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ORDER POLAR TOPOLOGIES

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In honor and memoriam of Prof. P.K. Kamthan

1. INTRODUCTION

The duality theory of locally convex spaces is one of the most important and useful theories in the study of topological vector spaces and therefore while considering the dual pairs of Riesz spaces or ordered vector spaces, it is natural to investigate the impact of ordering on topological structure and vice-versa. Such study on normed vector lattices was initiated during the period between 1937 and 1948 and has now developed considerably through the pioneering work of several mathematicians, namely Krein, Freudenthal, Namioka, Schaeffer, Luxemburg, Zaanen etc., cf. [6], [13], [15], [16] and references given therein. This study has further been generalized to locally convex spaces and we now have several books dealing with the literature of ordered topological vector spaces and locally convex solid Riesz spaces, cf. [1], [13], [16].

In this note, our investigations are based on this interrelationship of ordering with the duality of locally convex topological vector spaces. Indeed, we condider here the polar topologies which are named as order polar topologies, for the concepts of ordering as well as duality are involved in dealing with such topologies. These are the topologies $\tau_c(X,Y), \tau_{sc}(X,Y)$ and $T_{op}(X^*,X)$ which are different from the ones, namely $O(X,Y)$ and $\tau_s(X,Y), \tau_s(X,Y)^*$ considered earlier by Peressini [13] and Duhoux [4] respectively. In this paper, whereas the Section two makes the presentation reasonably self-contained by including basic results, definitions etc. which are to be used in this study, Sections three and four are devoted to the study of these topologies. Several examples from the theory of sequence spaces have also been constructed in support of definitions and hypotheses assumed in several results.

2. BASIC RESULTS AND TERMINOLOGY

For the convenience of readers, we present in this section the rudiments from the theories of Riesz spaces, locally convex spaces, locally solid Riesz spaces and sequence spaces. However, for details of these theories, we refer to the texts [1], [6], [9], [10], [11], [12], [14], [15], [16] and [17].

We denote throughout by X a real vector space which is a vector lattice or a Riesz space ordered by a cone K. We write X' for the algebraic dual of X and in case, X is equipped with a locally convex topology τ, its topological dual is denoted by X^*. For $x \in X$, the symbols x^+, x^- and $|x|$ have their usual meaning, i.e., $x^+ = x \vee \theta, x^- = (-x)^+$ and $|x| =$

$x^+ + x^-$. An increasing (resp. a decreasing) net $\{x_\alpha : \alpha \in \Lambda\} \equiv \{x_\alpha\}$ in X is denoted by x_α (resp. x_α) and if supremum (resp. infimum) of $\{x_\alpha\}$ is x, we write $x_\alpha \uparrow x$ (resp. $x_\alpha \downarrow x$).

Following [1] and [13], we have

Definition 2.1. *In a Riesz space X, (i) a net $\{x_\alpha\}$ is said to* order converge *to an element x of X,* written *as $x_\alpha \xrightarrow{(0)} x$ in x, if there exists a net $\{y_\alpha\}$ such that $y_\alpha \downarrow \theta$ and $|x_\alpha - x| \leq y_\alpha$, for each α and in this case x is said to be* order limit *of $\{x_\alpha\}$ or $x = 0 - \lim_\alpha x_\alpha$; (ii) a set B is said to be (a)* order closed *if it contains all its order limits, and (b)* solid *if $y \in B$ whenever $|y| \leq |x|$ for some $x \in B$; and (iii) an* ideal *in X is solid subspace of X, whereas a* band *of X is an order closed ideal of X. A Riesz space X is said to be (iv)* order complete *(resp.* σ-order complete*) if every majorized subset (resp. every countable majorized set) in X has a supremum in X; and (v)* order separable *if every subset A of X that has a supremum in X contains a countable subset A' such that* $\sup(A) = \sup(A')$.

Definition 2.2. *A linear operator T from a Riesz space X to another Riesz space Y is said to be (i)* order bounded *if it maps each order bounded set in X to an order bounded subset of Y; (ii)* order continuous *if $Tx_\alpha \xrightarrow{(0)} Tx$ in Y whenever $x_\alpha \xrightarrow{(0)} x$ in X; and* sequentially order continuous *if $Tx_n \xrightarrow{(0)} Tx$ in Y whenever $x_n \xrightarrow{(0)} x$ in X.*

We denote by $\mathscr{L}^b(X,Y)$, the class of all order bounded operators from X to Y and the subspaces of $\mathscr{L}^b(X,Y)$ containing order continuous and sequentially order continuous linear operators are respectively denoted by $\mathscr{L}^c(X,Y)$ and $\mathscr{L}^{so}(X,Y)$. In particular, for $Y \equiv \mathbb{R}$ we write $X^b \equiv \mathscr{L}^b(X,\mathbb{R}), X^{so} \equiv \mathscr{L}^{so}(X,\mathbb{R})$ and $X^c \equiv \mathscr{L}^c(X,\mathbb{R})$, which are respectively known as the order dual, sequential order dual and continuous order dual of X.

Concerning these spaces, we have [1]:

Theorem 2.3. *Let X and Y be Riesz spaces with Y as an order complete space. Then $\mathscr{L}^b(X,Y)$ is an order complete Riesz space ordered by the cone $\tilde{K} = \{T \in \mathscr{L}^b(X,Y) : T(x) \geq \theta, \forall x \in K\}$, where for T in $\mathscr{L}^b(X,Y)$ and $x \in X, |T|$ in $\mathscr{L}^b(X,Y)$ is defined by $|T|(|x|) = \sup\{|T(y)| : |y| \leq |x|\}$. Moreover, $\mathscr{L}^{so}(X,Y)$ and $\mathscr{L}^c(X,Y)$ are bands in $\mathscr{L}^b(X,Y)$.*

Proposition 2.4. *Every sequentially order continuous linear operator T from an* order separable *Riesz space X into an Archimedean Riesz space Y is order continuous, i.e., $\mathscr{L}^{so}(X,Y) \subset \mathscr{L}^c(X,Y)$.*

For a subset A of X' and $x \in X$, we write

$$p_A(x) = \sup_{f \in A} | < x, f > |$$

Depending on the behaviour of p_A, let us recall the following concepts from [3], [4].

Definition 2.5. *A subset A of X' is said to satisfy condition (i)* A_1 *if $A \subset X^{so}$ and $p_A(x_n) \to 0$ for every sequence $\{x_n\}$ in X, which order converges to θ in X; (ii) A_2 if $A \subset X^c$ and $p_A(x_\alpha) \to 0$ for every net $\{x_\alpha\}$ in X, order converging to θ in X; and (iii)* A_3 *if $A \subset X^b$ and $p_A(x_n - x_m) \xrightarrow[n,m\to\infty]{} 0$ for every majorized increasing sequence $\{x_n\}$ in X.*

The sets satisfyng conditions A_1, A_2 and A_3 are also known as the equi-σ-continuous, equicontinuous and equi-l^1-continuous sets respectively; cf. [2].

For a locally convex space (X, τ) with dual X^*, the notations $\sigma(X^*, X), \tau(X^*, X)$, $\beta(X^*, X)$ and $\lambda(X^*, X)$ respectively denote the weak topology, Mackey topology, strong topology and the topology of uniform convergence on all precompact subsets of X.

Relating the topology $\lambda(X^*, X)$ with $\sigma(X^*, X)$, we have the famous «Banach Dieudonne theorem» contained in [9].

Theorem 2.6. *Let X be a metrizable locally convex space with dual X^*. Then the finest locally convex topology on X^* which induces on every equicontinuous subset of X^* the same topology as $\sigma(X^*, X)$, is the topology of uniform convergence on precompact subsets of X, i.e., $\lambda(X^*, X)$.*

For the theory of locally solid Riesz spaces, main references are [1], [6], [16]. Let us begin with

Definition 2.7. *(i) A linear topology τ on X is said to be a* locally solid *(l.s.)* topology *if it has a neighbourhood basis consisting of solid sets; in addition if τ is also a locally convex topology, it is called a* locally convex solid *(l.c.s.) topology, (ii) a l.s. space (X, τ) satisfies (a)* σ-Lebesgue property *(or A_1) if $x_n \downarrow \theta$ in X implies $x_n \xrightarrow{\tau} \theta$; (b)* Lebesgue property *(or A_2) if $x_\alpha \downarrow \theta$ in X implies $x_\alpha^\tau \theta$; (c)* pre-Lebesque *(or A_3) if $\theta \leq x_n \uparrow \leq x$ in X yields that $\{x_n\}$ is a τ-Cauchy sequence in X.*

One can easily verify the following characterization of a locally solid topology.

Proposition 2.8. *A locally convex topology on X is locally solid if and only if X^* is an ideal in X^b and the solid hull of an equicontinuous set in X^* is equicontinuous.*

Concerning the locally solid character of Mackey topology on X, we have [3]:

Proposition 2.9. *For a Riesz space X, (i) the Mackey topology $\tau(X, X^b)$ is always locally solid; and (ii) $\tau(X, X^{so})$ (resp. $\tau(X, X^c)$) is locally solid if and only it satisfies A_1 (resp. A_2).*

If $< X, Y >$ is a dual pair of Riesz spaces where Y is an ideal in X^b, the symbols $O(X, Y)$ or $|\sigma|(X, Y)$ denote the topology of uniform convergence on all order intervals of Y, or equivalently, it is the topology defined by the seminorms $\{p_y : y \in Y\}$, where $p_y(x) = <|x|, |y|>$.

We need the following from [1].

Proposition 2.10. *The dual of X equipped with $O(X,Y)$ is Y.*

Theorem 2.11. *In a Hausdorff l.c.s. Riesz space $(X,\tau), \sigma(X,X^*) = |\sigma|(X,X^*)$ if and only if every order interval of X^* is contained in a finite dimensional vector space.*

Concerning the order topology τ_O on X, we have [13].

Definition 2.12. *The* order topology τ_O *on a Riesz space X, is the finest locally convex topology for which every order bounded set is topologically bounded.*

Proposition 2.13. *(i) The dual of (X,τ_O) is X^b; and (ii) if (X,τ) is a complete metrizable locally solid Riesz space, then τ coincides with the order topology τ_O on X.*

Now, we mention from [5], the following concepts in a l.s. Riesz space.

Definition 2.14. *A set A in (X,τ) is said to be quasi-order precompact (resp. order precompact) if for each neighbourhood U of θ, there exists a positive element x in X (resp. in the ideal generated by A in X) such that $A \subset [-x,x] + U$.*

Clearly, every order precompact set is quasi-order precompact, but the converse is not necessarily true, e.g. the set $\{e^n : n \geq 1\}$ [cf. (2.16) for definition] is quasi-order precompact, but not order precompact in l^∞. However, we have [5].

Proposition 2.15. *The topology τ of a l.c.s. Riesz space (X,τ) is pre-Lebesgue if and only if every quasi-order precompact set in (X,τ) is order precompact.*

We follow [10] for various notations and results in the theory of sequence spaces. As we are condidering real vector spaces in this paper, we denote by ω the space of all real valued sequences and φ, the subspace of ω, spanned by $\{e^n : n \geq 1\}$, where

$$e^n = \{0,0,\ldots,\underset{n^{th}\ place}{1},0,0,\ldots\}. \tag{2.16}$$

By a sequence space, we mean a subspace λ of ω, containing φ. The symbol

$$\lambda^x = \{\{\beta_n\} \in \omega : \sum_1^\infty |\alpha_n\beta_n| < \infty, \forall\{\alpha_n\} \in \lambda\},$$

is used to denote the Köthe dual of λ. The duality between λ and λ^x is given by the bilinear form, <, >, where $<\overline{\alpha},\overline{\beta}> = \sum_1^\infty \alpha_n\beta_n$; for $\overline{\alpha} = \{\alpha_n\} \in \lambda$ and $\overline{\beta} = \{\beta_n\}$ in λ^x. The weak topology $\sigma(\lambda, \lambda^x)$ and the normal topology $\eta(\lambda, \lambda^x)$ are generated by the family $\{q_{\overline{\beta}} : \overline{\beta} \in \lambda^x\}$ and $\{p_{\overline{\beta}} : \overline{\beta} \in \lambda^x\}$ of seminorms where

$$q_{\overline{\beta}}(\overline{\alpha}) = |\sum_{n=1}^{\infty} \alpha_n \beta_n| \text{ and } p_{\overline{\beta}}(\overline{\alpha}) = \sum_{n=1}^{\infty} |\alpha_n \beta_n|.$$

A sequence space λ is *monotone* (resp. *normal*) if $\{\alpha_n x_n\} \in \lambda$ whenever $\{x_n\} \in \lambda$, and $\alpha_n = 0$ or 1 (resp. $|\alpha_n| \leq 1, \forall n \geq 1$).

Concerning the order structure of sequence spaces, we know that they have natural coordinatewise ordering induced from the ordering of $\mathbf{R}$. We make this assumption throughout this paper. We now have [13].

Proposition 2.17. *Every sequence space λ is an order separable ordered vector space.*

Proposition 2.18. *Let a sequence space λ be an ideal in w which is a Riesz space with respect to coordinatewise ordering. Then a linear functional $\overline{f}$ on λ is sequentially order continuous if and only if there is a unique $\overline{u} = \{u_n\} \in \lambda^x$ such that*

$$\overline{f}(\overline{x}) = <\overline{x}, \overline{u}> = \sum_{n=1}^{\infty} x_n u_n,$$

for all $\overline{x} = \{x_n\}$ in λ.

Note 2.19. *In view of Propositions 2.4 and 2.17, it follows from Proposition 2.18 that $\lambda^x \equiv \lambda^{so} \equiv \lambda^c$.*

3. POLAR TOPOLOGIES $\tau_c(X, Y)$ AND $\tau_{so}(X, Y)$

Throughout this section, we assume that X is in duality with X^c and hence with, X^{so} and X^b. Then the conditions A_2 and A_1 of sets in X^c and X^{so} yield the Hausdorff polar topologies $\tau_c(X, Y)$ and $\tau_{so}(X, Y)$ on X, introduced in

Definition 3.1. *Let $< X, Y >$ be a dual pair of Riesz spaces. If Y is an ideal in X^c, we write ζ_c for the collection of all solid, convex, $\sigma(Y, X)$ relatively compact subsets of Y which satisfy A_2; and if Y is an ideal in X^{so}, we set ζ_{so} for the collection of all solid, convex, $\sigma(Y, X)$ relatively compact subsets of Y which satisfy A_1. Then the* topologies $\tau_c(X, Y)$ *and $\tau_{so}(X, Y)$ are respectively the topologies of uniform convergence on members of ζ_c and ζ_{so}.*

Concerning these topologies, we have

Propositon 3.2. *$\tau_c(X,Y)$ (resp. $\tau_{so}(X,Y)$) on X is the finest locally convex solid topology compatible with the dual pair $< X,Y >$ satisfying Lebesgue (resp. σ-Lebesque) property.*

Proof. We prove the result for $\tau_c(X,Y)$; as the result for $\tau_{so}(X,Y)$ would follow analogously on replacing X^c by X^{so}, nets by sequences and the condition A_2 by A_1.

Observe that the topology $\tau_c(X,Y)$ is clearly a l.c.s. topology as the members of ζ_c are solid, convex and $\sigma(Y,X)$ bounded subsets of Y. Further, the inclusions $O(X,Y) \subset \tau_c(X,Y) \subset \tau(X,Y)$ yield the compatibility of $\tau_c(X,Y)$ with $< X,Y >$; cf. Proposition 2.10 and [9], p. 205.

In order to show that it is the finest locally convex solid topology having all the properties announced above, consider a locally convex solid topology τ on X which is compatible with the dual pair $< X,Y >$ satisfying Lebesque property. In view of Proposition 2.8 and l.c.s. nature of τ, we may assume that the collection ζ_τ generating the topology τ is comprised of τ-equicontinuous, convex, solid sets. The proof would follow if we show that each A in ζ_τ satisfies A_2. So consider a set A in ζ_τ and a net $\{x_\alpha : \alpha \in \wedge\}$ in X such that $x_\alpha \xrightarrow{(0)} \theta$ in X. Then, from the equicontinuity of A, we can find a τ-neighbourhood U of θ such that $A \subset U^\circ$; and $x_\alpha \xrightarrow{(0)} \theta$ yields a net $\{y_\alpha : \alpha \in \wedge\}$ with $y_\alpha \downarrow \theta$ in X, such that $|x_\alpha| \leq y_\alpha, \forall \alpha \in \wedge$. As τ is Lebesque, for any $\varepsilon > 0$, we can find $\alpha_\circ \in \wedge$ such that $y_\alpha \in \varepsilon U, \forall \alpha \geq \alpha_\circ$. Consequently, $p_A(x_\alpha) \leq \varepsilon, \forall \alpha \geq \alpha_\circ$ since A is solid. Thus $\zeta_\tau \subset \zeta_c$ and the result follows.

In spaces where $\sigma(X^c,X)$ (resp. $\sigma(X^{so},X)$) bounded sets are also order bounded, one can easily verify that the topologies $\tau_c(X,X^c), O(X,X^c)$ and $\beta(X,X^c)$ (resp. τ_{so} $(X,X^{so}), O(X,X^{so})$ and $\beta(X,X^{so})$) coincide.

In the following examples we respectively illustrate the space X, where $\sigma(X^c,X)$ bounded sets in X^c are order bounded, the indispensability of this condition for the equality $\beta(X,X^c) = \tau_c(X,X^c)$ to hold good and the condition is not a necessary one for the topologies to coincide.

Example 3.3. Consider the space $X = \varphi$. Then $X^c = w$ by Note 2.19. Hence a $\sigma(w,\varphi)$ bounded set, being pointwise bounded, is order bounded; cf. [10], p. 104.

Example 3.4. Let $X = l^\infty$. Using once again the Note 2.19, $(l^\infty)^c \equiv (l^\infty)^{so} \equiv (l^\infty)^x \equiv l^1$. Since $\beta(l^\infty, l^1)$ is the norm topology of l^∞ and $(l^\infty, \beta(l^\infty, l^1))^* \neq l^1$, cf. [10], p. 129, it follows from Proposition 3.2 that $\tau_c(l^\infty, l^1) \neq \beta(l^\infty, l^1)$ bounded, but it is not order bounded.

Example 3.5. Take the space X as l^2, for which $(l^2)^c \equiv (l^2)^{so} \equiv (l^2)^x \equiv l^2$, cf. Note 2.19. Observe that the set $A \equiv \left\{ \frac{1}{\sqrt{n}} e^n : n \geq 1 \right\}$ is $\sigma(l^2, l^2)$-bounded, but it is not order bounded in l^2. However, $\beta(l^2, l^2) \equiv \tau_\circ$ on l^2 and so $(l^2)^b \equiv l^2$ by Proposition 2.13. Consequently, by Propositions 2.9 and 3.2, $\tau_q(l^2, l^2) \equiv \tau(l^2, l^2) \equiv \beta(l^2, l^2)$.

For proving rest of the results in this section, we need restrict the Riesz space X as introduced in

Definition 3.6. *A Riesz space X is said to be (i) an* O-space *if for any sequence $\{x_n : n \geq 1\}$ in X, $x_n \xrightarrow{(0)} \theta$ in X if $f(x_n) \to 0$ for each f in X^c, and (ii) a* σ-O space *if for any sequence $\{x_n\} \subset X, x_n \xrightarrow{(0)} \partial$ in X if $f(x_n) \to 0$, for every f in X^{so}.*

Clearly, every O-space is a σ-O space and these two concepts coincide when $X^{so} = X^c$. It would be *interesting to know examples of σ-O-spaces which are not O-spaces*. However, illustrating O-space we have

Example 3.7. Consider the Riesz space w for which $w^c = \varphi = w^{so}$ by Note 2.19. Observe that any $\sigma(w, \varphi)$-null sequence $\{\overline{x}^n : n \geq 1\}$ in w, is $\sigma(w, \varphi)$ bounded and so we can find a sequence $\{y_i : i \geq 1\} \in w$ such that

$$|x_i^n| \leq y_i, \forall n \geq 1,$$

cf. [10], p. 104. Hence $\overline{x}^n \xrightarrow{(0)} \theta$, cf. [13] (see also [8] for the vector valued sequence spaces). Thus w is a $\sigma - O$ as well as an O-space.

The spaces considered in the following three examples are not σ-O-spaces and hence they are not O-spaces.

Example 3.8. Consider the Riesz space $X = C[0,1]$ with usual pointwise ordering. It is well known that $X^{so} = X^c = \{\theta\}$ for this space, cf. [17], p. 153. Hence, obviously it cannot be an O-space.

Example 3.9. The Riesz space $X = l^1$ for which $(l^1)^c = (l^1)^{so} = (l^1)^x = l^\infty$, is not σ-O space; for the sequence $\overline{x}^n = \left\{\frac{1}{n}e^n : n \geq 1\right\}$, does not O-converge to zero, but $\overline{f}(\overline{x}^n) \to 0$, for each $\overline{f}$ in l^∞.

Example 3.10. Here we consider the space $X = L^1[0,1]$ formed by the equivalence classes of Lebesque integrable almost everywhere equal functions. It is a well known fact that $L^1[0,1]$ is an order complete Riesz space for the ordering defined as, $[f] \leq [g]$ in $L^1[0,1]$ if and only if $f(x) \leq g(x)$ almost everywhere. Since the space $L^1[0,1]$, equipped with the normed topology is a Banach lattice which is an abstract L-space, it is a Lebesgue and hence a σ-Lebesgue space. Consequently,

$$(*) \qquad L^\infty = (L^1[0,1])^* \subseteq (L^1[0,1])^c \subseteq (L^1[0,1])^{so} \subset (L^1[0,1])^b = (L^1[0.1])^* = L^\infty,$$

where L^∞ is the space of equivalence classes of essentially bounded functions on [0,1], cf. [1] p. 71 and 112.

For constructing the required non-o-convergent sequence $\{[f_n]\}$ in $L^1[0,1]$, let us recall the sets E_n's used in the construction of a Cantor set, namely

$$E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right];$$

$$E_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right];$$

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If P is the Cantor set, write

$$A_o = P; A_1 = E_1^c \text{ and } A_n = (A_{n-1} \cup E_n)^c, n \geq 2.$$

Then we have

$$(i)\ [0,1] = \bigcup_{i \geq 0} A_i; (ii)\ \mu(A_o) = 0 \text{ and } \mu(A_n) = \frac{2^{n-1}}{3^n}, n \geq 1; and$$

$$(iii)\ A_i \cap A_j = \varphi, i \neq j.$$

Now define the sequence $\{f_n : n \geq 0\}$ as follows:

$$f_o = X_{A_o}; \text{ and } f_n = \frac{3^n}{n2^n} X_{A_n}, n \geq 1.$$

Then from (*), we have for $[\varphi] \in L^\infty$

$$|<[f_n],[\varphi]>| = |\int_o^1 f_n(x)\emptyset(x)dx|$$

$$\leq \frac{M}{2^n} \to 0 \text{ as } n \to \infty,$$

where M is an essential bound of $\emptyset$. But the sequence $\{[f_n] : n \geq 0\}$ is not order bounded in $L^1[0,1]$.

It is obvious from the construction of $\tau_c(X, X^c)$ that every closed set in $(X, \tau_c(X, X^c))$ is order closed; however for O-spaces we have

Proposition 3.11. *Let X be an O-space. Then a sequentially order closed set in X is $\tau_c(X, X^c)$ sequentially closed.*

Proof. Immediate from Proposition 3.2 and the definition of O-space.

In case of σ-O spaces, we have

Proposition 3.12. *For a σ-O space X, the $\tau_{so}(X, X^{so})$ sequentially closed and sequentially order closed sets are the same.*

Proof. Straightforward.

A characterization of sequentially continuous linear operators on $(X, \tau_c(X, X^c))$ in terms of its order-continuity is contained in

Proposition 3.13. *Let X be an O-space. A linear operator T from $(X, \tau_c(X, X^c))$ into itself is sequentially continuous if and only if it is sequentially order continuous.*

Proof. The result follows from Proposition 3.2 and the definition of O-space.

Remark. Replacing O-spaces by σ-O spaces in the above Proposition, we have analogous result for the topology $\tau_{so}(X, X^{so})$.

The restriction of O-spaces in the above Proposition is indispensable. Before we pass on to the precise example, let us recall from [10] a few concepts contained in

Definition 3.14. *Let λ and μ be two sequence spaces and $T \equiv [t_{ij}]$ be an infinite matrix. Then T is a* matrix transformation *from λ into μ if (a) for each $\bar{x} = (x_j) \in \lambda$, the series $\sum_{j=1}^{\infty} t_{ij} x_j$ converges absolutely for each i; and (b) for each $x = (x_j) \in \lambda$, the sequence $\bar{y} = (y_i)$ defined by $y_i = \sum_{j=1}^{\infty} t_{ij} x_j$ is an element of μ.*

Definition 3.15. *A sequence space λ equipped with a locally convex topology τ is known as a* Fréchét AK*-space if (λ, τ) is a complete metrizable space such that for each $\bar{x} = \{x_i\} \in \lambda$, $\bar{x}^{(n)} = \sum_{i=1}^{n} x_i e^i \to \bar{x}$ in τ and τ is finer than the co-ordinatewise convergence topology.*

We now prove a general result for real sequence spaces.

Proposition 3.16. *Let an ideal λ in ω, equipped with a locally convex solid topology τ, be a Fréchét AK-space. Then, every matrix transformation on λ is $\tau_q(\lambda, \lambda^c) - \tau_q(\lambda, \lambda^c)$ continuous.*

Proof. In view of Proposition 2.13, Note 2.19 and [10], p. 52, 60, we have

$$\lambda^{x} \equiv \lambda^{so} \equiv \lambda^{c} \equiv \lambda^{b}.$$

Hence $\tau_q(\lambda, \lambda^c) \equiv \tau(\lambda, \lambda^c)$ by Proposition 2.9 and 3.2. Consequently, any matrix transformation on λ is $\tau_q(\lambda, \lambda^c) - \tau_q(\lambda, \lambda^c)$ continuous, cf. [10], p. 205 and [9] p. 257.

Example 3.17. Consider the space l^2 and the operator $T : l^2 \to l^2$, given by the matrix $[a_{pq}]$, where

$$a_{pq} = \begin{cases} \frac{1}{pq} & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$$

Then by Proposition 3.16, T is $\tau_q(l^2, l^2) - \tau_q(l^2, l^2)$ continuous, but it is not order continuous; cf. [13], p. 170.

4. TOPOLOGY OF UNIFORM CONVERGENCE ON QUASI ORDER PRECOMPACT SETS

In this section, we consider an order polar topology on the dual of a locally convex solid Riesz space, defined with the help of quasi-order precompact sets in X. Indeed, let us introduce

Definition 4.1. *For a l.c.s. Riesz space (X, τ), let ζ_p denote the collection of all convex, solid and quasi-order precompact subsets of (X, τ). Then $T_{op}(X^*, X)$ is defined as the* topology of uniform convergence on meembers *of ζ_p.*

Since the solid, convex hull of a quasi-order precompact set is quasi-order precompact, we may replace ζ_p by ζ_h which is the collection of convex solid hulls of all quasi-order precompact subsets of (X, τ).

Theorem 4.1. *On τ-equicontinuous subsets of X^*, $T_{op}(X^*, X)$ coincides with the absolute weak topology $|\sigma|(X^*, X)$.*

Proof. Since $|\sigma|(X^*, X) = 0(X^*, X)$ and intervals are members of ζ_p, it suffices to prove $T_{op}(X^*, X)|_M \subset |\sigma|(X^*, X)|_M$, for any equicontinuous subset M of X^*. Therefore, consider a net $\{f_\alpha : \alpha \in \lambda\}$ in M such that $f_\alpha \to \theta$ in $|\sigma|(X^*, X)$; here we may assume that $\theta \in M$. Choose a τ-neighbourhood U of θ in X such that $M \subset U^\circ$. Then for any given $\varepsilon > 0$ and for a quasi-order precompact set A in X, there exists $x \geq \theta$ such that $A \subseteq [-x, x] + \varepsilon U$. Hence, for $y \in A$, the inequality

$$| < y, f_\alpha > | \leq < x, |f_\alpha| > + \varepsilon,$$

along with the fact that $< x, |f_\alpha| > \to 0$, yields that $f_\alpha \to \theta$ in $T_{op}(X^*, X)|_M$. This completes the proof.

It is known [5] that a precompact subset of a locally solid Riesz space is always quasi-order precompact, but the converse is not necessarily true; for we have

Example 4.2. Consider $X = l^\infty$, equipped with its norm topology. Then the order interval $[-\overline{e}, \overline{e}]$, where $\overline{e} = \{1, 1, 1, \ldots\}$ in l^∞ is quasi-order precompact but not precompact.

However, converse holds in the form of

Proposition 4.3. *Let (X, τ) be a l.c.s. Riesz space such that $\sigma(X^*, X) = |\sigma|(X^*, X)$. Then any quasi-order precompact subset of X is precompact.*

Proof. In view of Theorem 2.11, let us first note that any order interval in X is contained in a finite dimensional subspace of X.

Now consider a quasi-order precompact subset A of (X, τ) and U, a solid τ-neighbourhood of θ. Then we can find a solid neighbourhood V of θ such that $V + V \subseteq U$ and a positive element x in X such that $A \subseteq [-x, x] + V$. As $[-x, x]$ is a bounded set in a finite dimensional subspace of X, it is precompact. Consequently, A is precompact.

Illustrating the Riesz space for which $\sigma(X^*, X) = |\sigma|(X^*, X)$, we have

Example 4.4. Consider the Riesz space w equipped with the absolute weak topology $|\sigma|(w, \varphi)$, then $(w, |\sigma|(w, \varphi))^* = \varphi$ and $\sigma(w, \varphi) = \eta(w, \varphi) = |\sigma|(w, \varphi)$. (cf. [10]).

The final result of this note, which makes use of Dieudonné theorem, is contained in

Theorem 4.5. *Let (X, τ) be a metrizable l.c.s. Riesz space satisfying the condition $\sigma(X^*, X) = |\sigma|(X^*, X)$. Then the topology $T_{op}(X^*, X)$ is the finest locally convex topology on X^*, which induces on every equicontinuous subset of X^*, the same topology as $|\sigma|(X^*, X)$.*

Proof. By Proposition 4.3, $T_{op}(X^*, X) = \lambda(X^*, X)$. Now apply Theorem 2.6, in order to get the result.

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