|V}$ denotes the retriction to V).*

We notice here that in the above theorem the space V need not be complete or metric, the conditions inadvertently imposed in [AS, 1967]. A simple proof of the modified version of the result is repeated below in a more general setting of Theorem 2.5.

We introduce the *solid domain* Δ_{KL} of K relative to L and the *solid domain* $\Delta_{\tilde{K}L}$ of $\tilde{K}$ relative to L:

$$\Delta_{KL} = \{u \in D_K : K[u] \subset L\}, \qquad \Delta_{\tilde{K}L} = \{u \in \tilde{D}_K : \tilde{K}[u] \subset L\}.$$

Clearly, Δ_{KL} and $\Delta_{\tilde{K}L}$ are solid.

Proposition 2.2. *For every $u \in \Delta_{KL}$, the set $K[u]$ is bounded in L. For every $u \in \Delta_{\tilde{K}L}$, the set $\tilde{K}[u]$ is bounded in L.*

Proof. We prove the second statement. Let $V_u = \{v \in L^0 : v \in [\alpha u] \text{ for some } \alpha > 0\}$ for u in $\Delta_{\tilde{K}L}$. With the unit ball $[u]$, V_u is a solid Banach space contained in $\tilde{D}_K$. By solidity of $\tilde{D}_K$, $[u]$ is bounded in $\tilde{D}_K$ and the inclusion $V_u \subset \tilde{D}_K$ is continuous. The continuity of $\tilde{K}$ from $\tilde{D}_K$ into L^0 and the continuity of the inclusion of L in L^0 imply, by using the closed graph theorem, that $\tilde{K} : V_u \to L$ is continuous and that $\tilde{K}[u]$ is bounded in L. The proof of the first part is obtained in the same way, replacing $\tilde{K}$ by K and $\tilde{D}_K$ by D_K.

It is obvious that $\Delta_{KL} \subset \Delta_{\tilde{K}L}$.

We now make the *nonsingularity* assumption: *$\Delta_{\tilde{K}L}$ is order dense in $L^0(S)$.*

This is equivalent to a seemingly stronger assumption that Δ_{KL} is order dense in L^0. Indeed, let $u_0 \in D_K$ be such that $u_0 > 0$ a.e. Then the set $\{\min(\alpha u_0, |v|) \operatorname{sign} v;\ v \in \Delta_{\tilde{K}L}, \alpha > 0\}$ is contained in Δ_{KL} and is order dense in L^0 if $\Delta_{\tilde{K}L}$ is.

Similarly to (2.1), we define the following group seminorms on L^0:

$$d_{\tilde{K}L}(u) = \sup\{\rho_L(\tilde{K}v) : v \in [u] \cap \Delta_{\tilde{K}L}\}, \quad d_{KL} = \sup\{\rho_L(Kv) : v \in [u] \cap \Delta_{KL}\}.$$

Theorem 2.3. *The group norms $\tilde{\rho}_{KL} = \rho_S + d_{KL}$ and $\rho_{\tilde{K}L} = \rho_S + d_{\tilde{K}L}$ are solid and complete on L^0. $\tilde{\rho}_{KL} \leq \rho_{\tilde{K}L}$. Both norms define on L^0 topologies stronger than that given by $\tilde{\rho}_K$.*

Proof. The completeness proof is the same as for $\tilde{\rho}_K$ (see [AS, 1967]) and we omit it. The inequality $d_{KL} \leq d_{\tilde{K}L}$ is obvious. If $d_{KL}(u_n) \to 0$, then for any sequence (v_n) such that $v_n \in D_K \cap [u_n]$, we have $\rho_L(Kv_n) \to 0$ and, by continuity of the inclusion of L in L^0, it follows that $d_K(u_n) \to 0$. This proves the last statement.

Proposition 2.4. *$(\Delta_{\tilde{K}L}, \rho_{\tilde{K}L})$ is an F-space.*

Proof. The continuity of multiplication $\alpha \to \alpha u$, $u \in D_{\tilde{K}L}$, follows from Proposition 2.2. If (u_n) is a Cauchy sequence in $\Delta_{\tilde{K}L}$, then (u_n) converges in $\tilde{D}_K$ to a limit u and, by continuity, $(\tilde{K}u_n)$ converges to $\tilde{K}u$ in L^0. The inequality $\rho_L(K(u_n - u_m)) \leq d_{\tilde{K}L}(u_n - u_m)$ implies that $(\tilde{K}u_n)$ is a Cauchy sequence in L and that $\tilde{K}u \in L$. If $v \in [u]$, then we apply the same argument to the sequence $(v_n), v_n = \min(|u_n|, |v|)$ sign v, to conclude that $v \in \tilde{D}_K$ and that $\tilde{K}v \in L$. It follows that $u \in \Delta_{\tilde{K}L}$.

Theorem 2.5 (maximality relative to L). *Let V be a solid topological vector subspace of L^0 such that $V \cap \Delta_{\tilde{K}L}$ is dense in V and that $\tilde{K} : V \cap \Delta_{\tilde{K}L} \to L$ is V-continuous. Then V is continuously contained in $\Delta_{\tilde{K}L}$.*

Proof. We show that on $V \cap D_{\tilde{K}L}$ the topology of V is stronger than the topology of $\Delta_{\tilde{K}L}$. For given $\varepsilon > 0$, we find a solid neighborhood N of the origin in V such that $\rho_L(\tilde{K}v) < \varepsilon$ for all $v \in N \cap \Delta_{\tilde{K}L}$. Then $d_{\tilde{K}L}(v) \leq \varepsilon$ for all $v \in N \cap \Delta_{\tilde{K}L}$.

A more direct extension of K relative to L is obtained by taking the closure of Δ_{KL} in $(L^0, \tilde{\rho}_{KL})$, which we denote by $\tilde{\Delta}_{KL}$. The nonsingularity assumption implies that $\tilde{\Delta}_{KL}$ contains a function which is positive a.e. The space $(\tilde{\Delta}_{KL}, \tilde{\rho}_{KL})$ is a solid F-space with the following maximality property.

Theorem 2.6. *Let V be a solid topological vector subspace of L^0 such that $V \cap \Delta_{KL}$ is dense in V and $K : V \cap \Delta_{KL} \to L$ is V-continuous. Then V is continuously contained in $\tilde{\Delta}_{KL}$.*

We note that if the space D_{KL} is not solid, then it is not contained in $\Delta_{\tilde{K}L}$. In fact, if $f \in D_{KL}$ is such that $[f]$ is not contained in D_{KL}, then $f \notin \Delta_{\tilde{K}L}$. Otherwise the order interval $[f]$ would be contained in $\Delta_{\tilde{K}L} \cap D_K$ and $\tilde{K}[f] = K[f]$ would be contained in L, contrary to the assumption.

Remark. The hypotheses of Theorem 2.6 are equivalent to K being $V - L$ semiregular. In fact, under these hypotheses, we have $\Delta_{KL} \cap V = D_K \cap V$.

The inclusion $\Delta_{KL} \cap V \subset D_K \cap V$ is obvious. To see that $D_K \cap V \subset \Delta_{KL} \cap V$, let $u \in D_K \cap V$ and consider $v_n \in \Delta_{KL} \cap V$ such that $v_n \to u$ in V. By solidity of V, we may assume that $v_n \in [u]$. Then $Kv_n \to K_{VL}(u)$ in L and $Kv_n \to Ku$ a.e. It follows that $Ku = K_{VL}(u) \in L$ and $u \in D_{KL}$. The same argument shows that $[u] \subset D_{KL}$ and $u \in \Delta_{KL}$.

We recall that for a solid subspace V of L^0, the space $V^\sharp$ is defined by

$$V^\sharp = \{u \in L^0 : [u] \cap V \text{ is bounded in } V\}.$$

Proposition 2.7. $\tilde{\Delta}_{KL} \subset \Delta_{\tilde{K}L} \subset \tilde{\Delta}^{\natural}_{KL}$ *and* $d_{KL} = d_{\tilde{K}L}$ *on* $\tilde{\Delta}_{KL}$.

Proof. Since K is $\tilde{\Delta}_{KL} - L$ semiregular, the first inclusion follows from Theorem 2.5. If $u \in \Delta_{\tilde{K}L}$ and $\varepsilon > 0$ then, by Proposition 2.2, for some $\lambda > 0, \rho_L(\lambda K v) < \varepsilon$ for all $v \in [u]$. It follows that $d_{KL}(\lambda v) \leq \varepsilon$ for all $v \in [u] \cap \Delta_{KL}$ which implies the second inclusion. The last equality is obvious.

The inclusion in the above Proposition can be reversed under additional conditions on L introduced in the next Section.

Remark. The constructions as in (2.3) can be carried out for F-norms ρ_L with special properties (e.g. norms, p-norms). d_{KL} may then inherit properties of ρ_L.

3. SPECIAL CLASSES OF RANGE SPACES L

We now consider some assumptions on the space L which allow to obtain inclusions of the extended domains, mentioned in the preceding section and to carry out the construction of the extended domain relative to L, using a sublattice of D_K rather than its solid subspace.

Conditions imposed on L involve C-sequences. We recall that a sequence $(x_n) \subset L$ is a *C-sequence in L* if, for every numerical sequence (a_n) convergent to 0, the series $\Sigma a_n x_n$ is convergent in L.

We will use the «if» part of the following standard result.

Proposition 3.1. *A sequence (x_n) is an arbitrary F-space X is a C-sequence iff the set of all finite sums $\Sigma a_n, x_n, |a_n| \leq 1$, is bounded in X.*

It is known, [O, 1951], that for every C-sequence (x_n) in L^0, the series Σx_n is convergent in L^0. Hence, by the continuity of the inclusion of L in L^0, for eery C-sequence (x_n) in L, the sum Σx_n exists in L^0.

The conditions we impose on L are the following:

(c_w) If (x_n) is a C-sequence in L, then $\Sigma x_n \in L$ (it is not required that the series be convergent in L).

(c) If (x_n) is a C-sequence in L, then Σx_n is convergent in L.

We remark that (c_w) is weaker than (c) and that (c_w) is satisfied in many concrete function spaces, in particular in any dual space in the sense of Köthe. An intermediate condition between (c_w) and (c) was introduced in [D, 1974] under the name of bounded Fatou property. By [K, 1975], L satisfies (c) iff it does not contain a linearly homeomorphic copy of c_0.

The following statement is immediate and is listed for the sake of reference.

The space L satisfies (c) iff for every C-sequence (x_n) in $L, x_n \to 0$ in L.

We now show how (c_w) allows one to replace Δ_{KL}, in the construction of $\tilde{\Delta}_{KL}$, by a sufficiently large sublattice of D_{KL}.

Let $\Delta \subset D_{KL}$ be a vector sublattice of L^0. We consider the following conditions on Δ:

(3.1) Δ is order dense in L^0. For every $u \in \Delta$, the set $K([u] \cap \Delta)$ is bounded in L.

Given such Δ, one can carry out the construction of the F-norm d_{KL} as follows. First d_{KL} is defined on Δ by the formula (2.3), the least upper bound being extended over $[u] \cap \Delta$. Once this is accomplished, d_{KL} is extended to L^0 by monotonicity. The latter step requires some additional hypotheses on L related to the condition (c_w). The resulting construction supersedes the one in Section 2 with seemingly weaker hypotheses. Although useful for some purposes, this weakening of the requirements on Δ is illusory, as shown in the next Proposition.

Proposition 3.2. *If L satisfies (c_w) and if D_{KL} contains a lattice Δ satisfying (3.1), then the solid hull of Δ is contained in D_{KL}. In particular, Δ_{KL} is order dense in L^0.*

Proof. It is sufficient to prove that for every $u \in \Delta$, $[u]$ is contained in D_{KL}. If $v \in [u]$, we write $v = v_+ - v_-$ (a sum of four terms in the complex case) and prove that v_+ and v_- are in D_{KL}. By the hypothesis, there is a sequence $v_0 = 0, v_n \in \Delta, n \in \mathbf{N}$, such that $v_n \uparrow v_+$. The order boundedness of K implies that the sequence $(Kv_{n+1} - Kv_n)$ is a C-sequence in L with the sum Kv_+. It follows from (c_w) that $Kv_+ \in L$.

The proof of the proposition shows also that if L satisfies (c_w) and if D_{KL} is order dense but not solid in L^0, then D_{KL} can not be a lattice. In particular, if K is the Fourier transform, then the space D_{KL}, considered in Section 1 *is not a lattice.*

Let V be a topological subspace L^0. Recall that:

V has the σ*-Levi property* if, for every sequence (v_n) bounded in V, $v_n \geq 0$, and $v \in L^0$, such that $v_n \uparrow v$ a.e., it follows that $v \in V$.

V has the σ*-Lebesgue property* if the conditions $v_n \in V, v_n \downarrow 0$ a.e imply that $v_n \to 0$ in V.

The spaces D_K and $\tilde{D}_K$ are known to have both the σ-Levi and the σ-Lebesgue properties.

Proposition 3.3. *If L satisfies (c_w), then $\Delta_{\tilde{K}L}$ has the σ-Levi property.*

Proof. Let (u_n) be a bounded sequence in $\Delta_{\tilde{K}L}, u_n \geq 0, u_0 = 0$ and suppose that $u_n \uparrow u$ a.e. Since $\tilde{D}_K$ has the σ-Levi property and (u_n) is bounded in $\tilde{D}_K$, it follows that $u \in \tilde{D}_K$

and that $\tilde{K}u = \lim \tilde{K}u_n$ in L^0. Also, the sequence $(u_n - u_{n-1})$ is a C-sequence in $\Delta_{\tilde{K}L}$ (by Proposition 3.1) and $(Ku_n - Ku_{n-1})$ is a C-sequence in L (by Proposition 2.2) and $Ku = \sum(Ku_n - Ku_{n-1}) \in L$. If $v \in [u]$, then the same argument applied to the sequence $(v_n), v_n = \min(u_n, |v|)$ sign v, shows that $\tilde{K}v \in L$ and hence $u \in \Delta_{\tilde{K}L}$.

Theorem 3.4. *If L satisfies (c_w), then $\tilde{\Delta}^\sharp_{KL} = \Delta_{\tilde{K}L}$.*

Proof. If $u \geq 0, u \in \tilde{\Delta}^\sharp_{KL}$, then by the nonsingularity assumption, we can find a sequence $u_n \uparrow u$ such that $u_n \in \Delta_{KL}$ and, by Proposition 3.3, $u \in \Delta_{\tilde{K}L}$. The definition of $\tilde{\Delta}^\sharp_{KL}$ implies that (u_n) is bounded in $\tilde{\Delta}_{KL}$ and hence in $\Delta_{\tilde{K}L}$. The reverse inclusion is given by Proposition 2.7.

If $\Delta_{\tilde{K}L} = \tilde{\Delta}_{KL}$, then Proposition 3.3 implies that $\tilde{\Delta}_{KL}$ has the σ-Levi property. This equality does not seem to follow from (c_w) alone.

Theorem 3.5. *If L satisfies (c), then $\Delta_{\tilde{K}L} = \tilde{\Delta}_{KL}$ and $\tilde{\Delta}_{KL}$ has the σ-Levi property. Also, in this case $\tilde{\Delta}_{KL}$ has the σ-Lebesgue property.*

Proof. We first prove that $\tilde{\Delta}_{KL}$ has the σ-Lebesgue property. If $u_n \in \Delta_{\tilde{K}L}, u_n \downarrow 0$ and if $d_{\tilde{K}L}(u_n) \not\to 0$, then the series $\Sigma(u_n - u_{n-1})$ cannot converge in $\Delta_{\tilde{K}L}$, or else it would have to converge to u_1 and (u_n) would have to converge to 0 in $\Delta_{\tilde{K}L}$. Choosing, if necessary, a subsequence and using the definition of $d_{\tilde{K}L}$, we find $v_n \in [u_n - u_{n+1}]$ such that $\rho_L(\tilde{K}v_n) \not\to 0$. Since (v_n) is a C-sequence in $\Delta_{\tilde{K}L}, (\tilde{K}v_n)$ is a C-sequence in L and the series $\Sigma \tilde{K}v_n$ converges in L. We get a contradiction.

To complete the proof, suppose that $u \geq 0, u \in \Delta_{\tilde{K}L}$. Since Δ_{KL} is order dense in L^0, we find $u_n \in \Delta_{KL}, u_n \geq 0$, such that $u_n \uparrow u$ a.e. Then, by the σ-Lebesgue property, $d_{KL}(u - u_n) \leq d_{\tilde{K}L}(u - u_n) \to 0$ with $n \to \infty$ and $u \in \tilde{\Delta}_{KL}$.

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Received September 19, 1990 and in revised form, December 11, 1992

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