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REMARKS ON THE WEYL QUANTIZEED RELATIVISTIC HAMILTONIAN

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Dedicated to the memory of Professor Gottfried Köthe

1. INTRODUCTION

In this note we consider the same Weyl quantized relativistic Hamiltonian H_A of a spinless particle as in [7], [8], but with a *singular* magnetic vector potential $A(x)$, corresponding to the classical relativistic Hamiltonian (e.g. Landau-Lifschitz [14])

$$\sqrt{c^2(p-A(x))^2+m^2c^4}, \qquad (p,x)\in \mathbb{R}^d\times\mathbb{R}^d, \tag{1.1}$$

where $c>0$ is the light velocity and $m\geq 0$ is the mass of the particle. We assume for the magnetic vector potential $A:\mathbf{R}^d\to\mathbf{R}^d, A(x)=(A_1(x),\dots,A_d(x))$, that

$$A(x) \text{ is in } L^{2+\delta}_{\text{loc}} \text{ for some } \delta>0. \tag{1.2}$$

Then the *Weyl quantized relativistic Hamiltonian with magnetic fields* or *relativistic magnetic Schrödinger operator* $H_A = H_A^{c,m}$ corresponding to the classical symbol (1.1) is defined through

$$\begin{aligned}([H_A-mc^2]u)(x) &= -\lim_{r\downarrow 0}\int_{|y|>r}[e^{-iyA(x+y/2)}u(x+y)-u(x)]n(dy)\\ &= -\lim_{r\downarrow 0}\int_{|y|>r}[e^{-iyA(x+y/2)}u(x+y)-u(x)\\ &\quad - I_{\{|y|<1\}}y(\partial_x-iA(x))u(x)]n(dy), \quad u\in C_0^\infty(\mathbb{R}^d).\end{aligned} \tag{1.3}$$

Here the $r\downarrow 0$ limit will be taken in L^2. $I_{\{|y|<1\}}$ is the indicator function of the set $\{|y|<1\}$, and $n(dy) = n^{c,m}(dy)$ is a σ-finite measure on $\mathbb{R}^d\backslash\{0\}$ dependent on the light velocity $c>0$ and mass $m\geq 0$, called the *Lévy measure.* It behaves as $O(|y|^{-(d+1)})dy$ near $y=0$ and is a bounded measure on $\{|y|\geq 1\}$, and is, in fact, given by

$$n(dy) = \begin{cases} 2(2\pi)^{-\frac{d+1}{2}}c(mc)^{\frac{d+1}{2}}|y|^{-\frac{d+1}{2}}K_{(d+1)/2}(mc|y|)\,dy, & m>0,\\ \pi^{-\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}\right)c|y|^{-(d+1)}dy, & m=0,\end{cases} \tag{1.4}$$

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where $K_\nu(z)$ is the modified Bessel function of the third kind of order ν and $\Gamma(z)$ the gamma function. The second equality in (1.3) is due to rotational invariance of the Lévy measure $n(dy)$.

The aim of this note is to discuss the nonrelativistic limit ($c \to \infty$) and zero-mass limit ($m \downarrow 0$) for $H_A - mc^2$. The usual factor $1/c$ in front of $A(x)$ in (1.1) is omitted (cf. [4]) so that it can be kept fixed in the limit $c \to \infty$.

In Section 2, justifying the definition of the Weyl quantized relativistic Hamiltonian H_A, we state the main results, and, in Section 3, give their proof. Section 4 gives some further results on H_A defined through the quadratic form.

2. RESULTS

We assume that $A(x)$ satisfies the condition (1.2), unless otherwise specified.

The following proposition justifies the definition of (1.3).

Proposition 2.1. *The $r \downarrow 0$ limit in (1.3) exists in the sense of convergence of L^2, and H_A defines a symmetric operator in $L^2(\mathbb{R}^d)$ with domain $C_0^\infty(\mathbb{R}^d)$ which is bounded from below by mc^2. Moreover, if*

(2.1)
$$A(x) \text{ is in } L^{2+\delta}_{\text{loc}} \text{ for some } \delta > 0 \text{ and } \int_{0<|y|<1} y(A(x+y/2) - A(x))\, n(dy) \text{ is in } L^2_{\text{loc}},$$

then

$$([H_A - mc^2]u)(x) = -\int_{|y|>0} [e^{-iyA(x+y/2)}u(x+y) - u(x) - I_{\{|y|<1\}} y(\partial_x - iA(x))u(x)]\, n(dy), \quad u \in C_0^\infty(\mathbb{R}^d). \tag{2.2}$$

Notice that the condition (1.2) implies by the Calderon-Zygmund theorem (e.g. [22]) that

$$\lim_{r\downarrow 0} \int_{r<|y|<1} y(A(x+y/2) - A(x))\, n(dy) \tag{2.3}$$

exists in the sense of convergente of $L^{2+\delta}_{\text{loc}}$ and so of L^2_{loc}. Therefore (2.1) is a slightly stronger requirement than (1.2), in the sense that (2.1) assumes integrability of $y(A(x+y/2) - A(x))$ with respect to $n(dy)$, which turns out to yield integrability of the integrand on the right-hand side of (2.2).

The definition of H_A through (2.2) has been given first in [7], [8], when

$$A(x) \text{ and } \int_{0<|y|<1} |A(x+y/2) - A(x)||y| n(dy) \text{ are locally bounded}, \tag{2.4}$$

together with the proof of its essential selfadjointness on $C_0^\infty(\mathbb{R}^d)$. It is not yet known whether H_A defined by (1.3) and/or (2.2) is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$, although it is the case, as to be shown in a forthcoming paper [10], if $A(x)$ satisfies a condition somewhat stronger than (2.1) but weaker than (2.4), i.e. that $A(x)$ is in $L^{2+\delta}_{\mathrm{loc}}$ for some $\delta > 0$ and $\int_{0<|y|<1} |y(A(x+y/2) - A(x))| n(dy)$ is in L^2_{loc}.

For $A(x) \equiv 0$, (1.3) and (2.2) become

$$\begin{aligned} (H_0 u)(x) &\equiv (\sqrt{-c^2\Delta + m^2c^4}\, u(x))(x) \\ &= mc^2 u(x) - \int_{|y|>0} [u(x+y) - u(x) - I_{\{|y|<1\}} y \partial_x u(x)] n(dy), \end{aligned} \tag{2.5}$$

which is, by Fourier transform, equivalent to the Lévy-Khinchin formula

$$\sqrt{c^2p^2 + m^2c^4} = mc^2 - \int_{|y|>0} [e^{ipy} - 1 - I_{\{|y|<1\}} ipy] n(dy). \tag{2.6}$$

for the conditionally negative definite function $\sqrt{c^2p^2 + m^2c^4} - mc^2$ (e.g. [11], [18]). Here note that $I_{\{|y|<1\}}$ may be replaced by $I_{\{|y|<r\}}$ for any $r > 0$. The last member of (2.5) exists also for bounded $u \in C^\infty(\mathbb{R}^d)$.

When $A(x)$ is sufficiently smooth with bounded derivatives $|\partial^\alpha A(x)| \le C_\alpha, \alpha = (\alpha_1, \ldots, \alpha_d), |\alpha| = \alpha_1 + \ldots + \alpha_d \ge 1$, H_A may be defined as the Weyl pseudo-differential operator H_A^w:

$$\begin{aligned} (H_A^w u)(x) &= \\ &= (2\pi)^{-d} \iint e^{i(x-y)p} \sqrt{c^2 \left(p - A\left(\frac{x+y}{2}\right)\right)^2 + m^2c^4}\, u(y)\, dy dp, \quad u \in \mathscr{S}(\mathbb{R}^d), \end{aligned} \tag{2.7}$$

where the integral on the right is an oscillatory integral (e.g. [19]). Of course, H_A agrees with H_A^w, when (2.7) makes sense.

Now we consider the nonrelativistic limit problem. For the nonrelativistic magnetic Schrödinger operator

$$H_A^{NR} = (2m)^{-1}(-i\partial - A(x))^2 \tag{2.8}$$

Leinfelder-Simader [15] (cf. [1, p. 11]) proved its essential selfadjointness on $C_0^\infty(\mathbf{R}^d)$, if

(2.9) $\qquad A(x)$ is in L^4_{loc} and $\partial A(x) = \partial_1 A_1(x) + \ldots + \partial_d A_d(x)$ is in L^2_{loc}.

This is a definitive result in the sense that the condition (2.9) is minimal to assure that H_A^{NR}, (2.8), defines a linear operator in $L^2(\mathbf{R}^d)$ with domain $C_0^\infty(\mathbf{R}^d)$. The second half of the condition (2.9) may be thought of as the nonrelativistic limit ($c \to \infty$) of that of the condition (2.1). In fact, should it hold uniformly for $c \geq 1$ that

$$\int_{0<|y|<1} y(A(x+y/2) - A(x))n^{c,m}(dy)$$
$$= \int_{0<|y|<c} \frac{y(A(x+y/2c) - A(x))}{1/c} n^{1,m}(dy) \text{ is in } L^2_{\text{loc}},$$

with a suitable integrability condition on the integrand, we should get $\partial A(x) \in L^2_{\text{loc}}$ by tending $c \to \infty$.

Theorem 2.2. *(Nonrelativistic limit $c \to \infty$). Put $m = 1$ and write H_A as H_A^c. Assume that $A(x)$ satisfies (2.9). Then as $c \to \infty$,*

(2.10) $\qquad [H_A^c - c^2]u \to H_A^{NR}u \quad \text{in } L^2, \qquad \text{for } u \in C_0^\infty(\mathbf{R}^d).$

Next we consider the zero-mass limit problem.

Theorem 2.3. *(Zero-mass limit $m \downarrow 0$). Put $c = 1$ and write H_A as H_A^m. Assume $A(x)$ satisfies (1.2). Then*

(2.11) $\qquad \|[H_A^m - m]u - H_A^0 u\| \leq 2m\|u\|, \qquad \text{for } u \in C_0^\infty(\mathbf{R}^d).$

where $\|\cdot\|$ is the L^2 norm. Moreover, as $m \downarrow 0$,

(2.12) $\qquad ([H_A^m - m]u, u) \uparrow (H_A^0 u, u), \qquad \text{for } u \in C_0^\infty(\mathbf{R}^d).$

Remark 1. Some consequences of Theorems 2.2 and 2.3 are mentioned, when H_A is essentially selfadjoint on $C_0^\infty(\mathbf{R}^d)$. Denote the unique selfadjoint extension of H_A by the same H_A and that of H_A^{NR} by the same H_A^{NR}. Then by Kato [12, VIII, Cor. 1.6, p. 429], the convergence of (2.10) in the nonrelativistic limit implies the strong resolvent convergence

(2.13) $\qquad ([H_A^c - c^2] - \lambda)^{-1} \underset{s}{\to} (H_A^{NR} - \lambda)^{-1}, c \to \infty,$

for every nonreal λ, which is, by Kato [12, IX, Theorem 2.16, p. 504], equivalent to the convergence of the semigroup and unitary group

$$(2.14) \qquad \exp\{-t[H_A^c - c^2]\} \underset{s}{\to} \exp[-tH_A^{NR}], \qquad t \geq 0,$$

$$(2.15) \qquad \exp\{-it[H_A^c - c^2]\} \underset{s}{\to} \exp[-itH_A^{NR}], \qquad t \in \mathbf{R},$$

as $c \to \infty$, on $L^2(\mathbb{R}^d)$ uniformly on bounded intervals of t.

The same is true for the convergence in the zero-mass limit. (2.11) implies

$$(2.16) \qquad ([H_A^m - m] - \lambda)^{-1} \underset{s}{\to} (H_A^0 - \lambda)^{-1}, \qquad m \downarrow 0,$$

for every nonreal λ, which is equivalent to

$$(2.17) \qquad \exp\{-t[H_A^m - m]\} \underset{s}{\to} \exp[-tH_A^0], \qquad t \geq 0,$$

$$(2.18) \qquad \exp\{-it[H_A^m - m]\} \underset{s}{\to} \exp[-itH_A^0], \qquad t \in \mathbb{R},$$

as $m \downarrow \infty$, on $L^2(\mathbb{R}^d)$ uniformly on bounded intervals of t.

Remark 2. When $A(x)$ is sufficiently smooth and bounded together with its derivatives of sufficiently higher order, Ichinose [5] (cf. [6]) showed (2.14) (and hence (2.15)), using the path integral representation of the semigroup $\exp\{-t[H_A^c - c^2]\}$ established in [9] to prove its convergence in the nonrelativistic limit to the Feynmann-Kac-Itô formula (e.g. [20]) of the semigroup $\exp[-tH_A^{NR}]$.

Remark 3. Nagase-Umeda ([16], [17]) proved, for $A(x)$ sufficiently smooth with $|\partial^\alpha A(x)| \leq C_\alpha, |\alpha| \geq 1$, an estimate slightly weaker than (2.11) for the pseudo-differential operator H_A^w as well as its essential selfadjointness.

Remark 4. In Section 4 we refer to the definition of the Weyl quantized relativistic Hamiltonian or relativistic magnetic Schrödinger operator H_A through the corresponding quadratic form. But our H_A differs from the square root

$$(2.19) \qquad \sqrt{c^2(-i\partial - A(x))^2 + m^2c^4}$$

of the nonegative selfadjoint operator $c^2(-i\partial - A(x))^2 + m^2c^4$, whether both H_A and $c^2(-i\partial - A(x))^2 + m^2c^4$ are defined as operators or through quadratic forms.

For the nonrelativistic limit for (2.19), De Angelis and Serva [2] has made a probabilistic treatment.

3. PROOFS

First we collect here the notations to be used in the following proofs of Proposition 2.1, Theorems 2.2 and 2.3.

By $\|f\|_p$ we denote the L^p-norm of a function $f(x)$ in $\mathbf{R}^d$, while the L^2-norm simply by $\|f\|$. For a compact set K in $\mathbf{R}^d$, $|K|$ stands for the volume of K, and for $r > 0$, put $K_r = \{x \in \mathbf{R}^d; dist(x, K) \leq r\}$. Put $\|f\|_{p,K} = \|f\|_{L^p(K)}, 1 \leq p \leq \infty$. For $\alpha > 0$ put

$$n_\alpha = \int_{0<|y|<1} |y|^{1+\alpha} n(dy), \tag{3.1}$$

$$N_\alpha = \int_{|y|\geq 1} |y|^\alpha n(dy). \tag{3.2}$$

We see from the behavior (1.4) of the Lévy measure $n(dy)$ that $n_\alpha, \alpha > 0$, and N_0 are finite. Note that N_0 is denoted in [8] by n_∞.

In the following, when we need to emphasize the c- and/or m-dependence, we shall write $n(dy) = n(y)\,dy, n_\alpha$ and N_α as $n^{c,m}(dy) = n^{c,m}(y)\,dy, n_\alpha^{c,m}$ and $N_\alpha^{c,m}$, respectively.

Proof of Proposition 2.1. We assume (1.2). We may suppose that $0 < \delta \leq 2$. Let $u \in C_0^\infty(\mathbf{R}^d)$ and let K be the support of u, which is compact.

First we show H_A is a linear operator in $L^2(\mathbf{R}^d)$ with domain $C_0^\infty(\mathbf{R}^d)$. Rewrite (1.3) as

$$\begin{aligned}
(H_A u)(x) = {} & \Big\{ mc^2 u(x) - \int_{|y|>0} [u(x+y) - u(x) - I_{\{|y|<1\}} y\partial_x u(x)]\, n(dy) \Big\} \\
& + \int_{|y|\geq 1} -(e^{-iyA(x+y/2)} - 1)\, u(x+y)\, n(dy) \\
& + \int_{0<|y|<1} -(e^{-iyA(x+y/2)} - 1 + iyA(x+y/2))\, u(x+y)\, n(dy) \\
& + \int_{0<|y|<1} iyA(x+y/2)(u(x+y) - u(x))\, n(dy) \\
& + \lim_{r\downarrow 0} \int_{r<|y|<1} iy(A(x+y)/2) - A(x))\, u(x)\, n(dy) \\
\equiv {} & (H_0 u)(x) + \sum_{j=1}^{4} (I_j u)(x).
\end{aligned} \tag{3.3}$$

By (2.5), $H_0 u$ is in L^2. So we must show that the $I_j u$ belong to $L^2(\mathbf{R}^d), j = 1, 2, 3, 4$.

We can obtain by the Schwarz and Hölder inequalities

(3.4)
$$
\begin{aligned}
&\|I_1 u\| \le 2 N_0 \|u\|,\\
&\|I_2 u\| \le 3 n_{\delta/2} \|A\|_{2+\delta,K_1}^{(2+\delta)/2} \|u\|_\infty,\\
&\|I_3 u\| \le n_1 \|A\|_{2,K_1} \|\partial u\|_\infty \le |K_1|^{\delta/2(2+\delta)} n_1 \|A\|_{2+\delta,K_1} \|\partial u\|_\infty,\\
&\|I_4 u\| \le C_K \|A\|_{2,K_1} \|u\|_\infty \le C_K |K_1|^{\delta/2(2+\delta)} \|A\|_{2+\delta,K_1} \|u\|_\infty.
\end{aligned}
$$

Here, to get the estimate for $I_2 u$, use is made of

$$|e^{-it} - 1 + it| \le 3|t|^{(2+\delta)/2}, \quad 0 < \delta \le 2.$$

In the first inequality for $I_4 u$ we have used the Calderno-Zygmund theorem (e.g. [22]); $C_K > 0$ is a constant independent of A and dependent on K. When $A(x)$ satisfies (2.1) rather than (1.2), we see the expression for $I_4 u$ is valid with $\lim_{r\downarrow 0} \int_{r<|y|<1}$ replaced by $\int_{0<|y|<1}$ and have $\|I_4 u\| \le C(A, K)\|u\|_\infty$, with a finite constant

$$C(A,K) = \left(\int_K \left| \int_{0<|y|<1} y(A(x+y/2) - A(x))n(dy) \right|^2 dx \right)^{1/2}.$$

Next to see that H_A is a symmetric operator on $C_0^\infty(\mathbf{R}^d)$, put

(3.5)
$$
\begin{aligned}
(H_A u)(x) &= L^2 - \lim_{r\downarrow 0} (H_{A,r} u)(x)\\
&\equiv L^2 - \lim_{r\downarrow 0} \left\{ mc^2 u(x) - \int_{|y|\ge r} [e^{-iyA(x+y/2)} u(x+y) - u(x)] n(dy) \right\},\\
&u \in C_0^\infty(\mathbf{R}^d),
\end{aligned}
$$

Hence $(H_A u, v) = \lim_{r\downarrow 0} (H_{A,r} u, v) = \lim_{r\downarrow 0} (u, H_{A,r} v) = (u, H_A v)$, for $u, v \in C_0^\infty(\mathbf{R}^d)$.

Finally, we show that $H_A - mc^2$ is nonegative. Let $u \in C_0^\infty(\mathbf{R}^d)$ and $u_\varepsilon(x) = \sqrt{|u(x)|^2 + \varepsilon^2}, \varepsilon > 0$. Then u_ε is C^∞ and bounded. Note that $-|u(x)||u(x+y)| + |u(x)|^2 \ge -u_\varepsilon(x) u_\varepsilon(x+y) + u_\varepsilon(x)^2$, and $\partial |u(x)|^2 = \partial u_\varepsilon(x)^2$. By taking a subsequence $r \downarrow 0$ if necessary, we have for a.e. x (for simplicity, writing $((H_A - mc^2)u)(x)$

and $((H_0 - mc^2)u_\varepsilon)(x)$ as $(H_A - mc^2)u(x)$ and $(H_0 - mc^2)u_\varepsilon(x)$, respectively)

$$
\begin{aligned}
&\mathrm{Re}[\overline{u(x)}(H_A - mc^2)u(x)] \\
&= 2^{-1}\{\overline{u(x)}(H_A - mc^2)u(x) + u(x)\overline{(H_A - mc^2)u(x)}\} \\
&= 2^{-1}\lim_{r\downarrow 0}\{\overline{u(x)}(H_{A,r} - mc^2)u(x) + u(x)\overline{(H_{A,r} - mc^2)u(x)}\} \\
&\geq \lim_{r\downarrow 0}\int_{|y|>r}[-|u(x)||u(x+y)| + |u(x)|^2 + 2^{-1}I_{\{|y|<1\}}y\partial|u(x)|^2]n(dy) \qquad (3.6)\\
&\geq \int_{|y|>0}[-u_\varepsilon(x)u_\varepsilon(x+y) + u_\varepsilon(x)^2 + 2^{-1}I_{\{|y|<1\}}y\partial u_\varepsilon(x)^2]n(dy) \\
&= u_\varepsilon(x)(H_0 - mc^2)u_\varepsilon(x).
\end{aligned}
$$

Since $w = u_\varepsilon - \varepsilon$ is in C_0^∞, $(H_0 - mc^2)u_\varepsilon = (H_0 - mc^2)w$ is in L^2, so that the last member of (3.6) equals

$$
w(x)(H_0 - mc^2)w(x) + \varepsilon(H_0 - mc^2)w(x).
$$

Therefore, integrating the inequality between the first and last member of (3.6), we have

$$
(u, (H_A - mc^2)u) \geq (w, (H_0 - mc^2)w) \geq 0
$$

proving Proposition 2.1.

In connection with Proposition 2.1 we should like to insert here a comment on [8, Lemma 2.3, pp. 273-277]. The former extends part of the latter, since the latter assumes that $A(x)$ satisfies (2.4), a less general condition than (1.2). However, the proof of this lemma contains some erroneous arguments, although all of its statements are correct. In fact, to establish the estimate $\|i_1(\varepsilon)\|_{2,K} \leq C(K_1)\|u\|_{2,K_2}$ [8, (2.20), p. 275], we cannot make such a change of the integration variables $x + y = x'$. The argument in [8, p. 275, lines 1-7 from the top] should read as follows: We use the Schwarz inequality to get

$$
\begin{aligned}
\|i_1(\varepsilon)\|_{2,K} \leq \Bigg\{\int_{K_1} dx \Bigg(\int_{\varepsilon\leq|y|<1} &[2^{-1}|y|^2|A(x+y/2)|^2 + \\
&+ |y||A(x+y/2) - A(x+y)|]n^m(dy)\Bigg) \\
&\times \int_{\varepsilon\leq|y|<1}[2^{-1}|y|^2|A(x+y/2)|^2 \\
&+ |y||A(x+y/2) - A(x+y)|]|\varphi(x+y)u(x+y)|n^m(dy)\Bigg\}^{1/2}
\end{aligned}
$$

and hence obtain

$$\|i_1(\varepsilon)\|_{2,K} \le [2^{-1}a(K_2)^2 n_1^m + (b(K_1) + \widehat{b}(K_1))]\|\varphi u\|_2 \le C(K_1)\|u\|_{2,K_2},$$

with [8, (2.5, a, b)] as well as the fact that [8, (2.5b)] implies

$$\widehat{b}(K) \equiv \sup_{x\in K} \int_{0<|y|<1} |A(x+y) - A(x)||y|n^m(dy) < \infty$$

for every compact set K. The same care should be taken in showing

$$\|i_1(\varepsilon)\|_{\infty,K_1} \le C(K_1)\|u\|_\infty, \qquad \text{and} \qquad \|i_2(\varepsilon)\|_{\infty,K_1} \le C_K\left[\|u\|_\infty + \sum_{j=1}^d \|\partial_j u\|_\infty\right],$$

[8, p. 277, lines 3-5 from the top] and [8, (3.46), p. 287]. However, a simpler proof of this lemma can be given, using the same decomposition (3.3) of $H_A u$ as in the proof of Proposition 2.1.

To prove Theorems 2.2 and 2.3 we need some properties of the Lévy measure $n(dy) = n^{c,m}(dy) = n^{c,m}(y)dy$, (1.4), where $n^{c,m}(y)$ is the density function of $n^{c,m}(dy)$ with respect to the Lebesgue measure dy, as in the following lemma.

Lemma 3.1. *For $m > 0$,*

$$\begin{aligned} &\int_{|y|>0} y_j^2 n^{c,m}(dy) = 1/m, \qquad 1 \le j \le d, \\ &\int_{|y|>0} |y|^2 n^{c,m}(dy) = d/m. \end{aligned} \tag{3.7}$$

As $c \to \infty$,

$$N_2^{c,m} = \int_{|y|\ge 1} |y|^2 n^{c,m}(dy) \to 0, \tag{3.8}$$

$$n_2^{c,m} = \int_{0<|y|<1} |y|^3 n^{c,m}(dy) \to 0. \tag{3.9}$$

(ii) For $c > 0$, the function $n^{c,m}(y)$ is increasing as $m \downarrow 0$.
(iii) For $c > 0$,

$$\int_{|y|>0} [n^{c,m}(y) - n^{c,0}(y)]dy = -mc^2. \tag{3.10}$$

Proof. (i) We show the second half of (3.7), (3.8) and (3.9); the first half of (3.7) follows from its second half. We have

$$\begin{aligned}\int_{|y|>0} |y|^2 n^{c,m}(dy) &= C_d S_d m^{-1} \int_0^\infty \rho^{(d+1)/2} K_{(d+1)/2}(\rho)\, d\rho \\ &= 2(2\pi)^{-(d+1)/2} S_d m^{-1} 2^{(d-1)/2} \pi^{1/2} \Gamma\left(\frac{d}{2}+1\right) \\ &= \pi^{-d/2}\Gamma\left(\frac{d}{2}+1\right) S_d m^{-1} = d/m,\end{aligned}$$

where $C_d = 2(2\pi)^{-(d+1)/2}$ and $S_d = 2\pi^{d/2}\Gamma\left(\frac{d}{2}\right)^{-1}$ is the area of the $(d-1)$-dimensional unit sphere. In the second equality we have used an identity for $K_\nu(z)$ [3, Chap. 7, 7.7.3, (27), p. 51]. Similarly we have

$$\int_{|y|\geq 1} |y|^2 n^{c,m}(dy) = C_d S_d m^{-1} \int_{mc}^\infty \rho^{(d+1)/2} K_{(d+1)/2}(\rho)\, d\rho,$$

which converges to zero as $c \to \infty$, showing (3.8). We get (3.9), since

$$\int_{0<|y|<1} |y|^3 n^{c,m}(dy) = C_d S_d m^{-2} c^{-1} \int_0^{mc} \rho^{(d+3)/2} K_{(d+1)/2}(\rho)\, d\rho$$

converges to zero as $c \to \infty$, because the integral on the right is bounded by $2^{(d+1)/2}\Gamma\left(\frac{d+3}{2}\right)$, by use of the same identity for $K_\nu(z)$ as used above.

(ii) By (1.4) we have

$$n^{c,m}(y) = C_d c^{d+2} m^{d+1} (mc|y|)^{-(d+1)/2} K_{(d+1)/2}(mc|y|).$$

Therefore, for $|y| > 0$,

$$\begin{aligned}C_d^{-1} \frac{d}{dm} n^{c,m}(y) &= c^{d+2} \Big\{ (d+1) m^d (mc|y|)^{-(d+1)/2} K_{(d+1)/2}(mc|y|) \\ &\quad + m^{d+1} c|y| \frac{d}{d(mc|y|)} [(mc|y|)^{-(d+1)/2} K_{(d+1)/2}(mc|y|)] \Big\} \\ &= c^{d+2} m^d (mc|y|)^{-(d+1)/2} [(d+1) K_{(d+1)/2}(mc|y|) \\ &\quad - mc|y| K_{(d+3)/2}(mc|y|)] \\ &= -c^{d+2} m^d (mc|y|)^{-(d-1)/2} K_{(d-1)/2}(mc|y|) < 0.\end{aligned}$$

Here we have used, in the second equality, the identity

$$z^{-1}(d/dz)[z^{-\nu}K_\nu(z)] = -z^{-(\nu+1)}K_{\nu+1}(z)$$

[3, Chap. 7, 7.11, (22), p. 79] and, in the last equality, the identity

$$2\nu K_\nu(z) - zK_{\nu+1}(z) = -zK_{\nu-1}(z)$$

[3, Chap. 7, 7.11, (25), p. 79]. This proves that $n^{c,m}(y)$ is decreasing as m increases or the desired assertion.

(iii) The assertion is trivial for $m = 0$. For $m > 0$, we obtain from the proof of (ii)

$$\begin{aligned} n^{c,m}(y) - n^{c,0}(y) &= \int_0^m \frac{d}{ds} n^{c,s}(y)\,ds \\ &= -C_d c^{d+2} \int_0^m s^d (sc|y|)^{-(d-1)/2} K_{(d-1)/2}(sc|y|)\,ds. \end{aligned}$$

Then

$$\begin{aligned} &\int_{|y|>0} [n^{c,m}(y) - n^{c,0}(y)]\,dy \\ &\qquad = -C_d S_d c^2 \int_0^m ds \int_0^\infty \rho^{(d-1)/2} K_{(d-1)/2}(\rho)\,d\rho. \\ &\qquad = -C_d S_d c^2 m 2^{(d-3)/2} \pi^{1/2} \Gamma\left(\frac{d}{2}\right) = -mc^2, \end{aligned}$$

where in the last equality we have again used the same identity for $K_\nu(z)$ as above [3, Chap. 7, 7.7.3, (27), p. 51].

Proof of Theorem 2.2. Write $n(dy)$ as $n^c(dy)$. Let $u \in C_0^\infty(\mathbf{R}^d)$ and let K be the support of u. Since

$$(H_A^{NR}u)(x) = -2^{-1}[\partial_x^2 - i(\partial_x A)(x) - 2iA(x)\partial_x - A(x)^2]u(x),$$

we have with (3.7)

$$\begin{aligned} (3.11)\quad &([H_A^c - c^2 - H_A^{NR}]u)(x) \\ &= \left\{ -\int_{|y|>0} [u(x+y) - u(x) - I_{\{|y|<1\}} y\partial_x u(x)]\,n^c(dy) + 2^{-1}\partial_x^2 u(x) \right\} \\ &+ \left\{ -\int_{|y|\geq 1} [e^{-iyA(x+y/2)} - 1]\,u(x+y)\,n^c(dy) \right\} \\ &+ \left\{ -\int_{0<|y|<1} [e^{-iyA(x+y/2)} - 1 + \right. \\ &+ \left. iyA(x+y/2)]\,u(x+y)\,n^c(dy) - 2^{-1}A(x)^2 u(x) \right\} \end{aligned}$$

$$+\left\{\int_{0<|y|<1} iyA(x+y/2)(u(x+y)-u(x))n^c(dy) - iA(x)\partial_x u(x)\right\}$$

$$+\left\{\lim_{r\downarrow 0}\int_{r<|y|<1} iy(A(x+y/2)-A(x))u(x)n^c(dy) - 2^{-1}i(\partial_x A)(x)u(x)\right\}$$

$$\equiv \sum_{j=1}^{5}(\Delta_j u)(x).$$

We want to show that all $\Delta_j u$, $1 \le j \le 5$, on the right of (3.11) converge to 0 as $c \to \infty$. We use the notations at the beginning of this section.

For $\Delta_1 u$: This term refers to the difference between the free relativistic and nonrelativistic Schrödinger operators. We have

$$\|\Delta_1 u\| = \|(\sqrt{-c^2\Delta + c^4} - c^2)u + 2^{-1}\Delta u\|. \tag{3.12}$$

which is by Fourier transform equal to

$$\|[(\sqrt{c^2p^2+c^4}-c^2) - 2^{-1}p^2]\hat{u}\| = \left\|\left(\frac{p^2}{\sqrt{(p/c)^2+1}+1} - \frac{p^2}{2}\right)\hat{u}\right\|,$$

tending to zero as $c \to \infty$, where $\hat{u}(p)$ is the Fourier transform of

$$u(x) : \hat{u}(p) = (2\pi)^{-d/2}\int e^{ipy}u(x)\,dx$$

For $\Delta_2 u$: By the Schwarz inequality we can show

$$\|\Delta_2 u\| \le 2N_0^c\|u\|, \tag{3.13}$$

which tends to zero as $c \to \infty$, because $N_0^c \le N_2^c \to 0$, by Lemma 3.1, (3.8).

For $\Delta_3 u$: Decompose it into three terms

$$\begin{aligned}
&(\Delta_3 u)(x)\\
&= -\int_{0<|y|<1}[(e^{-iyA(x+y/2)} - 1 + iyA(x+y/2))\\
&+ 2^{-1}(yA(x))^2]u(x+y)n^c(dy)\\
&+ 2^{-1}\int_{0<|y|<1}(yA(x))^2(u(x+y)-u(x))n^c(dy) \qquad (3.14a)\\
&- 2^{-1}\int_{|y|\ge 1}(yA(x))^2u(x)n^c(dy)\\
&\equiv \sum_{k=1}^{3}(\Delta_{3k}u)(x).
\end{aligned}$$

Here we have used not only (3.7) but also $\int_{|y|>0} y_j y_k n^c(dy) = 0$ for $j \neq k$, $1 \leq j, k \leq d$, so that $\int_{|y|>0} (yA(x))^2 n^c(dy) = A(x)^2$. We estimate these three $\Delta_{3k}u$. As for $\Delta_{31}u$, we first make the change of variables $y = y'/c$ (write y again instead of y'), noting $\int_{0<|y|<1} f(y) n^c(dy) = c^2 \int_{0<|y|<c} f(y) n^1(dy)$, and then apply the mean value theorem to get

$$
\begin{aligned}
&(\Delta_{31}u)(x) \\
&= \int_{0<|y|<c} [(yA(x+y/2c))^2 \int_0^1 (1-\theta) e^{-i(\theta/c)yA(x+y/2c)} d\theta - 2^{-1}(yA(x))^2] \\
&\qquad\qquad \times u(x+y/c) n^1(dy).
\end{aligned}
$$

Hence we obtain by the Schwarz inequality with (3.7)

$$
\begin{aligned}
||\Delta_{31}u|| \leq d^{1/2} ||u||_\infty \Big\{ &\int_{0<|y|<c} |y|^2 n^1(dy) \\
\times &\int_{K_1} dx \Big| (\tilde{y}A(x+y/2c))^2 \cdot \\
&\cdot \int_0^1 (1-\theta) e^{-i(\theta/c)yA(x+y/2c)} d\theta - 2^{-1}(\tilde{y}A(x))^2 \Big|^2 \Big\}^{1/2}
\end{aligned}
\tag{3.14b}
$$

with $\tilde{y} = y/|y|$. The dx-integral over K_1 on the right of (3.14b), which is a function of y, is bounded for all y and c with $|y| < c$ and $c \geq 1$, and convergent to 0 as $c \to \infty$, because, as $y \to 0$, $A(x + y/2)$ is convergent to $A(x)$ in L^4_{loc} as well as a.e. Since $|y|^2 n^1(dy)$ is by (3.7) a finite measure on $\mathbf{R}^d \backslash \{0\}$, it follows by the Lebesgue bounded convergence theorem that the $|y|^2 n^1(dy)$-integral on the right of (3.14b) tends to zero as $c \to \infty$. For the other $\Delta_{32}u$ and $\Delta_{33}u$ we can also show

$$
||\Delta_{32}u|| \leq 2^{-1} n_2^c ||A||^2_{4,K_1} ||\partial u||_\infty, \tag{3.14c}
$$

and

$$
||\Delta_{33}u|| \leq 2^{-1} N_2^c ||A||^2_{4,K} ||\partial u||_\infty, \tag{3.14d}
$$

both of which tend to zero as $c \to \infty$, because $N_2^c \to 0$ and $n_2^c \to 0$, by Lemma 3.1. Thus with (3.14abcd) we have shown $\Delta_3 u \to 0$ in L^2 as $c \to \infty$.

For $\Delta_4 u$: Decompose it into three terms:

$$
\begin{aligned}
&(\Delta_4 u)(x) \\
&= i \int_{0<|y|<1} yA(x+y/2)(u(x+y) - u(x) - (y\partial_x)u(x))\, n^c(dy) \\
&+ i \int_{0<|y|<1} y(A(x+y/2) - A(x))(y\partial_x)u(x) n^c(dy) \\
&- i \int_{|y|\geq 1} (yA(x))(y\partial_x)u(x) n^c(dy) \\
&\equiv \sum_{k=1}^{3} (\Delta_{4k} u)(x),
\end{aligned}
$$

where we have used (3.7) and $\int_{|y|>0} y_j y_k n^c(dy) = 0$ for $j \neq k, 1 \leq j,k \leq d$, or $\int_{|y|>0} (yA(x))(y\partial_x)u(x) n^c(dy) = A(x)\partial_x u(x)$. By the Schwarz inequality we have

$$
\text{(3.15b)} \qquad ||\Delta_{41} u|| \leq 2^{-1} d^2 n_2^c ||A||_{2,K_1} \sup_{1\leq j,k\leq d} ||\partial_j \partial_k u||_\infty,
$$

$$
\text{(3.15c)} \qquad
\begin{aligned}
||\Delta_{42} u|| \leq d^{1/2} \Big(&\int_{0<|y|<c} |y|^2 n^1(dy) \cdot \\
&\cdot \int_K dx |A(x+y/2c) - A(x)|^2 \Big)^{1/2} ||\partial u||_\infty,
\end{aligned}
$$

$$
\text{(3.15d)} \qquad ||\Delta_{43} u|| \leq N_2^c ||A||_{2,K} ||\partial u||_\infty,
$$

where to get (3.15c) we have used (3.7) and made the change of variables $y = y'/c$. It is clear that as $c \to \infty$, $\Delta_{41} u$ and $\Delta_{43} u$ tend to zero, because n_2^c and $N_2^c \to 0$. To see $\Delta_{42} u \to 0$, we apply analogous arguments used for $\Delta_{31} u$ in (3.14b). The dx-integral over K on the right-hand side of (3.15c), which is a function of y, is bounded for all y and c with $|y| < c$ and $c \geq 1$, and convergent to 0 as $c \to \infty$, because, as $y \to 0$, $A(x + y/2)$ is convergent to $A(x)$ in $L^4_{\rm loc}$ and hence in $L^2_{\rm loc}$ as well as a.e. Since $|y|^2 n^1(dy)$ is by (3.7) a finite measure on $\mathbf{R}^d \setminus \{0\}$, its integral on the right-hand side of (3.15c) converges to 0, as $c \to \infty$, by the Lebesgue bounded convergence theorem, yielding $\Delta_{42} u \to 0$. Thus we have shown $\Delta_4 u \to 0$.

For $\Delta_5 u$: Let $\chi(x)$ be a nonnegative C^∞ function with compact support such that $\chi(x) = 1$ on K_1 and supp $\chi \subset K_2$. Note that if $\partial A(x)$ is in $L^2_{\rm loc}, \partial(\chi(x)A(x))$ is in L^2. Then we have

$$(3.16) \qquad \|\Delta_5 u\| \le \delta^c(K)\|u\|_\infty,$$

with

$$\delta^c(K) \equiv \lim_{r\downarrow 0}\left(\int_K \left|\int_{r<|y|<1} iy(A(x+y/2) - A(x))n^c(dy) - 2^{-1} i\partial A(x)\right|^2 dx\right)^{1/2},$$

where note that $A(x) \in L^4_{\rm loc}$ implies $A(x) \in L^2_{\rm loc}$, so that the limit (2.3) exists in the sense of convergence of $L^2_{\rm loc}$. With $(\chi A)(x) = \chi(x)A(x)$ we obtain

$$\begin{aligned}
&\delta^c(K)^2 \\
&= \lim_{r\downarrow 0}\int_K \left|\int_{r<|y|<1} iy((\chi A)(x+y/2) - (\chi A)(x))n^c(dy) - 2^{-1} i\partial(\chi A)(x)\right|^2 dx \\
&\le \lim_{r\downarrow 0}\int_{\mathbf{R}^d} \left|\int_{r<|y|<1} iy((\chi A)(x+y/2) - (\chi A)(x))n^c(dy) - 2^{-1} i\partial(\chi A)(x)\right|^2 dx,
\end{aligned}$$

which is, by the Parseval formula, equal to

$$\begin{aligned}
&\lim_{r\downarrow 0}\int_{\mathbf{R}^d} \left|\int_{r<|y|<1} i[e^{ipy/2} - 1]((\widehat{\chi A})(p)n^c(dy) + 2^{-1}p(\widehat{\chi A})(p)\right|^2 dp \\
&= \int_{\mathbf{R}^d} \left|\int_{0<|y|<1} i[e^{ipy/2} - 1]y((\widehat{\chi A})(p)n^c(dy) + 2^{-1}p(\widehat{\chi A})(p)\right|^2 dp,
\end{aligned}$$

because the integral $\int_{0<|y|<1} i[e^{ipy} - 1]yn^c(dy)$ exists for each fixed p. Since we have from (2.6)

$$\begin{aligned}
&\int_{0<|y|<1} i[e^{ipy/2} - 1]yn^c(dy) = \\
&= -\frac{c^2 p/2}{\sqrt{(cp/2)^2 + c^4}} - \int_{|y|\ge 1} ie^{ipy/2} yn^c(dy),
\end{aligned}$$

we get by the triangle inequality and the Parseval formula

$$\delta^c(K) \le 2^{-1}\left(\int\left|\left(\frac{c^2}{\sqrt{(cp/2)^2+c^4}}-1\right)p(\widehat{\chi A})(p)\right|^2 dp\right)^{1/2}$$

$$+\left(\int\left|\int_{|y|\ge 1} ie^{ipy/2}y(\widehat{\chi A})(p)\,n^c(dy)\right|^2 dp\right)^{1/2}$$

$$\le 2^{-1}\left(\int\left|\left(\frac{1}{\sqrt{(p/2c)^2+1}}-1\right)\partial(\widehat{\chi A})(p)\right|^2 dp\right)^{1/2} + N_1^c\|(\widehat{\chi A})\|,$$

$$= 2^{-1}\|[(1-(2c)^{-2}\Delta)^{-1/2}-1]\partial(\chi A)\| + N_1^c\|\chi A\|,$$

which tends to zero as $c \to \infty$, by the Lebesgue dominated convergence theorem or the strong convergence $[(1-(2c)^{-2}\Delta)^{-1/2}-1] \underset{s}{\to} 0$, and because $N_1^c \le N_2^c \to 0$, by (3.8). It follows with (3.16) that $\Delta_5 u \to 0$. This completes the proof of Theorem 2.2.

Proof of Theorem 2.3. First we show the first assertion. Let $u \in C_0^\infty(\mathbf{R}^d)$. By Lemma 3.1 (ii), $n^{(0,m)}(dy) \equiv n^0(dy) - n^m(dy)$ is a positive measure on $\mathbf{R}^d\backslash\{0\}$ if $m > 0$. We have

$$\|H_A^0 u - [H_A^m - m]u\|^2$$

$$= \lim_{r\downarrow 0}\int dx\left|\int_{|y|>r}[e^{-iyA(x+y/2)}u(x+y) - u(x)\right.$$

$$\left.- I_{\{|y|<1\}}y(\partial_x - iA(x))u(x)]n^{(0,m)}(dy)\right|^2$$

$$= \liminf_{r\downarrow 0}\int\left|\int_{|y|>r}[e^{-iyA(x+y/2)}u(x+y) - u(x)]n^{(0,m)}(dy)\right|^2 dx$$

$$\le \liminf_{r\downarrow 0}\int\left|\int_{|y|>r}(|u(x+y)| + |u(x)|)n^{(0,m)}(dy)\right|^2 dx$$

$$\le \liminf_{r\downarrow 0}\int dx\left|\int_{|y|>r}n^{(0,m)}(dy)\int_{|y|>r}(|u(x+y)| + |u(x)|)^2 n^{(0,m)}(dy)\right|.$$

It follows with (3.10) that

$$\|H_A^0 u - [H_A^m - m]u\| \le \liminf_{r\downarrow 0} 2\int_{|y|>r} n^{(0,m)}(dy)\|u\| \le 2m\|u\|.$$

Next we show the second assertion. Let $0 \le m \le m'$. The proof will proceed analogously with the arguments used to prove $H_A \ge mc^2$ in the proof of Proposition 2.1.

Let $u \in C_0^\infty(\mathbf{R}^d)$ and $u_\varepsilon(x) = \sqrt{|u(x)|^2 + \varepsilon^2}, \varepsilon > 0$. Then u_ε is C^∞ and bounded. By taking a subsequence $r \downarrow 0$ if necessary, we have, for a.e. x,

(3.17)

$$\begin{aligned}&\mathrm{Re}[\overline{u(x)}(H_A^m - m)u(x) - \overline{u(x)}(H_A^{m'} - m')u(x)]\\ &= \lim_{r\downarrow 0} 2^{-1}\int_{|y|>r} (-\overline{u(x)}[e^{-iyA(x+y/2)}u(x+y) - u(x) - I_{\{|y|<1\}}y(\partial_x - iA(x))u(x)]\\ &+ u(x)[e^{iyA(x+y/2)}\overline{u(x+y)} - \overline{u(x)} - I_{\{|y|<1\}}y(\partial_x + iA(x))\overline{u(x)}])n^{(m,m')}(dy),\end{aligned}$$

where $n^{(m,m')}(dy) \equiv n^m(dy) - n^{m'}(dy)$ is a positive measure on $\mathbf{R}^d\backslash\{0\}$, by Lemma 3.1. (ii). It follows that the right-hand side of (3.17) is larger than or equal to

$$\begin{aligned}&\lim_{r\downarrow 0}\int_{|y|>r} [-|u(x)||u(x+y)| + |u(x)|^2 + 2^{-1}I_{\{|y|<1\}}y\partial|u(x)|^2]n^{(m,m')}(dy)\\ &\ge \int_{|y|>0} [-u_\varepsilon(x)u_\varepsilon(x+y) + u_\varepsilon(x)^2 + 2^{-1}I_{\{|y|<1\}}y\partial u_\varepsilon(x)^2]n^{(m,m')}(dy)\\ &= u_\varepsilon(x)(H_0 - m)u_\varepsilon(x) - u_\varepsilon(x)(H_0 - m')u_\varepsilon(x).\end{aligned}$$

Thus

$$\begin{aligned}&\mathrm{Re}[\overline{u(x)}(H_A^m - m)u(x) - \overline{u(x)}(H_A^{m'} - m')u(x)]\\ &\ge u_\varepsilon(x)[(H_0 - m) - (H_0 - m')]u_\varepsilon(x).\end{aligned}$$

Integrating both sides we get

$$(u,(H_A^m - m)u) - (u,(H_A^{m'} - m')u) \ge (u_\varepsilon,[(H_0^m - m) - (H_0^{m'} - m')]u_\varepsilon) \ge 0,$$

ending the proof of Theorem 2.3.

4. NOTES

If the magnetic vector potential $A : \mathbf{R}^d \to \mathbf{R}^d$ is in $L^{1+\delta}_{\mathrm{loc}}$ for some $\delta > 0$, we can define the Weyl quantized relativistic Hamiltonian with magnetic fields or relativistic magnetic Schrödinger operator, denoted by $H_A^{c,m}$ again, as the selfadjoint operator associated with the closed quadratic form

$$\begin{aligned}&h_A^{c,m}[u,u] = mc^2\|u\|^2\\ (4.1)\qquad &+ \frac{1}{2}\iint_{|x-y|>0} |e^{-i(x-y)A(2^{-1}(x+y))}u(x) - u(y)|^2 \cdot n^{c,m}(x-y)\,dx\,dy,\\ &u \in Q[h_A^{c,m}],\end{aligned}$$

with domain $Q[h_A^{c,m}]$, which is the subspace of $L^2(\mathbb{R}^d)$ of the functions u such that the integral on the right-hand side of (4.1) is finite. Here $n^{c,m}(s)$ is the density function of the Lévy measure $n^{c,m}(dz)$, (1.4): $n^{c,m}(dz) = n^{c,m}(z)dz$. In view of the elementary inequality $|e^{-it} - 1| \leq 2|t|^{(1+\delta)/2}, 0 < \delta \leq 1$, it can be shown that $C_0^\infty(\mathbb{R}^d)$ is not only a subspace of $Q[h_A^{c,m}]$ but also a form core of $H_A^{c,m}$.

If $A(x)$ is in L^2_{loc}, the nonrelativistic magnetic Schrödinger operator H^{NR} can also defined through the quadratic form

$$h_A^{NR}[u,u] = (2m)^{-1}||(-i\partial - A)u||^2 \tag{4.2}$$

(See Kato [13], Simon [21] and also [15], [1, p. 8]).

For the nonrelativistic limit ($c \to \infty$) and zero-mass limit ($m \downarrow 0$) for $h_A^{c,m}$, it will be shown that if $A(x)$ is in L^2_{loc}, then

$$h_A^{c,1}[u,u] - c^2||u||^2 \to h_A^{NR}[u,u], \quad \text{as} \quad c \to \infty (m = 1),$$

for $u \in C_0^\infty(\mathbb{R}^d)$, and if $A(x)$ is $L^{1+\delta}_{\text{loc}}$ for some $\delta > 0$, then

$$h_A^{1,m}[u,u] - m||u||^2 \uparrow h_A^{1,0}[u,u], \quad \text{as} \quad m \downarrow 0 (c = 1),$$

for $u \in C_0^\infty(\mathbb{R}^d)$, with

$$0 \leq h_A^{1,0}[u,u] - [h_A^{1,m}[u,u] - m||u||^2] \leq 2m||u||^2, \quad \text{for } u \in C_0^\infty(\mathbb{R}^d).$$

Here together with Theorems 2.2 and 2.3, we see that the convergente in the zero-mass limit is monotone as quadratic forms, while this does not seem to be valid for the convergence in the nonrelativistic limit.

Note added in proof. In another forthcoming paper: T. Ichinose and T. Tsuchida, *On essential selfadjointness of the Weyl quantized relativistic Hamiltonian*, it has been proved that H_A in (1.3) is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$ under the assumption (1.2), so that all the assertions in Remark 1 to Theorems 2.2 and 2.3 are now true.

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