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ON GROUPS WITH MANY SUBNORMAL SUBGROUPS

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Abstract. *The structure of groups in which every subgroup is either subnormal or self-normalizing is investigated.*

1. INTRODUCTION

A well-known theorem of Roseblade [15] states that a group in which all subgroups are subnormal with bounded defect is nilpotent with bounded class. On the other hand, Heineken and Mohamed constructed in 1968 the first example of a (metabelian) group with trivial centre having all subgroups subnormal (see [6]), and in the last few years several authors have investigated the structure of groups in which every subgroup is subnormal (see for instance [1], [2], [10], [11], [12], [13], [16]). In particular Möhres [13] has recently proved that all groups with this property are soluble.

The aim of this paper is to study a class of groups in which many subgroups are subnormal. More precisely, a group G will be called an *ES-group* if each subgroup of G either is subnormal or coincides with its normalizer. Moreover, if n is a positive integer, we shall say that G is an *ES_n-group* if every subgroup of G either is self-normalizing or is subnormal in G with defect at most n. In particular ES_1-groups are the groups in which every subgroup either is normal or coincides with its normalizer. Such groups were described in [4] (see also [3]).

Our first result deals with the case of ES-groups which are not Baer groups. It shows in particular that a non-periodic ES-group is a Baer group (recall that a group is a *Baer group* if it is generated by its abelian subnormal subgroups).

Theorem A. *Let G be a locally graded group which is not a Baer group. Then the following statements are equivalent:*

(a) G is an ES-group.

(b) G is an ES_n-group for some positive integer n.

(c) $G = \langle x \rangle \ltimes Q$, where x is an element with order a power of a prime p, and Q is a periodic nilpotent group without elements of order p such that $C_Q(x) = 1$ and $C_{\langle x \rangle}(Q) = \langle x^p \rangle$.

In the above theorem the assumption that the group G is locally graded cannot be removed, as Tarski p-groups (i.e. infinite simple groups whose proper non-trivial subgroups have the same prime order p) are clearly ES-groups.

Theorem B. *Let G be a Baer ES-group having no non-trivial perfect sections. Then every subgroup of G is subnormal and G is soluble.*

It follows immediately from Roseblade's theorem that a group whose finitely generated subgroups are subnormal with bounded defect is nilpotent. Hence every Baer ES_n-group is nilpotent. Also, using the results in [10],[11] and [13], we obtain the following consequence of Theorem B.

Corollary. *Let G be a Baer ES-group having no non-trivial perfect sections. Then:*
(1) If G has finite exponent, then it is nilpotent.
(2) If G is periodic, then it is a Fitting group.
(3) If G is torsion-free, then it is hypercentral.

Finally, it will be shown that in an arbitrary ES-group the join of two subnormal subgroups is subnormal.

We refer to [9] for properties of subnormal subgroups, and to [14] for general facts and notation. In particular:

A group G is *locally graded* if every finitely generated non-trivial subgroup of G has a proper subgroup of finite index.

A group G is *subsoluble* if it has an ascending subnormal series with abelian factors. If G is a group, the subgroup generated by its abelian subnormal subgroups is a Baer group, which is called the *Baer radical* of G.

If G is a group, $\pi(G)$ is the set of prime divisors of the orders of elements of G.

2. PROOF OF THE THEOREMS

The first lemma gives an elementary property of finite ES-groups.

Lemma 1. *Let G be a finite ES-group. Then G is soluble.*

Proof. Since every group of square-free order is soluble, we may suppose that p^2 divides the order of G for a certain prime p. Then the ES-group G has a subgroup of order p^2, and hence also a subnormal subgroup H of order p. The normal closure H^G of H is properly contained in G, so that the groups H^G and G/H^G are soluble by induction on the order of G. Therefore G is soluble. □

Our next lemma shows that locally finite ES-groups have large Baer radical.

Lemma 2. *Let G be a locally finite ES-group which is not a Baer group. Then the Baer radical B of G has prime index.*

Proof. Every finite subgroup of G is soluble by Lemma 1, so that G contains a cyclic subgroup which is properly contained in its normalizer, and hence is subnormal. Thus the Baer radical B is a proper non-trivial subgroup. Let x and y be two elements of $G\backslash B$. Then the subgroups $\langle x\rangle$ and $\langle y\rangle$ are self-normalizing, and hence they are Carter subgroups of the finite soluble group $\langle x, y\rangle$. Thus $\langle x\rangle$ and $\langle y\rangle$ are conjugate in $\langle x, y\rangle$, and in particular the cosets xB and yB have the same order. Moreover, if $\langle xB\rangle \neq \langle yB\rangle$, the subgroup $\langle xB, yB\rangle$ is not abelian. It follows that the factor group G/B has prime exponent p, and has no subgroups of order p^2. Therefore G/B has order p. □

Lemma 3. *Let G be a non-periodic ES-group. Then G is a Baer group.*

Proof. Let u be an element of infinite order of G. If p_1 and p_2 are distinct primes, the subgroups $\langle u^{p_1}\rangle$ and $\langle u^{p_2}\rangle$ are properly contained in their normalizers, and hence are subnormal in G. Thus also $\langle u\rangle = \langle u^{p_1}, u^{p_2}\rangle$ is a subnormal subgroup of G. Therefore it is enough to prove that G is generated by its elements of infinite order. Assume that this is false, so that G contains two elements of finite order x and y such that $z = xy$ has infinite order. Since $\langle z\rangle$ is subnormal in G, the element z belongs to the Baer radical B of $K = \langle x, y\rangle$. Clearly $K = \langle x, B\rangle$, so that K/B is finite and B is finitely generated. It follows that B is nilpotent, and K is polycyclic. In particular K is a finite extension of a torsion-free nilpotent group. Since K is not nilpotent, it contains a torsion-free nilpotent normal subgroup N such that K/N is a finite non-nilpotent group (see [5]). As the elements of infinite order generate subnormal subgroups, there exists an element a of K, whose order is a power of a prime number q, such that the coset aN does not belong to the Fitting subgroup F/N of K/N. The factor group $\langle a, N\rangle/N^q$ is a finite q-group, and in particular $\langle a, N^q\rangle$ is a subnormal subgroup of $\langle a, N\rangle$. As N is a finitely generated torsion-free nilpotent group, we have that $N^q \neq N$, and hence also $\langle a, N^q\rangle \neq \langle a, N\rangle$, since $\langle a\rangle \cap N = 1$. Therefore $\langle a, N^q\rangle$ is properly contained in its normalizer, and so is subnormal in G. Thus $\langle a, N\rangle$ is a subnormal subgroup of K, which is impossible, as aN does not belong to F/N. □

Proof of Theorem A. Let G be an ES-group. Then G is periodic by Lemma 3, and in order to prove that G is locally finite, we may assume by contradiction that G is a finitely generated infinite group. If the finite residual R or G has finite index, then R is finitely generated and has no proper subgroups of finite index, contrary to the assumption that G is locally graded. Therefore G/R is infinite, and we may suppose that G is residually finite. Assume first that G is residually (finite nilpotent), so that in particular every finite subgroup of G is nilpotent. Let h be an element of the Hirsch-Plotkin radical H of G, and let N be a normal subgroup of finite index of G such that $N \cap \langle h\rangle = 1$. Clearly the factor group G/H is infinite, and hence there exists an element $x \in N\backslash H$. The subgroup $\langle x, H\rangle$ is locally finite, so that $\langle x, h\rangle$ is finite, and thus nilpotent. On the other hand, $\langle x\rangle$ is not subnormal in G, so that $N_G(\langle x\rangle) = \langle x\rangle$, and hence $\langle x, h\rangle = \langle x\rangle$. Therefore $h = 1$ and G has trivial Hirsch-Plotkin radical. In particular every cyclic non-trivial subgroup of G is self-normalizing, and so each finite non-trivial subgroup of G has prime order. Let p be a prime in the set $\pi(G)$. Every quotient of G which is a finite p-group has exponent p, and it follows from a result of Kostrikin that its order is bounded by a function $d = d(r, p)$, where r is the minimum number of generators of G (see [8]). Therefore we may consider a normal subgroup M of G such that G/M is a finite p-group of maximal order. Let $(M_i)_{i\in I}$ be a system of G-invariant subgroups of M such that every G/M_i is a finite nilpotent group and $\bigcap_{i\in I} M_i = 1$. If y is an element of G of order p, we obtain that $[y, M] \leq M_i$ for all i, since M/M_i has order prime to p. Then $[y, M] \leq \bigcap_{i\in I} M_i = 1$, so that $\langle y\rangle$ is properly contained in its normalizer, and this contradiction shows that G is not residually (finite nilpotent). Since G is countable, there exists a descending chain $(K_n)_{n\in \mathbb{N}}$ of normal subgroups of G such that every G/K_n is a finite non-nilpotent group and $\bigcap_{n\in N} K_n = 1$. If F_n/K_n is the Fitting subgroup of G/K_n, then F_n is contained in F_m for each $m \leq n$. Since every F_n has prime index in G by Lemma 2, we obtain that $F_n = F_1$ for each positive integer n, and so F_1/K_n is nilpotent for all n. Therefore the finitely generated group F_1 is residually (finite nilpotent), and it follows from the first part of the proof that F_1 is finite. Thus G is finite, and this contradiction proves that G is locally finite.

If B is the Baer radical of G, the factor group G/B has prime order p by Lemma 2. Thus $G = \langle x, B\rangle$, where x is an element of order a power of p and x^p belongs to B. Let P be the unique Sylow p-subgroup of B. Then $\langle x, P\rangle$ is a locally finite p-group, and $N_{\langle x,P\rangle}(\langle x\rangle) = \langle x\rangle$, since $\langle x\rangle$ is not subnormal in G. Therefore $\langle x, P\rangle = \langle x\rangle$, and $\langle x\rangle$ is a Sylow p-subgroup of G. Clearly $B = P \times Q$, where Q is a G-invariant p'-subgroup, and hence $G = \langle x, B\rangle = \langle x, Q\rangle = \langle x\rangle \ltimes Q$. Moreover $C_{\langle x\rangle}(Q) = \langle x^p\rangle$, and $C_Q(x) = 1$ since $N_G(\langle x\rangle) = \langle x\rangle$. Then x acts on Q as a fixed-point-free automorphism of order p, and it follows from a result of G. Higman that the locally nilpotent group Q is nilpotent (see [7], Theorem 3).

Suppose now that $G = \langle x\rangle \ltimes Q$, where x has order a power of a prime p, and Q is a periodic nilpotent group without elements of order p such that $C_Q(x) = 1$ and $C_{\langle x\rangle}(Q) = \langle x^p\rangle$. Clearly the Sylow p-subgroups of G are conjugate to $\langle x\rangle$. Moreover $N_G(\langle x\rangle) = \langle x\rangle \times (Q\cap N_G(\langle x\rangle))$, and hence $N_G(\langle x\rangle) = \langle x\rangle$. Thus a standard argument shows that every subgroup of G containing a Sylow p-subgroup is self-normalizing. Let K be a subgroup of G which is properly contained in its normalizer. Then K contains no Sylow p-subgroups of G, and hence it lies in the nilpotent normal subgroup $\langle x^p, Q\rangle$ of G. If c is the nilpotency class of $\langle x^p, Q\rangle$, it follows that K is subnormal in G with defect at most $c + 1$. Therefore G is an ES_{c+1}-group. □

The following lemma uses an argument introduced by Casolo in [2].

Lemma 4. *Let G be a subsoluble group whose hyperabelian subnormal subgroups are soluble. Then G is soluble.*

Proof. Clearly it is enough to prove that G is hyperabelian. If N is a normal subgroup of G having an ascending G-invariant series with abelian factors, the hypotheses are inherited by the factor group G/N. Hence we have only to show that G contains a non-trivial abelian normal subgroup. Let c be the minimum positive integer such that G has a non-trivial abelian subnormal subgroup H with defect c. Since H has defect $c-1$ in its normal closure H^G, we may suppose by induction on c that H^G has a non-trivial abelian normal subgroup. Then the Fitting subgroup F of H^G is a non-trivial soluble group. The smallest non-trivial term of the derived series of F is an abelian normal subgroup of G. The lemma is proved. □

Proof of Theorem B. Suppose first that G is hyperabelian, and assume by contradiction that G contains a subgroup K which is not subnormal. The commutator subgroup $L = K'$ of K is properly contained in K, so that $L < N_G(L)$ and L is subnormal in G. Let

$$L = L_0 \triangleleft L_1 \triangleleft \dots \triangleleft L_n = G$$

be the standard series of L in G, and let $i < n$ be the maximum such that L_i is contained in K. Then L_{i+1} is not contained in K, and K is a proper subgroup of $\langle K, L_{i+1}\rangle$. As L is normal in K, it is well-known that K normalizes every term of the standard series of L. Hence L_i is normal in $K\,L_{i+1}$, and the proper subgroup K/L_i of $K\,L_{i+1}/L_i$ is not subnormal, since $N_G(K) = K$. Thus we may suppose that K is abelian, so that every proper subgroup of K is subnormal in G. It follows that K cannot be the product of two proper subgroups, and in particular each homomorphic image of K is indecomposable. Assume that $K^q \neq K$ for a certain prime q. Then the factor group K/K^q has order q, and $K = K^q\langle x\rangle$ is subnormal, since G is a Baer group. This contradiction shows that K is radicable, and hence K is a group of type p^∞ for a certain prime p.

As K is properly contained in the subgroup T of all elements of finite order of G, we may suppose that the group G is periodic. Let

$$1 = G_0 \leq G_1 \leq \dots \leq G_\tau = G$$

be an ascending normal series with abelian factors of G, and let $\alpha \leq \tau$ be the least ordinal such that G_α is not contained in K. Clearly α is not a limit ordinal, and $G_{\alpha-1}$ lies in K. Thus $K/G_{\alpha-1}$ is a self normalizing proper subgroup of $K\,G_\alpha/G_{\alpha-1}$, and hence without loss of generality we may suppose that $G = AK$, where A is an abelian normal subgroup of G. Thus the intersection $A \cap K$ is a normal subgroup of G, and the factor group $G/(A \cap K)$ is a counterexample, so that it can also be assumed that $A \cap K = 1$. Let a be a non-trivial element of A. As G is locally nilpotent, a does not belong to the normal subgroup $[a, K]$ of G. Moreover, from $K[a,K] \cap A = [a,K]$ it follows that a is not in $K[a,K]$. Clearly $K[a,K]/[a,K]$ is centralized by a, so that $K[a,K]$ is properly contained in its normalizer, and hence is subnormal in G. Since $K[a,K]/[a,K] \simeq K$ is a subgroup of type p^∞, its normal closure $M/[a,K]$ in $G/[a,K]$ is a radicable abelian group (see [14] Part 1, Lemma 4.46). As $G/[a,K]$ is a periodic Baer group, it follows that $M/[a,K]$ lies in the centre of $G/[a,K]$ (see [14] Part 1, Lemma 3.13). Therefore $G' = [A,K] \leq [a,K]$, so that a does not belong to G', and $A \cap G' = 1$. Hence G is abelian, and this contradiction proves that each subgroup of G is subnormal when G is hyperabelian.

In the general case, every hyperabelian subgroup of G has all its subgroups subnormal, and hence is soluble by Satz 7 of [13] (see also [1]). Since a Baer group is obviously subsoluble, it follows from Lemma 4 that G is soluble. □

Finally, we note the following property of arbitrary ES-groups.

Proposition 5. *Let G be an ES-group. Then the join of every pair of subnormal subgroups of G is subnormal.*

Proof. Let H and K be subnormal subgroups of G. If the commutator subgroup $[H,K]$ is properly contained in $\langle H,K\rangle$, it follows that $[H,K] < N_G([H,K])$, and hence $[H,K]$ is subnormal in G. Thus also $\langle H,K\rangle$ is subnormal in G (see [9], Theorem 1.2.3). Suppose that $[H,K] = \langle H,K\rangle$. Then $[H,K] = H = K$, since H and K are subnormal, so that $\langle H,K\rangle = H = K$ is subnormal in G. □

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