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A NON-REALIZABLE LOPSIDED SUBSET OF THE 7-CUBE

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Abstract. *A lopsided set is a subset of the vertices of the n-cube that has a combinatorial property shared by the collection of orthants intersected by a convex subset of* $\mathbb{R}^n$. *The connection between lopsided sets and oriented matroids is exploited to obtain a lopsided subset of the 7-cube that cannot be realized as the collection of orthants intersected by a convex subset of* $\mathbb{R}^7$.

1. INTRODUCTION

Let E be a finite set and let $V(E)$ denote the set of all real valued functions f on E. Let $C(E)$ be the subset of $V(E)$ consisting of all functions f from E to $\{-1, 1\}$. A set $L \subseteq C(E)$ is *lopsided* if for every pair of subsets A and B of E, with $A \cap B = \emptyset, A \cup B = E$, and every function $f \in C(E)$, either:

(a) All or none of the functions in $C(E)$ that agree with f on A are in L; or

(b) There is a function g that agrees with f on A so that the function $g' \in C(E)$, given by $g'(e) = g(e)$ for $e \in A, g'(e) = -g(e)$ for $e \in B$, is not in L.

The set of f in $C(E)$ will also be called the set of vertices of the cube $C(E)$. The set of f of $C(E)$ that agree on a subset A of E with a function g of $C(E)$ will be called a face of $C(E)$. A subset L of the vertices of $C(E)$ is lopsided if for every face of the cube $C(E)$, either L contains all or none of the vertices of the face or L contains an asymmetric subset of the vertices of the face.

Lopsided sets were introduced by Lawrence [L]. Many lopsided sets arise as collections of orthants of $V(E)$ intersected by convex sets, in the following manner. Let $K \subseteq V(E)$ be convex. Define a set $L \subseteq C(E)$ by $f \in L$ iff there exists $g \in K$ with $g(e)f(e) \geq 0$ for all $e \in E$. Thus, for example, $L = C(E)$ when $0 \in K$. It was shown in [L] that every set L arising in this way is lopsided. In this case, L is said to be *realized* by K. An example of a lopsided subset of the 8-cube that is not realizable by any convex subset of $\mathbb{R}^8$ is given by Lawrence in [L], and he asks if there are any non-realizable lopsided subsets of the 7-cube. It has been conjectured that lopsided sets correspond precisely by a duality map to the shellable subcollections of faces of the n-dimensional cross-polytope.

In this paper, a non-realizable lopsided subset of the 7-cube will be constructed similarly to the construction of the non-realizable subset of the 8-cube given in [L].

2. LOPSIDED SETS AND ORIENTED MATROIDS

Many lopsided sets, including those realizable by convex subsets of $\mathbb{R}^n$, arise as subcollections of the sets of topes of simple oriented matroids. This was proved in [L], and a brief description of the connection will be given here. The basic definitions of oriented matroids will be given next. See $[BLV]$, $[FL]$, or $[M]$ for more on oriented matroids.

Let E be a finite set. A *signed set* X on E is an ordered pair (X^-, X^+) of disjoint subsets of E. The set underlying a signed set $X = (X^-, X^+)$ is the set $\underline{X} = X^- \cup X^+$. The opposite of $X = (X^-, X^+)$ is the signed set $-X = (X^+, X^-)$.

Definition. *An* oriented matroid *is a pair* $(E, \mathscr{C})$*, where* E *is a finite set and* $\mathscr{C}$ *is a collection of signed sets on* E *satisfying:*
(C1) $\emptyset \notin \mathscr{C}$*, and* $C \in \mathscr{C}$ *implies* $-C \in \mathscr{C}$.
(C2) $C_1, C_2 \in \mathscr{C}, \underline{C}_1 \subseteq \underline{C}_2$ *implies* $C_1 = C_2$ *or* $C_1 = -C_2$.
(C3) $C_1, C_2 \in \mathscr{C}, C_1 \neq -C_2$*, and* $e \in (C_1^+ \cap C_2^-) \cup (C_1^- \cap C_2^+)$ *imply there exists* $C_3 \in \mathscr{C}$ *with* $C_3^+ \subseteq (C_1^+ \cup C_2^+) \backslash \{e\}, C_3^- \subseteq (C_1^- \cap C_2^-) \backslash \{e\}$.

An oriented matroid $(E, \mathscr{C})$ is *simple* if there is an integer $m \geq 0$ so that $|\underline{C}| = m + 1$ for all $C \in \mathscr{C}$, and $\bigcup_{C \in \mathscr{C}} \underline{C} = E$. The *rank* of a simple oriented matroid is m. If $(E, \mathscr{C})$ is an oriented matroid, then $\mathscr{C}$ is the set of *circuits* of $(E, \mathscr{C})$. A *tope* (This term is due to Mandel [M]) of a simple oriented matroid $(E, \mathscr{C})$ is a signed set (T^-, T^+) with $\underline{T} = E$, such that there exists circuits $C_1, C_2, \ldots, C_r$ of $\mathscr{C}$ with $T^- = C_1^- \cup C_2^- \cup \ldots \cup C_r^-$ and $T^+ = C_1^+ \cup C_2^+ \cup \ldots \cup C_r^+$. The set of topes of $(E, \mathscr{C})$ will be denoted $\mathscr{T}(E, \mathscr{C})$.

For any simple oriented matroid $(E, \mathscr{C})$, define the set $L(E, \mathscr{C}) \subseteq C(E)$ by: for all $f \in C(E), f \in L(E, \mathscr{C})$ iff the signed set $(f^{-1}(-1), f^{-1}(1)) \in \mathscr{T}(E, \mathscr{C})$. The set $L(E, \mathscr{C})$ is a symmetric subset of $C(E)$, so it will only be lopsided if $L(E, \mathscr{C}) = C(E)$. On the other hand, $[L]$ showed that the restriction of $L(E, \mathscr{C})$ to any proper face of $C(E)$ is lopsided.

Theorem 1. ($[L]$) *Let* $(E, \mathscr{C})$ *be a simple oriented matroid. Let* $f \in C(E)$ *and let* A *and* B *be disjoint subsets of* E *so that* $A \cup B = E$*, and* $A \neq \emptyset$*. Let* L_B *be the set of* $g \in C(B)$ *such that there exists a function* $h \in L(E, \mathscr{C})$ *that agrees with* f *on* A *and with* g *on* B*. Then* L_B *is a lopsided subset of* $C(B)$.

It is not known if every lopsided det arises in this way. Thus the connection between oriented matroids and lopsided sets is only partially understood.

Next we will outline the connection between oriented matroids and sphere systems, which will be used to illustrate properties of our oriented matroid that gives rise to a non-realizable lopsided set. The connection between oriented matroids and sphere systems was first proved in $[FL]$, and then elaborated upon and improved by Edmonds and Mandel $[M]$. The next definitions come from $[M]$.

Definition. *Let* $S^n = \{x \in \mathbf{R}^{n+1} : x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1\}$ *and identify* S^{n-1} *with* $\{x \in S^n : x_{n+1} = 0\}$*. An* n-sphere *is a topological space homeomorphic to* S^n*. A subset* K

of an n-sphere S is a hypersphere *of S if it is the image of S^{n-1} by a homeomorphism of S^n to S; the components of S-K are called the sides of K.*

Definition. *A* sphere system *is a triple $(S, E, \mathscr{H})$, where S is a topological sphere, E is a finite set, and $\mathscr{H} = \{H_e^j : e \in E, j \in \{0, +, -\}\}$ is a family of subsets of S satisfying:*

(a) For each $e \in E$, either $(H_e^0, H_e^+, H_e^-) = (S, \emptyset, \emptyset)$ or S_e^0 is a hypersphere of S with sides S_e^+, S_e^-,

(b) For every nonempty subset A of E, the subspace $\bigcap\{S_e^0 : e \in A\}$ is a sphere. This is called a flat *of the system.*

(c) For every flat F and hypersphere S_e^0 not containing it, $S_e^0 \cap F$ is a hypersphere of F with sides $S_e^+ \cap F$ and $S_e^- \cap F$.

For a sphere system $(S, E, \mathscr{H})$, define the *signed support* of a point $x \in S$ to be the signed set (X_x^-, X_x^+) on E by $e \in X_x^-$ iff $x \in H_e^-, e \in X_x^+$ iff $x \in H_e^+$. The set of (X_x^-, X_x^+) for points $x \in S$ that are maximal intersections of collections of H_e^0 is the set of circuits of an oriented matroid. The representation theorem says that every oriented matroid can be represented this way.

Let $\{x_e : e \in E\}$ be a subset of $\mathbb{R}^m$ indexed by a finite set E, and let $S^{m-1} = \{y \in \mathbb{R}^m : y^T y = 1\}$. For $e \in E$, define $H_e^- = \{y \in S^{m-1} : x_e^T y < 0\}, H_e^0 = \{y \in S^{m-1} : x_e^T y = 0\}, H_e^+ = \{y \in S^{m-1} : x_e^T y > 0\}$. The triple $(S^{m-1}, E, \mathscr{H})$, where $\mathscr{H} = \{(H_e^-, H_e^0, H_e^+) : e \in E\}$ is a sphere system, and the set $\{x_e : e \in E\}$ is said to realize the oriented matroid obtained from $(S^{m-1}, E, \mathscr{H})$ as above.

Finally, let $(E, \mathscr{C})$ be an oriented matroid, let $e \in E$, and $E' = E \backslash \{e\}$. Let $\mathscr{C}' = \{C \in \mathscr{C} : e \notin \underline{C}\}$. Then $(E', \mathscr{C}')$ is called the oriented matroid obtained from $(E, \mathscr{C})$ by contracting e. Given a sphere system representing $(E, \mathscr{C})$, the sphere system $(H_e^0, E', \{\{H_f^j \cap H_e^0, j \in \{0, +, -\}\}\})$ represents $(E', \mathscr{C}')$.

3. A NON-REALIZABLE ORIENTED MATROID

The oriented matroid which will give us a non-realizable lopsided subset of the 7-cube will be constructed in this section. Let the vectors x_e in $\mathbb{R}^4$ be given by the columns of the matrix

$$\begin{pmatrix} 4 & -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & 4 & -1 & -1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 4 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 4 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $E = \{s_1, s_2, s_3, s_4, t_1, t_2, t_3, t_4\}$ and the elements of E ordered this way index the vectors x_e as they appear in the matrix. The circuits of the oriented matroid $(E, \mathscr{C})$ realized by $\{x_e : e \in E\}$ are given below, in abbreviated form. The circuit $(\{s_1, t_1\}, \{s_2, t_2\})$, for example, is written as $\overline{s}_1 s_2 \overline{t}_1 t_2$. A complete list of the circuits of $(E, \mathscr{C})$ would include the negatives of those given below.

$\overline{s}_1 s_2 \overline{t}_1 t_2$	$s_1 \overline{s}_2 \overline{s}_3 \overline{s}_4 t_1$	$s_1 \overline{s}_2 \overline{s}_3 t_1 t_4$	$\overline{s}_1 \overline{s}_3 s_4 t_2 t_4$	$s_1 \overline{s}_4 t_1 t_2 t_3$
$\overline{s}_1 s_3 \overline{t}_1 t_3$	$\overline{s}_1 s_2 \overline{s}_3 \overline{s}_4 t_2$	$\overline{s}_1 s_2 \overline{s}_3 t_2 t_4$	$s_2 \overline{s}_3 \overline{s}_4 t_1 t_2$	$\overline{s}_1 s_4 t_2 t_3 t_4$
$\overline{s}_1 s_4 \overline{t}_1 t_4$	$\overline{s}_1 \overline{s}_2 s_3 \overline{s}_4 t_3$	$\overline{s}_1 \overline{s}_2 s_3 t_3 t_4$	$\overline{s}_2 s_3 \overline{s}_4 t_1 t_3$	$s_2 \overline{s}_3 t_1 t_2 t_4$
$\overline{s}_2 s_3 \overline{t}_2 t_3$	$\overline{s}_1 \overline{s}_2 \overline{s}_3 s_4 t_4$	$s_1 \overline{s}_2 \overline{s}_4 t_1 t_3$	$\overline{s}_2 \overline{s}_3 s_4 t_1 t_4$	$\overline{s}_2 s_3 t_1 t_3 t_4$
$\overline{s}_2 s_4 \overline{t}_2 t_4$	$s_1 t_1 t_2 t_3 t_4$	$\overline{s}_1 s_2 \overline{s}_4 t_2 t_3$	$s_1 \overline{s}_2 t_1 t_3 t_4$	$s_2 \overline{s}_4 t_1 t_2 t_3$
$\overline{s}_3 s_4 \overline{t}_3 t_4$	$s_2 t_1 t_2 t_3 t_4$	$\overline{s}_1 \overline{s}_2 s_4 t_3 t_4$	$\overline{s}_1 s_2 t_2 t_3 t_4$	$\overline{s}_2 s_4 t_1 t_3 t_4$
	$s_3 t_1 t_2 t_3 t_4$	$s_1 \overline{s}_3 \overline{s}_4 t_1 t_2$	$s_1 \overline{s}_3 t_1 t_2 t_4$	$s_3 \overline{s}_4 t_1 t_2 t_3$
	$s_4 t_1 t_2 t_3 t_4$	$\overline{s}_1 s_3 \overline{s}_4 t_2 t_3$	$\overline{s}_1 s_3 t_2 t_3 t_4$	$\overline{s}_3 s_4 t_1 t_2 t_4$

Mandel $[M]$ introduced a technique, called surgery, to transform one oriented matroid into another. This technique is used here to transform $(E, \mathscr{C})$ into a simple, nonrealizable oriented matroid $(E, \mathscr{C}')$. Form the set $\mathscr{C}'$ from $\mathscr{C}$ by replacing

$\overline{s}_1 s_2 \overline{t}_1 t_2$ by $\{\overline{s}_1 s_2 s_3 \overline{t}_1 t_2,\ \overline{s}_1 s_2 \overline{s}_4 \overline{t}_1 t_2,\ \overline{s}_1 s_2 \overline{t}_1 t_2 \overline{t}_3,\ \overline{s}_1 s_2 \overline{t}_1 t_2 t_4\}$,

$\overline{s}_1 s_3 \overline{t}_1 t_3$ by $\{\overline{s}_1 \overline{s}_2 s_3 \overline{t}_1 t_3,\ \overline{s}_1 s_3 s_4 \overline{t}_1 t_3,\ \overline{s}_1 s_3 \overline{t}_1 t_2 t_3,\ \overline{s}_1 s_3 \overline{t}_1 t_3 \overline{t}_4\}$,

$\overline{s}_1 s_4 \overline{t}_1 t_4$ by $\{\overline{s}_1 s_2 s_4 \overline{t}_1 t_4,\ \overline{s}_1 \overline{s}_3 s_4 \overline{t}_1 t_4,\ \overline{s}_1 s_4 \overline{t}_1 \overline{t}_2 t_4,\ \overline{s}_1 s_4 \overline{t}_1 t_3 t_4\}$,

$\overline{s}_2 s_3 \overline{t}_2 t_3$ by $\{s_1 \overline{s}_2 s_3 \overline{t}_2 t_3,\ \overline{s}_2 s_3 \overline{s}_4 \overline{t}_2 t_3,\ \overline{s}_2 s_3 \overline{t}_1 \overline{t}_2 t_3,\ \overline{s}_2 s_3 \overline{t}_2 t_3 t_4\}$,

$\overline{s}_2 s_4 \overline{t}_2 t_4$ by $\{\overline{s}_1 \overline{s}_2 s_4 \overline{t}_2 t_4,\ \overline{s}_2 s_3 s_4 \overline{t}_2 t_4,\ \overline{s}_2 s_4 t_1 \overline{t}_2 t_4,\ \overline{s}_2 s_4 \overline{t}_2 \overline{t}_3 t_4\}$, and

$\overline{s}_3 s_4 \overline{t}_3 t_4$ by $\{s_1 \overline{s}_3 s_4 \overline{t}_3 t_4,\ \overline{s}_2 \overline{s}_3 s_4 \overline{t}_3 t_4,\ \overline{s}_3 s_4 \overline{t}_1 \overline{t}_3 t_4,\ \overline{s}_3 s_4 t_2 \overline{t}_3 t_4\}$.

The negatives of the replaced circuits of $\mathscr{C}$ also have to be replaced by the negatives of the signed sets above to get all of $\mathscr{C}'$. By Mandel's surgery, $(E, \mathscr{C}')$ is an oriented matroid. Clearly, $(E, \mathscr{C}')$ is simple. $(E, \mathscr{C}')$ is also non-realizable. In fact, $(E, \mathscr{C}')$ is isomorphic to the oriented matroid $FMR(8)$ from $[RS]$.

4. A NON-REALIZABLE SUBSET OF THE 7-CUBE

Let $(E, \mathscr{C}')$ be as defined in section 3. Define $E' = E/\{s_4\}$. For every tope $T \in \mathscr{T}(E, \mathscr{C}')$ such that $s_4 \in T^+$, define a function f_T of $C(E)$ by $(T^-, T^+/\{s_4\}) = (f_T^{-1}(-1), f_T^{-1}(1))$.

The set $L = \{f_T : T \in \mathscr{T}(E, \mathscr{C}), s_4 \in T^+\}$ is a lopsided subset of $C(E')$, by theorem 1. The goal of this section is to show that it is non-realizable. The strategy is to pick a set of four vertices of L and show that any lopsided subset of L containing these four vertices must contain all of L. We start by listing all of the topes T of $\mathscr{T}(E, \mathscr{C}')$ with $s_4 \in T^+$. Along with each tope, we describe the type of region of S^3 it would correspond to in a representation of $(E, \mathscr{C}')$ as a sphere system.

1. $s_1 s_2 s_3 t_1 t_2 t_3 t_4$ simplex, sides s_1, s_2, s_3, s_4.
2. $\bar{s}_1 s_2 \bar{s}_3 t_1 t_2 \bar{t}_3 \bar{t}_4$ simplex, sides s_1, s_4, t_1, t_4.
3. $\bar{s}_1 \bar{s}_2 s_3 \bar{t}_1 t_2 t_3 \bar{t}_4$ simplex, sides s_2, s_4, t_2, t_4.
4. $s_1 \bar{s}_2 \bar{s}_3 t_1 \bar{t}_2 t_3 \bar{t}_4$ simplex, sides s_3, s_4, t_3, t_4.
5. $\bar{s}_1 s_2 s_3 \bar{t}_1 t_2 t_3 \bar{t}_4$ prism, sides s_2, s_4, t_2, t_3, t_4.
6. $\bar{s}_1 s_2 s_3 \bar{t}_1 t_2 \bar{t}_3 \bar{t}_4$ prism, sides s_3, s_4, t_2, t_3, t_4.
7. $\bar{s}_1 s_2 \bar{s}_3 \bar{t}_1 t_2 \bar{t}_3 \bar{t}_4$ 6-sided, sides $s_1, s_3, s_4, t_1, t_2, t_4$.
8. $s_1 s_2 \bar{s}_3 t_1 t_2 \bar{t}_3 \bar{t}_4$ prism, sides s_1, s_4, t_1, t_2, t_4.
9. $s_1 s_2 \bar{s}_3 t_1 \bar{t}_2 \bar{t}_3 \bar{t}_4$ prism, sides s_2, s_4, t_1, t_2, t_4.
10. $s_1 \bar{s}_2 \bar{s}_3 t_1 \bar{t}_2 \bar{t}_3 \bar{t}_4$ 6-sided, sides $s_2, s_3, s_4, t_1, t_3, t_4$.
11. $s_1 \bar{s}_2 s_3 t_1 \bar{t}_2 t_3 \bar{t}_4$ prism, sides s_3, s_4, t_1, t_3, t_4.
12. $s_1 \bar{s}_2 s_3 \bar{t}_1 \bar{t}_2 t_3 \bar{t}_4$ prism, sides s_1, s_4, t_1, t_3, t_4.
13. $\bar{s}_1 \bar{s}_2 s_3 \bar{t}_1 \bar{t}_2 t_3 \bar{t}_4$ 6-sided, sides $s_1, s_2, s_4, t_2, t_3, t_4$.
14. $\bar{s}_1 s_2 s_3 \bar{t}_1 \bar{t}_2 t_3 \bar{t}_4$ prism, sides s_2, s_4, t_2, t_3, t_4.
15. $s_1 s_2 \bar{s}_3 \bar{t}_1 t_2 \bar{t}_3 \bar{t}_4$ prism, sides s_1, s_4, t_1, t_2, t_4.
16. $s_1 \bar{s}_2 s_3 t_1 \bar{t}_2 \bar{t}_3 \bar{t}_4$ prism, sides s_3, s_4, t_1, t_3, t_4.
17. $s_1 \bar{s}_2 s_3 \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_4$ cube, sides $s_1, s_3, s_4, t_1, t_3, t_4$.
18. $\bar{s}_1 \bar{s}_2 s_3 \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_4$ cube, sides $s_1, s_2, s_3, s_4, t_3, t_4$.
19. $\bar{s}_1 s_2 s_3 \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_4$ cube, sides $s_2, s_3, s_4, t_2, t_3, t_4$.
20. $\bar{s}_1 s_2 \bar{s}_3 \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_4$ cube, sides $s_1, s_2, s_3, s_4, t_2, t_4$.
21. $s_1 s_2 \bar{s}_3 \bar{t}_1 \bar{t}_2 \bar{t}_3 \bar{t}_4$ cube, sides $s_1, s_2, s_4, t_1, t_2, t_4$.

22. $s_1\bar{s}_2\bar{s}_3\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4$ cube, sides $s_1, s_2, s_3, s_4, t_1, t_4$.

23. $\bar{s}_1\bar{s}_2\bar{s}_3\bar{t}_1\bar{t}_2\bar{t}_3\bar{t}_4$ prism, sides s_1, s_2, s_3, s_4, t_4.

24. $s_1\bar{s}_2\bar{s}_3 t_1\bar{t}_2 t_3 t_4$ 6-sided, sides $s_1, s_3, s_4, t_2, t_3, t_4$.

25. $\bar{s}_1\bar{s}_2\bar{s}_3 t_1\bar{t}_2 t_3 t_4$ prism, sides s_1, s_3, t_1, t_2, t_3.

26. $\bar{s}_1\bar{s}_2\bar{s}_3\bar{t}_1\bar{t}_2 t_3 t_4$ prism, sides s_2, s_3, t_1, t_2, t_3.

27. $\bar{s}_1 s_2\bar{s}_3\bar{t}_1\bar{t}_2 t_3 t_4$ simplex, sides s_2, s_3, t_2, t_3.

28. $\bar{s}_1 s_2\bar{s}_3\bar{t}_1 t_2 t_3 t_4$ 6-sided, sides $s_2, s_3, s_4, t_1, t_2, t_3$.

29. $\bar{s}_1\bar{s}_2\bar{s}_3\bar{t}_1 t_2 t_3 t_4$ prism, sides s_2, s_3, t_1, t_2, t_3.

30. $\bar{s}_1\bar{s}_2\bar{s}_3 t_1 t_2 t_3 t_4$ cube, sides $s_1, s_2, s_3, t_1, t_2, t_3$.

31. $s_1\bar{s}_2\bar{s}_3 t_1 t_2 t_3 t_4$ cube, sides $s_1, s_2, s_3, s_4, t_2, t_3$.

32. $\bar{s}_1 s_2\bar{s}_3 t_1 t_2 t_3 t_4$ cube, sides $s_1, s_2, s_3, s_4, t_1, t_3$.

33. $s_1 s_2\bar{s}_3 t_1 t_2 t_3 t_4$ prism, sides s_1, s_2, s_3, s_4, t_3.

34. $\bar{s}_1 s_2 s_3\bar{t}_1 t_2\bar{t}_3 t_4$ prism, sides s_3, s_4, t_2, t_3, t_4.

35. $\bar{s}_1 s_2 s_3\bar{t}_1\bar{t}_2\bar{t}_3 t_4$ prism, sides s_2, s_3, t_2, t_3, t_4.

36. $\bar{s}_1\bar{s}_2 s_3\bar{t}_1\bar{t}_2\bar{t}_3 t_4$ 6-sided, sides $s_1, s_2, s_3, t_1, t_3, t_4$.

37. $\bar{s}_1\bar{s}_2 s_3 t_1\bar{t}_2\bar{t}_3 t_4$ simplex, sides s_1, s_3, t_1, t_3.

38. $s_1\bar{s}_2 s_3 t_1\bar{t}_2\bar{t}_3 t_4$ prism, sides s_1, s_3, t_1, t_3, t_4.

39. $s_1\bar{s}_2 s_3\bar{t}_1\bar{t}_2\bar{t}_3 t_4$ prism, sides s_1, s_3, t_1, t_3, t_4.

40. $s_1\bar{s}_2\bar{s}_3 t_1 t_2\bar{t}_3 t_4$ 6-sided, sides $s_1, s_2, s_4, t_1, t_2, t_3$.

41. $s_1, \bar{s}_2\bar{s}_3\bar{t}_1 t_2\bar{t}_3 t_4$ simplex, sides s_1, s_2, t_1, t_2.

42. $\bar{s}_1\bar{s}_2\bar{s}_3\bar{t}_1 t_2\bar{t}_3 t_4$ prism, sides s_1, s_2, t_1, t_2, t_3.

43. $\bar{s}_1\bar{s}_2 s_3\bar{t}_1 t_2 t_3 t_4$ 6-sided, sides $s_2, s_3, s_4, t_1, t_2, t_4$.

44. $\bar{s}_1\bar{s}_2\bar{s}_3 t_1 t_2\bar{t}_3 t_4$ prism, sides s_1, s_2, t_1, t_2, t_3.

45. $\bar{s}_1\bar{s}_2 s_3 t_1 t_2 t_3 t_4$ cube, sides $s_1, s_2, s_3, s_4, t_1, t_2$.

46. $s_1\bar{s}_2 s_3 t_1 t_2 t_3 t_4$ prism, sides s_1, s_2, s_3, s_4, t_2.

47. $\bar{s}_1 s_2 s_3\bar{t}_1\bar{t}_2 t_3 t_4$ prism, sides s_2, s_3, t_2, t_3, t_4.

48. $\bar{s}_1 s_2 \bar{s}_3 \bar{t}_1 \bar{t}_2 \bar{t}_3 t_4$ 6-sided, sides $s_1, s_2, s_3, t_2, t_3, t_4$.

49. $s_1 s_2 \bar{s}_3 \bar{t}_1 \bar{t}_2 \bar{t}_3 t_4$ prism, sides s_1, s_2, t_1, t_2, t_4.

50. $s_1 s_2 \bar{s}_3 t_1 \bar{t}_2 \bar{t}_3 t_4$ prism, sides s_2, s_4, t_1, t_2, t_4.

51. $\bar{s}_1 s_2 \bar{s}_3 t_1 t_2 \bar{t}_3 t_4$ 6-sided, sides $s_1, s_2, s_4, t_1, t_3, t_4$.

52. $\bar{s}_1 \bar{s}_2 \bar{s}_3 t_1 \bar{t}_2 \bar{t}_3 t_4$ prism, sides s_1, s_3, t_1, t_2, t_3.

53. $\bar{s}_1 \bar{s}_2 s_3 t_1 \bar{t}_2 t_3 t_4$ 6-sided, sides $s_1, s_3, s_4, t_1, t_2, t_3$.

54. $\bar{s}_1 s_2 s_3 t_1 t_2 t_3 t_4$ prism, sides s_1, s_2, s_3, s_4, t_1.

55. $s_1 \bar{s}_2 \bar{s}_3 \bar{t}_1 \bar{t}_2 \bar{t}_3 t_4$ 6-sided, sides $s_1, s_2, s_3, t_1, t_2, t_4$.

56. $s_1 \bar{s}_2 s_3 \bar{t}_1 \bar{t}_2 t_3 t_4$ prism, sides s_1, s_4, t_1, t_3, t_4.

57. $s_1 s_2 \bar{s}_3 \bar{t}_1 t_2 \bar{t}_3 t_4$ prism, sides s_1, s_2, t_1, t_2, t_4.

58. $s_1 s_2 \bar{s}_3 t_1 t_2 \bar{t}_3 t_4$ 7-sided, sides $s_1, s_2, s_4, t_1, t_2, t_3, t_4$.

59. $s_1 \bar{s}_2 s_3 t_1 \bar{t}_2 t_3 t_4$ 7-sided, sides $s_1, s_3, s_4, t_1, t_2, t_3, t_4$.

60. $\bar{s}_1 s_2 s_3 \bar{t}_1 t_2 t_3 t_4$ 7-sided, sides $s_2, s_3, s_4, t_1, t_2, t_3, t_4$.

61. $\bar{s}_1 \bar{s}_2 \bar{s}_3 \bar{t}_1 \bar{t}_2 \bar{t}_3 t_4$ 7-sided, sides $s_1, s_2, s_3, t_1, t_2, t_3, t_4$.

62. $s_1 \bar{s}_2 \bar{s}_3 t_1 \bar{t}_2 \bar{t}_3 t_4$ 8-sided

63. $\bar{s}_1 s_2 \bar{s}_3 \bar{t}_1 t_2 \bar{t}_3 t_4$ 8-sided

64. $\bar{s}_1 \bar{s}_2 s_3 \bar{t}_1 \bar{t}_2 t_3 t_4$ 8-sided

Lemma 1. *If a lopsided subset of $C(E')$ contained in L contains the vertices corresponding to topes 2, 3, 4, then it must contain all of the vertices corresponding to topes 2-23.*

Proof. The topes 2-23 are exactly those topes T for which $t_4 \in T^-$. In a representation of $(E, \mathscr{C}')$ by a sphere system, each of these topes would be bounded by $H^0_{s_4}$. The oriented matroid $(E', \mathscr{C}'')$ obtained from $(E', \mathscr{C}')$ by contracting s_4 has topes of the form $\hat{T}(T^-\backslash\{s_4\}, T^+\backslash\{s_4\})$, for topes T of $(E, \mathscr{C}')$ that are bounded by $H^0_{s_4}$. Thus, the set of topes 2-23 of $\mathscr{T}(E, \mathscr{C}')$ correspond to the topes $\hat{T}$ of $(E', \mathscr{C}'')$ for which $t_4 \in \hat{T}^-$. The oriented matroid $(E', \mathscr{C}'')$ can be represented as a sphere system, as in figure 1. Figure 1 shows the hemisphere $H^-_{t_4}$ of such a representation. The components labelled 2, 3 and 4 correspond to the topes 2, 3 and 4 of $(E, \mathscr{C}')$.

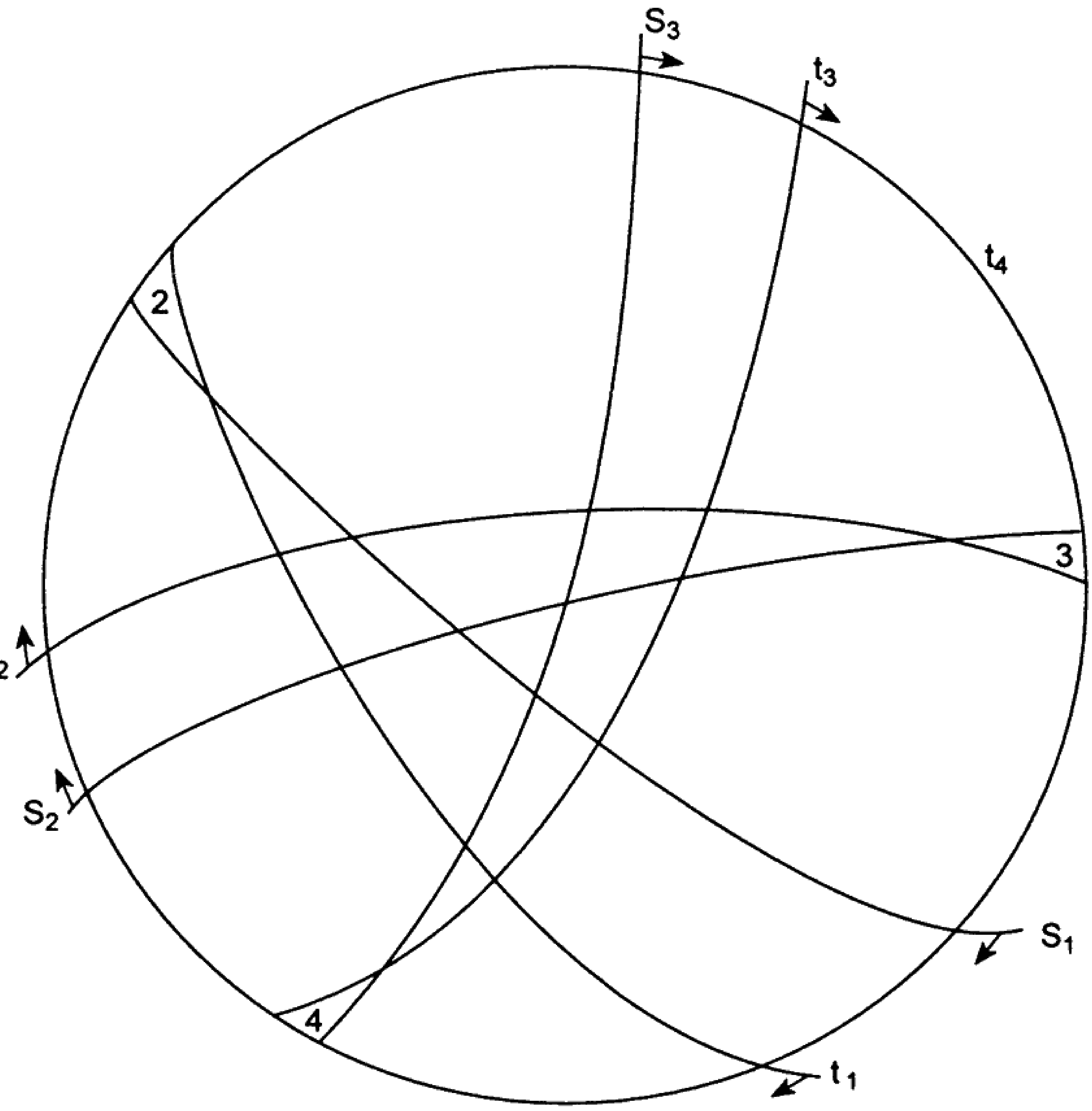

Figure 1

Now let $E'' = E' \backslash \{t_4\}$. If the set of topes T of $(E, \mathscr{C}')$ that have $s_4 \in T^+$ is to yield a lopsided set, then the set of topes of $(E, \mathscr{C}')$ that have $s_4 \in T^+, t_4 \in T^-$ must also give a lopsided set. This set contains the topes 2, 3 and 4. The argument of $[L]$ which proved the existence of a non-realizable lopsided subset of the 8-cube can be used to show that any lopsided subset of L that contains vertices corresponding to topes 2, 3 and 4, must also contain the vertices corresponding to topes 5-23.

Lemma 2 ([L]). *A lopsided subset L of a cube $C(E)$ must be connected, that is, if f, g are in L, then there is a sequence $f_1, \ldots, f_r$ of vertices in L with $f_1 = f, f_r = g$, such that for $i = 1, \ldots, r-1$, there is a unique $e_i \in E$ such that $f_i(e_i) \neq f_{i-1}(e_i)$*

Denote by J_1 the set of vertices of L that correspond to topes 1-23.

Lemma 3. *A lopsided subset of L that contains J_1 must contain the vertices corresponding to topes 33, 46, 54, 58, 59 and 60.*

Proof. Consider the set of vertices f of L for which $f(s_2) = f(s_3) = 1$. This set contains f_T for tope number 1. The set also contains f_T for several T in J_1 with $|T^-| \geq 3$. On the other hand, the only f_T in this set for which $0 < |T^-| < 3$ are the f_T corresponding to topes 54 and 60. By the connectivity lemma, a lopsided set containing J_1 and contained in L must also contain f_T for topes 54 and 60. Consideration of the set of vertices of L for which $f(s_1) = f(s_3) = 1$ leads by the same argument to the inclusion of f_T for topes 46 and 59, and consideration of the set of vertices of L for which $f(s_1) = f(s_2) = 1$ leads to the inclusion of f_T for topes 33 and 58.

Let J_2 be $J_1 \cup \{f_T$ for topes 33, 46, 54, 58, 59, 60$\}$.

Lemma 5. *If a lopsided subset of L contains the vertices of J_2, it must also contain f_T for topes 24, 31, 32, 43, 45, 51.*

Proof. Consider the set of vertices of L for which $f(s_1) = -1, f(s_2) = f(t_1) = f(t_2) = 1$. The f_T of L in this set are f_T for topes 2, 32, 51 and 54. Now f_T for topes 2 and 54 are in J_2, so by the connectivity lemma, any lopsided set contained in L, containing J_2, must contain f_T for topes 32 and 51. Consideration of $\{f \in L : f(s_2) = -1, f(s_3) = f(t_2) = f(t_3) = 1\}$ leads by the same argument to the inclusion of f_T for topes 43 and 45, and consideration of $\{f \in L : f(s_3) = -1, f(s_1) = f(t_1) = f(t_3) = 1\}$ leads to the inclusion of f_T for topes 24 and 31.

Let $J_3 = J_2 \cup \{f_T$ for topes 24, 31, 32, 43, 45, 54$\}$.

Lemma 6. *A lopsided subset of L that contains the vertices of J_3 must also contain the f_T for topes 28, 34, 40, 50, 53, 56, 62, 63, 64.*

Proof. Consider the set $\{f \in L : f(s_1) = -1, f(s_2) = f(t_2) = 1\}$. The vertices of J_3 that are in this set are f_T for topes 2, 5, 6, 7, 32, 51, 54 and 60. The vertices of $L \backslash J_3$ in this set are f_T for topes 28, 34 and 63. The restrictions of the set of f_T for topes 2, 5, 6, 7, 32, 51, 54, 60 to the set $F = E' \backslash \{s_1, s_2, t_2\}$ forms a symmetric subset of the cube $C(E)$. Therefore, a lopsided subset of L containing J_3 must contain at least one of the f_T for topes 28, 34, 63. Suppose it contains f_T for tope 28 as well as J_3. Then the lopsided set contains f_T for topes 7 and 28 from the set $\{f \in L : f(s_1) = f(s_3) = f(t_1) = -1, f(s_2) = f(t_2) = 1\}$. By connectivity, the only other member of L in this set, namely f_T for tope 63 must be in the lopsided set. On the other hand, suppose the lopsided set contained f_T for tope 34 as well as J_3. Then it would contain f_T for topes 34 and 51 from the set $\{f \in L : f(s_1) = f(t_3) = -1, f(s_2) = f(t_2) = f(t_4) = 1\}$. The only other member of L in this set, namely f_T for tope 63, would also have to be in the lopsided set. Finally, suppose that the lopsided set contained f_T for tope 63 as well as J_3. Then it would contain f_T for topes 60 and 63 from the set $\{f \in L : f(s_1) = f(t_1) = -1, f(s_2) = f(t_2) = f(t_4) = 1\}$. By connectivity, the lopsided subset would have to contain one of the other two members of L from this set, namely f_T for topes 28 and 34. We have now shown that a lopsided subset of L containing J_3 must also contain the set $\{f_T$ for topes 28 and 63$\}$ or the set $\{f_T$ for topes 34 and 63$\}$ or both.

Suppose we have a lopsided subset of L containing the f_T for topes 28 and 63 as well as J_3. Consider the set $\{f \in L : f(s_1) = f(t_1) = -1, f(s_2) = f(t_2) = 1\}$. The vertices of $J_3 \cup \{f_T$ for topes 28 and 63$\}$ that are in this set are $\{f_T$ for topes 28, 32, 51, 54, 60, 63$\}$. The restrictions of this set of f_T to the set $F' = \{s_3, t_3, t_4\}$ form a symmetric subset of $C(F')$. Thus the only other member of L in this set, namely f_T for tope 34, would have to be in the lopsided set.

On the other hand, suppose a lopsided subset of L contained the f_T for topes 34 and 63 as well as J_3. Consider the set $\{f \in L : f(s_1) = -1, f(s_2) = f(t_2) = f(t_4) = 1\}$. The vertices of $J_3 \cup \{f_T$ for topes 34 and 63$\}$ that are in this set are $\{f_T$ for topes 32, 34, 51, 54, 60, 63$\}$. The restrictions of this set of f_T to the set $F'' = \{s_3, t_1, t_3\}$ form a symmetric subset of $C(F'')$. Thus the only other member of L in this set, namely f_T for tope 28, would have to be in the lopsided set.

So far we have proved that a lopsided subset of L containing J_3 must also contain f_T for topes 28, 34 and 63. An entirely analogous argument shows that it must also contain f_T for topes 40, 50 and 62, and another analogous argument shows that it must also contain f_T for topes 53, 56 and 64.

Let $J_4 = J_3 \cup \{f_T$ for topes 28, 34, 40, 50, 53, 56, 62, 63, 64$\}$. The result of lemmas 1-6 is that any lopsided subset of L that contains f_T for topes 1-4 must contain J_4. Note that J_4 is the set of f_T for T such that $H^0_{s_4}$ bounds the region of S^3 that T corresponds to in a representation of $(E, \mathscr{C}')$ as a sphere system. The remaining functions of L are f_T for topes T that correspond to regions of S^3 in the interior of $H^+_{s_4}$. An argument used by Lawrence $[L]$, in the construction of a non-realizable subset of the 8-cube, shows that if a lopsided subset of L contains J_4, corresponding to the «boundary» topes of $H^+_{s_4}$, it must also contain the other vertices of L, corresponding to the «interior» topes of $H^+_{s_4}$. Thus any lopsided subset of L containing f_T for topes 1-4 must contain all of L.

Now suppose that L is realized by some subset $K \subseteq V(E)$.

For $i = 1, \ldots, 4$, let x^i be a point of K in the closed orthant $\{x \in V(E) : x(e) \leq 0$ if $e \in T^-, x(e) \geq 0$ if $e \in T^+$, for all $e \in E\}$. Let F be a 3-dimensional affine subspace of $V(E)$ containing the points $x^i, i = 1, \ldots, 4$. The lopsided set realized by $F \cap K$ must contain all of L, since it contains the f_T for topes 1-4. The intersections of the coordinate hyperplanes of $V(E)$ with F, together with a sphere $H^0_{s_4}$ «at infinity», would yield a realizable sphere system that represents the oriented matroid $(E, \mathscr{C}')$. But the oriented matroid $(E, \mathscr{C}')$ is not realizable. On the other hand, the number of 3-dimensional regions created by a set of 7 hyperplanes in $\mathbb{R}^3$ cannot be more than $|L|$, by Zaslavsky's formula (see [Z]). Thus L is not a realizable lopsided set.

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