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On Certain Fractional Differential Equations Involving Generalized Multivariable Mittag – Leffler Function

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Abstract. In this paper some fractional differential equations are solved by using the Laplace transform method. These fractional differential equations with Riemann – Liouville fractional derivative operators are very general in nature and involve the wide range of fractional differential equations and their solutions in terms of various functions related to Mittag – Leffler function.

Keywords: Fractional differential equations, Riemann–Liouville fractional derivative operator, Generalized Mittag–Leffler function, confluent hypergeometric function, Laplace transform

MSC 2000 classification: 33E12, 34A08

1 Introduction

The entire function of the form

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)} \tag{1}$$

where $\alpha \in C$; $Re(\alpha) > 0$; $z \in C$ defines the Mittag – Leffler function [2,3].
A generalization of Mittag – Leffler function $E_{\alpha}(z)$ of (1) is defined and studied by Wiman [11] as follows:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \tag{2}$$

where $\alpha, \beta \in C$; $Re(\alpha) > 0$; $Re(\beta) > 0$; $z \in C$.

A generalization of Mittag – Leffler function $E_{\alpha,\beta}(z)$ of (2) is introduced by Prabhakar [5, p.7] as follows:

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!} \tag{3}$$

where $\alpha,\beta,\gamma\in C$; $Re(\alpha)>0$; $Re(\beta)>0$; $z\in C$ and $(\lambda)_n$denotes the familiar Pochhammer symbol or the shifted factorial , since

$$(1)_n=n!\,(n\in N_0)$$

and

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1 & (n=0;\lambda\in C\setminus\{0\})\\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n\in N;\lambda\in C)\end{cases} \tag{4}$$

$$\left(N_0=N\bigcup\{0\}=\{0,1,2...\}\right)$$

Recently generalization of Mittag–Leffler function $E^{\gamma}_{\alpha,\beta}(z)$ of (3) studied by Srivastava and Tomovski [10] is defined as follows:

$$E^{\gamma,K}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{Kn}}{\Gamma(\alpha n+\beta)}\frac{z^n}{n!} \tag{5}$$

$$(z,\beta,\gamma\in C;Re(\alpha)>\max\{0,Re(K)-1\};Re(K)>0)$$

which, in the special case when

$$K=q\ \ (q\in(0,1)\cup N)\text{ and }\min\ \{Re(\beta),Re(\gamma)\}>0 \tag{6}$$

was considered earlier by Shukla and Prajapati [8].
A multivariable analogue of Mittag–Leffler function defined in (3) is very recently studied by Gautam [1] and Saxena et al. [7, p.536, Eq. 1.14] in the following form:

$$E^{(\gamma_j)}_{(\rho_j),\lambda}[z_1,...,z_r]=$$
$$E^{(\gamma_1,...,\gamma_r)}_{(\rho_1,...,\rho_r),\lambda}[z_1,...,z_r]=\sum_{k_1,...,k_r=0}^{\infty}\frac{(\gamma_1)_{k_1}...(\gamma_r)_{k_r}}{\Gamma(\lambda+k_1\rho_1+...+k_r\rho_r)}\frac{z_1^{k_1}...z_r^{k_r}}{(k_1)!...(k_r)!} \tag{7}$$

where λ, γ_j, $\rho_j\in C$; $Re(\rho_j)>0$; $j=1,2,...,r$.

If in (7) we take $\rho_1 = \rho_2 = ... = \rho_r = 1$then it reduces to the following confluent hypergeometric series [9, p. 34, Eq. (1.4(8))]

$$\Phi_2^{(r)}\left[\gamma_1, ..., \gamma_r\ ; \lambda; z_1, ..., z_r\right] = \frac{1}{\Gamma(\lambda)} \sum_{k_1,...,k_r=0}^{\infty} \frac{\prod_{i=1}^{r} (\gamma_i)_{k_i}}{(\mu)_{k_1+...+k_r}} \frac{z_1^{k_1}...z_r^{k_r}}{(k_1)!...(k_r)!} \tag{8}$$

where $\lambda,\ \gamma_j,\ z_j \in C\ (j = 1, 2, ..., r)$ and max $\{|z_1|, ..., |z_r|\} < 1; \lambda \notin Z_0^- = \{0, -1, -2, ...\}$.
A mild generalization of multivariable analogue of Mittag – Leffler function in (7) is also due to Saxena et al. defined as follows [7, p. 547, Eq.7.1]:

$$E_{(\rho_r),\lambda}^{(\gamma_r),(l_r)}(z_1, ..., z_r) = \sum_{k_1,...,k_r=0}^{\infty} \frac{(\gamma_1)_{k_1 l_1} ... (\gamma_r)_{k_r l_r}}{\Gamma(\lambda + k_1\rho_1 + ... + k_r\rho_r)} \frac{z_1^{k_1}...z_r^{k_r}}{(k_1)!...(k_r)!} \tag{9}$$

where $\lambda,\ \gamma_j, l_j,\ \rho_j \in C;\ Re(\rho_j) > 0;\ Re(l_j) > 0;\ \lambda \notin Z_0^- = \{0, -1, -2, ...\}; j = 1, 2, ..., r$.
We consider the following integral operator involving the above generalized multivariable Mittag – Leffler function in the kernel is defined and represented as follow:

$$\left(E_{(\rho_r),\mu,(\omega_r);a+}^{(\gamma_r),(l_r)} \Psi\right)(x) = \int_a^x (x-t)^{\mu-1} E_{(\rho_r),\mu}^{(\gamma_r),(l_r)} [\omega_1 (x-t)^{\rho_1}, ..., \omega_r (x-t)^{\rho_r}] \Psi(t)\, dt \tag{10}$$

with $x > a;\ \omega_j,\ \rho_j,\ \gamma_j,\ l_j,\ \mu \in C;\ Re(\rho_j) > 0;\ |\omega_j (x-t)^{\rho_j}| < 1;\ j = 1, 2, ..., r$.
Remark: At $l_1 = l_2 = ... = l_r = 1$ the operator defined in (10) reduces to the integral operator studied by Gautam [1] and Saxena et al. [7].

In the present paper we consider the following type of differential equations

$$\left(D_{0+}^{\alpha} y\right)(x) = \lambda \left(E_{(\rho_r),\mu,(\omega_r);0+}^{(\gamma_r),(l_r)}\right)(x) + f(x) \tag{11}$$

with the initial condition

$$\left(I_{0+}^{1-\alpha} y\right)(0+) = c$$

where c is an arbitrary constant and $(\alpha,\ \mu,\ \rho_j,\ \gamma_j,\ l_j,\ \omega_j \in C;\ Re(\alpha) > 0;\ Re(\rho_j) > 0;\ Re(\mu) > 0;\ Re(l_j) > 0; j = 1, 2, ..., r)$.
Here D_{0+}^{α} is Riemann – Liouville fractional derivative operator defined by [6]:

$$\left(D_{a+}^{\alpha} \Psi\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{a+1}^{n-\alpha} \Psi\right)(x) \tag{12}$$

$$\alpha \in C : Re(\alpha) > 0\, (n = |Re(\alpha)| + 1)\,.$$

where $(I_{a+}^{\alpha}\Psi)(x)$ is the Riemann – Liouville fractional integral operator defined by

$$\left(I_{a+}^{\alpha}\Psi\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\frac{\Psi(t)}{(x-t)^{1-\alpha}}dt \tag{13}$$

$$(\alpha\in C;Re(\alpha)>0)$$

For $a=0$ the operator $\left(D_{a+}^{\alpha}\Psi\right)(x)$ is represented by $\left(D_{0+}^{\alpha}\Psi\right)(x)$.
The Laplace transform of a function $f\,(x)$ is defined by

$$L\left[f(x);s\right]=\int_{0}^{\infty}e^{-sx}f(x)\,dx=F(s) \tag{14}$$

The Laplace transform of fractional derivative $\left(D_{0+}^{\alpha}f\right)(x)$ is given by [4]:

$$L\left[D_{0+}^{\alpha}f;s\right]=s^{\alpha}F(s)-\sum_{k=1}^{n}s^{k-1}D_{0+}^{\alpha-k}f(0+)\,(n-1<\alpha<n) \tag{15}$$

$$Re(s)>0$$

The Laplace transform of the function $E_{(\rho_r),\mu}^{(\gamma_r),(l_r)}$ [.] defined in (9) is easily obtainable in the following form also required here

$$L\left\{x^{\mu-1}E_{(\rho_r),\mu}^{(\gamma_r),(l_r)}\left[\omega_1x^{\rho_1},...,\omega_rx^{\rho_r}\right]\right\}(s)=s^{-\mu}\prod_{i=1}^{r}\left\{{}_1\Psi_0^{*}\left[\begin{array}{c}(\gamma_i,l_i)\\ -\end{array};\frac{\omega_i}{s^{\rho_i}}\right]\right\} \tag{16}$$

$(\mu,\rho_j,\gamma_j,l_j,\omega_j\in C;Re(s)>0;Re(\rho_j)>0;Re(\mu)>0;Re(l_j)>0;j=1,2,...,r)$

where ${}_1\Psi_0^{*}$ is the Fox – Wright hypergeometric function defined as follows [9]:

$${}_p\Psi_q\left[\begin{array}{cc}(a_1,A_1),...,(a_p,A_p) & ;\\ (b_1,B_1),...,(b_q,B_q) & ;\end{array}z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_{A_in}}{\prod_{j=1}^{q}(b_j)_{B_jn}}\frac{z^n}{n!} \tag{17}$$

The following integral involving the generalized Mittag – Leffler function in (9) is also required

$$\frac{1}{\Gamma(\sigma)}\int_{0}^{x}(x-t)^{\mu-1}t^{\sigma-1}E_{(\rho_r),\mu}^{(\gamma_r),(l_r)}\left[\omega_1(x-t)^{\rho_1},...,\omega_r(x-t)^{\rho_r}\right]dt$$

$$=x^{\mu+\sigma-1}E_{(\rho_r),\sigma+\mu}^{(\gamma_r),(l_r)}\left[\omega_1x^{\rho_1},...,\omega_rx^{\rho_r}\right] \tag{18}$$

The integral in (18) is established in view of (9) and elementary beta integral.

We also need the following familiar derivative formula for Laplace transform here

$$\frac{d^n}{ds^n}[L[y(x)(s)] = (-1)^n L[x^n y(x)](s) \tag{19}$$

2 Main Results

Theorem 1. *Let $a \in R_+$; α, μ, ρ_j, γ_j, l_j, $\omega_j \in C$; $Re(\alpha) > 0$; $Re(\rho_j) > 0$; $Re(\mu) > 0$; $Re(l_j) > 0$; $j = 1, 2, ..., r$. Then for $x > a$, there hold the relations.*

$$D^{\alpha}_{a+}\left[(t-a)^{\mu-1}E^{(\gamma_r),(l_r)}_{(\rho_r),\mu}[\omega_1(t-a)^{\rho_1}, ..., \omega_r(t-a)^{\rho_r}]\right](x)$$

$$= (x-a)^{\mu-\alpha-1}\, E^{(\gamma_r),(l_r)}_{(\rho_r),\mu-\alpha}\,[\omega_1(x-a)^{\rho_1}, ..., \omega_r(x-a)^{\rho_r}] \tag{1}$$

and

$$I^{\alpha}_{a+}\left[(t-a)^{\mu-1}E^{(\gamma_r),(l_r)}_{(\rho_r),\mu}[\omega_1(t-a)^{\rho_1}, ..., \omega_r(t-a)^{\rho_r}]\right](x)$$

$$= (x-a)^{\mu+\alpha-1}\, E^{(\gamma_r),(l_r)}_{(\rho_r),\mu+\alpha}\,[\omega_1(x-a)^{\rho_1}, ..., \omega_r(x-a)^{\rho_r}] \tag{2}$$

Theorem 2. *The following fractional differential equation :*

$$\left(D^{\alpha}_{0+}y\right)(x) = \lambda\left(E^{(\gamma_r),(l_r)}_{(\rho_r),\mu,(\omega_r);0+}\right)(x) + f(x) \tag{3}$$

($\alpha, \mu, \rho_j, \gamma_j, l_j, \omega_j \in C; Re(\alpha) > 0; Re(\rho_j) > 0; Re(\mu) > 0; Re(l_j) > 0; j = 1, 2, ..., r.$) with the initial condition $\left(I^{1-\alpha}_{0+}y\right)(0+) = c$ has its solution in the space $L(0,\infty)$ given by

$$y(x) = c\frac{x^{\alpha-1}}{\Gamma(\alpha)} + \lambda x^{\mu+\alpha}E^{(\gamma_r),(l_r)}_{(\rho_r),\mu+\alpha+1}\,(\omega_1 x^{\rho_1}, ..., \omega_r x^{\rho_r}) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\, dt \tag{4}$$

where c is an arbitrary constant .

Theorem 3. *The following fractional differential equation:*

$$\left(D^{\alpha}_{0+}y\right)(x) = \lambda\left(E^{(\gamma_r),(l_r)}_{(\rho_r),\mu,(\omega_r);0+}\right)(x) + px^{\mu}E^{(\gamma_r),(l_r)}_{(\rho_r),\mu+1}\,[\omega_1 x^{\rho_1}, ..., \omega_r x^{\rho_r}] \tag{5}$$

(α, μ, ρ_j, γ_j, l_j, $\omega_j \in C$; $Re(\alpha) > 0$; $Re(\rho_j) > 0$; $Re(\mu) > 0$; $Re(l_j) > 0$; $j = 1, 2, ..., r$) with the initial condition $\left(I^{1-\alpha}_{0+}y\right)(0+) = c$ has its solution in the space $L(0,\infty)$ given by

$$y(x) = c\frac{x^{\alpha-1}}{\Gamma(\alpha)} + (\lambda + p)\, x^{\mu+\alpha}E^{(\gamma_r),(l_r)}_{(\rho_r),\mu+\alpha+1}\,[\omega_1 x^{\rho_1}, ..., \omega_r x^{\rho_r}] \tag{6}$$

where c is an arbitrary constant .

Theorem 4. *The following fractional differential equation:*

$$x\left(D_{0+}^{\alpha}y\right)(x)=\lambda\left(E_{(\rho_r),\mu,(\omega_r);0+}^{(\gamma_r),(l_r)}\right)(x) \tag{7}$$

$(\alpha,\mu,\rho_j,\gamma_j,l_j,\omega_j\in C; Re(\alpha)>0; Re(\rho_j)>0; Re(\mu)>0; Re(l_j)>0; j=1,2,...,r)$ *with the initial condition* $\left(I_{0+}^{1-\alpha}y\right)(0+)=c$ *has its solution in the space* $L(0,\infty)$ *given by*

$$y(x)=c\frac{x^{\alpha-1}}{\Gamma(\alpha)}-\frac{\lambda}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}(x-t)^{\mu-1}E_{(\rho_r),\mu+1}^{(\gamma_r),(l_r)}\left[\omega_1(x-t)^{\rho_1},...,\omega_r(x-t)^{\rho_r}\right]dt \tag{8}$$

where c is an arbitrary constant.

Theorem 5. *The following fractional differential equation:*

$$\left(D_{0+}^{\alpha}y\right)(x)=\lambda\left(E_{(\rho_r),\mu,(\omega_r);0+}^{(\gamma_r),(l_r)}\right)(x)+\sum_{j=1}^{n}\left[p_jx^{\mu_j}E_{(\rho_r^{(j)}),\mu_j+1}^{(\gamma_r^{(j)}),(l_r^{(j)})}[\omega_1^{(j)}x^{\rho_1^{(j)}},..,\omega_r^{(j)}x^{\rho_r^{(j)}}]\right] \tag{9}$$

with the initial condition $\left(I_{0+}^{1-\alpha}y\right)(0+)=c$ *has its solution in the space* $L(0,\infty)$ *given by*

$$\begin{aligned} y(x)=&c\tfrac{x^{\alpha-1}}{\Gamma(\alpha)}+\lambda x^{\mu+\alpha}E_{(\rho_r),\mu+\alpha+1}^{(\gamma_r),(l_r)}\left(\omega_1x^{\rho_1},...,\omega_rx^{\rho_r}\right)\\ &+\sum_{j=1}^{n}\left[p_jx^{\mu_j+\alpha}E_{(\rho_r^{(j)}),\mu_j+\alpha+1}^{(\gamma_r^{(j)}),(l_r^{(j)})}\left(\omega_1^{(j)}x^{\rho_1^{(j)}},...,\omega_r^{(j)}x^{\rho_r^{(j)}}\right)\right] \end{aligned} \tag{10}$$

where c is an arbitrary constant.

Outline of Proofs

Proof of (1)**:** To prove the assertion (1) of Theorem-1, we denote its left hand side by Δ_1 i.e.

$$\Delta_1=D_{a+}^{\alpha}\left[(t-a)^{\mu-1}E_{(\rho_r),\mu}^{(\gamma_r),(l_r)}[\omega_1(t-a)^{\rho_1},...,\omega_r(t-a)^{\rho_r}]\right](x)$$

On using the definition of $E_{(\rho_r),\mu}^{(\gamma_r),(l_r)}[.]$ given in (9), we obtain

$$\Delta_1=D_{a_+}^{\alpha}\left[(t-a)^{\mu-1}\sum_{k_1,...,k_r=0}^{\infty}\prod_{i=1}^{r}\left[\frac{(\gamma_i)_{k_il_i}}{(k_i)!}(\omega_i)^{k_i}\right]\frac{(t-a)^{\sum_{i=1}^{r}\rho_ik_i}}{\Gamma\left(\mu+\sum_{i=1}^{r}\rho_ik_i\right)}\right](x)$$

i.e.

$$\Delta_1 = \sum_{k_1,\ldots,k_r=0}^{\infty} \prod_{i=1}^{r} \left[\frac{(\gamma_i)_{k_i l_i}}{(k_i)!} (\omega_i)^{k_i} \right] \frac{1}{\Gamma\left(\mu + \sum_{i=1}^{r} \rho_i k_i\right)} D_{a_+}^{\alpha} \left[(t-a)^{\mu + \sum_{i=1}^{r} \rho_i k_i - 1} \right](x)$$

On using the fractional derivative [6] $D_{a+}^{\alpha}\left[(t-a)^{\lambda}\right](x) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}(x-a)^{\lambda-\alpha}$ we have

$$\Delta_1 = \sum_{k_1,\ldots,k_r=0}^{\infty} \prod_{i=1}^{r} \left[\frac{(\gamma_i)_{k_i l_i}}{(k_i)!} (\omega_i)^{k_i} \right] \frac{(x-a)^{\mu + \sum_{i=1}^{r} \rho_i k_i - \alpha - 1}}{\Gamma\left(\mu - \alpha + \sum_{i=1}^{r} \rho_i k_i\right)}$$

On interpreting in view of the definition of $E_{(\rho_r),\mu}^{(\gamma_r)(l_r)}[.]$ given in (9), we at once arrive at the desired result in (1).

Proof of (2)**:** The assertion (2) of Theorem-2 is proved similarly following the lines as to prove the result in (1) and using the definition of fractional integral operator I_{a+}^{α} therein.

Proof of (4)**:** On using the definition of operator $\left(E_{(\rho_r),\mu,(\omega_r);a+}^{(\gamma_r),(l_r)}\Psi\right)(x)$ given in (10) with ($a = 0$ and $\Psi(x) = 1$) and formula (18) with ($\sigma = 1$) in equation (3), it takes the following form

$$\left(D_{0+}^{\alpha} y\right)(x) = \lambda x^{\mu} E_{(\rho_r),\mu+1}^{(\gamma_r),(l_r)}\left[\omega_1 x^{\rho_1}, \ldots, \omega_r x^{\rho_r}\right] + f(x) \tag{11}$$

By applying Laplace transform on both sides of (11) and then using formulae (15) with ($n = 1$), (16) and (14) we obtain

$$s^{\alpha} y(s) = c + \lambda s^{-\mu-1} \prod_{i=1}^{r} \left\{ {}_1\Psi_0^{*}\left[\begin{matrix} (\gamma_i, l_i) \\ - \end{matrix} ; \frac{\omega_i}{s^{\rho_i}} \right] \right\} + F(s)$$

It yields

$$y(s) = c s^{-\alpha} + \lambda s^{-\mu-\alpha-1} \prod_{i=1}^{r} \left\{ {}_1\Psi_0^{*}\left[\begin{matrix} (\gamma_i, l_i) \\ - \end{matrix} ; \frac{\omega_i}{s^{\rho_i}} \right] \right\} + F(s) \cdot s^{-\alpha}$$

in view of (17) , we have

$$y(s) = c s^{-\alpha} + \lambda \sum_{k_1,\ldots,k_r=0}^{\infty} \prod_{i=1}^{r} \left[\frac{(\gamma_i)_{k_i l_i}}{(k_i)!} (\omega_i)^{k_i} \right] s^{-\mu-\alpha-\sum_{i=1}^{r} \rho_i k_i - 1} + F(s) \cdot s^{-\alpha} \tag{12}$$

Now taking the inverse Laplace transform on both sides of (12), we find by means of the Laplace convolution theorem that

$$y(x) = c\frac{x^{\alpha-1}}{\Gamma(\alpha)}+\lambda \sum_{k_1,\ldots,k_r=0}^{\infty} \prod_{i=1}^{r} \left[\frac{(\gamma_i)_{k_i l_i}}{(k_i)!}(\omega_i)^{k_i}\right] L^{-1}\left[s^{-\mu-\alpha-\sum_{i=1}^{r}\rho_i k_i-1}\right]+L^{-1}[F(s)\cdot s^{-\alpha}]$$

i.e.

$$y(x) = c\frac{x^{\alpha-1}}{\Gamma(\alpha)} + \lambda \sum_{k_1,\ldots,k_r=0}^{\infty} \prod_{i=1}^{r} \left[\frac{(\gamma_i)_{k_i l_i}}{(k_i)!}(\omega_i)^{k_i}\right] \frac{(x)^{\mu+\sum_{i=1}^{r}\rho_i k_i+\alpha}}{\Gamma\left(\mu+\alpha+\sum_{i=1}^{r}\rho_i k_i+1\right)} + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\, dt$$

Now on interpreting second term in view of the definition of $E^{(\gamma_r),(l_r)}_{(\rho_r),\mu}$ [.] given in (9), we at once arrive at the desired result in (4).

Proof of (6)**:** On using the definition of operator $\left(E^{(\gamma_r),(l_r)}_{(\rho_r),\mu,(\omega_r);a+}\Psi\right)(x)$ given in (10) with ($a = 0$ and $\Psi(x) = 1$) and formula (18) with ($\sigma = 1$) in equation (5), it takes the following form

$$\left(D^{\alpha}_{0+}y\right)(x) = (\lambda+p)x^{\mu}E^{(\gamma_r),(l_r)}_{(\rho_r),\mu+1}\left[\omega_1 x^{\rho_1},\ldots,\omega_r x^{\rho_r}\right] \tag{13}$$

By applying Laplace transform on both sides of (13) and then using formulae (15) with ($n = 1$) and (16), we get

$$y(s) = cs^{-\alpha} + (\lambda+p)\, s^{-\mu-\alpha-1}\prod_{i=1}^{r}\left\{{}_1\Psi_0^{*}\left[\begin{array}{c}(\gamma_i, l_i)\\ -\end{array};\frac{\omega_i}{s^{\rho_i}}\right]\right\}$$

in view of (17) we obtain

$$y(s) = cs^{-\alpha} + (\lambda+p)\sum_{k_1,\ldots,k_r=0}^{\infty} \prod_{i=1}^{r} \left[\frac{(\gamma_i)_{k_i l_i}}{(k_i)!}(\omega_i)^{k_i}\right]\left[s^{-\mu-\alpha-\sum_{i=1}^{r}\rho_i k_i-1}\right] \tag{14}$$

On taking inverse Laplace transform on both sides of (14), we have

$$y(x) = c\frac{x^{\alpha-1}}{\Gamma(\alpha)} + (\lambda+p)\sum_{k_1,\ldots,k_r=0}^{\infty} \prod_{i=1}^{r} \left[\frac{(\gamma_i)_{k_i l_i}}{(k_i)!}(\omega_i)^{k_i}\right]\frac{x^{\mu+\alpha+\sum_{i=1}^{r}\rho_i k_i}}{\Gamma\left(\mu+\alpha+\sum_{i=1}^{r}\rho_i k_i+1\right)}$$

Now on interpreting with help of definition of $E^{(\gamma_r),(l_r)}_{(\rho_r),\mu}$[.] given in (9), we at once arrive at the desired result in (6).

Proof of (8)**:** On using the definition of operator $\left(E^{(\gamma_r),(l_r)}_{(\rho_r),\mu,(\omega_r);a+}\Psi\right)(x)$ given in (10) with ($a = 0$ and $\Psi(x) = 1$) and formula (18) with ($\sigma = 1$) in equation (7), it takes the following form

$$x\left(D^{\alpha}_{0+}y\right)(x) = \lambda x^{\mu} E^{(\gamma_r),(l_r)}_{(\rho_r),\mu+1}\left[\omega_1 x^{\rho_1}, ..., \omega_r x^{\rho_r}\right] \tag{15}$$

By applying Laplace transform on both sides of (15) and then using formulae (16) and (19), (15) with ($n = 1$), we get

$$\frac{d}{ds}y(s) + \frac{\alpha}{s}y(s) = -\lambda s^{-\mu-\alpha-1}\prod_{i=1}^{r}\left\{{}_1\Psi_0^*\left[\begin{array}{c}(\gamma_i, l_i)\\ -\end{array}; \frac{\omega_i}{s^{\rho_i}}\right]\right\}$$

in view of (17), we have

$$\frac{d}{ds}y(s) + \frac{\alpha}{s}y(s) = -\lambda\sum_{k_1,...,k_r=0}^{\infty}\prod_{i=1}^{r}\left[\frac{(\gamma_i)_{k_i l_i}}{(k_i)!}(\omega_i)^{k_i}\right]s^{-\mu-\alpha-\sum_{i=1}^{r}\rho_i k_i - 1}$$

This is a linear differential equation of first order and first degree. Hence

$$y(s) = -\lambda\sum_{k_1,...,k_r=0}^{\infty}\prod_{i=1}^{r}\left[\frac{(\gamma_i)_{k_i l_i}}{(k_i)!}(\omega_i)^{k_i}\right]\frac{s^{-\alpha-\mu-\sum_{i=1}^{r}\rho_i k_i}}{(\mu + \sum_{i=1}^{r}\rho_i k_i)} + cs^{-\alpha} \tag{16}$$

By applying inverse Laplace transform on both sides of (16), we obtain

$$y(x) = -\lambda\sum_{k_1,...,k_r=0}^{\infty}\prod_{i=1}^{r}\left[\frac{(\gamma_i)_{k_i l_i}}{(k_i)!}(\omega_i)^{k_i}\right]\cdot \frac{1}{(\mu + \sum_{i=1}^{r}\rho_i k_i)}L^{-1}\left[s^{-\mu-\sum_{i=1}^{r}\rho_i k_i}\cdot s^{-\alpha}\right] + CL^{-1}(s^{-\alpha})$$

On using Laplace convolution theorem, we obtain

$$y(x) = -\lambda\sum_{k_1,...,k_r=0}^{\infty}\prod_{i=1}^{r}\left[\frac{(\gamma_i)_{k_i l_i}}{(k_i)!}(\omega_i)^{k_i}\right]\cdot \frac{1}{\Gamma(\mu + 1 + \sum_{i=1}^{r}\rho_i k_i)}\frac{1}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}(x-t)^{\mu+\sum_{i=1}^{r}\rho_i k_i - 1}dt + C\frac{1}{\Gamma(\alpha)}x^{\alpha-1}$$

Now on changing the order of integration and summation, and then on interpreting the so obtained result in view of definition of $E^{(\gamma_r),(l_r)}_{(\rho_r),\mu}[.]$ given in (9), we at once arrive at the desired result in (8).

Proof of (10)**:** On using the definition of operator $\left(E^{(\gamma_r),(l_r)}_{(\rho_r),\mu,(\omega_r);a+}\Psi\right)(x)$ given in (10) with ($a=0$ and $\Psi(x)=1$) and formula (18) with ($\sigma=1$) in equation (9), it takes the following form

$$\left(D^{\alpha}_{0+}y\right)(x)=\lambda x^{\mu}E^{(\gamma_r),(l_r)}_{(\rho_r),\mu+1}\left[\omega_1 x^{\rho_1},...,\omega_r x^{\rho_r}\right]+\sum_{j=1}^{n}\left[p_j x^{\mu_j}E^{(\gamma_r^{(j)}),(l_r^{(j)})}_{(\rho_r^{(j)}),\mu_j+1}\left(\omega_1^{(j)}x^{\rho_1^{(j)}},...,\omega_r^{(j)}x^{\rho_r^{(j)}}\right)\right] \quad (17)$$

By applying Laplace transform on both sides of (17), and then using formulae (15) with ($n=1$) and (16) we obtain

$$y(s)=Cs^{-\alpha}+\lambda s^{-\mu-\alpha-1}\prod_{i=1}^{r}\left\{{}_1\Psi_0^*\left[\begin{array}{c}(\gamma_i,l_i)\\ -\end{array};\frac{\omega_i}{s^{\rho_i}}\right]\right\}+\sum_{j=1}^{n}\left[p_j s^{-\alpha-\mu_j-1}\prod_{i=1}^{r}\left\{{}_1\Psi_0^*\left[\begin{array}{c}(\gamma_i^{(j)},l_i^{(j)})\\ -\end{array};\frac{\omega_i^{(j)}}{s^{\rho_i^{(j)}}}\right]\right\}\right]$$

in view of (17), we obtain

$$y(s)=cs^{-\alpha}+\lambda\sum_{k_1,..,k_r=0}^{\infty}\prod_{i=1}^{r}\left[\frac{(\gamma_i)_{k_il_i}}{(k_i)!}(\omega_i)^{k_i}\right]s^{-\mu-\alpha-\sum_{i=1}^{r}\rho_ik_i-1}$$

$$+\sum_{j=1}^{n}p_j\left[\sum_{k_1,..,k_r=0}^{\infty}\prod_{i=1}^{r}\left(\frac{\left(\gamma_i^{(j)}\right)_{k_il_i^{(j)}}}{(k_i)!}(\omega_i^{(j)})^{k_i}\right)\right]s^{-\mu_j-\alpha-\sum_{i=1}^{r}\rho_i^{(j)}k_i-1} \quad (18)$$

On applying the inverse Laplace transform on both sides of (18) we have

$$y(x)=c\frac{x^{\alpha-1}}{\Gamma(\alpha)}+\lambda\sum_{k_1,...,k_r=0}^{\infty}\prod_{i=1}^{r}\left[\frac{(\gamma_i)_{k_il_i}}{(k_i)!}(\omega_i)^{k_i}\right]\frac{x^{\mu+\alpha+\sum_{i=1}^{r}\rho_ik_i}}{\Gamma\left(\mu+\alpha+\sum_{i=1}^{r}\rho_ik_i+1\right)}+\sum_{j=1}^{n}p_j\left[\sum_{k_1,..,k_r=0}^{\infty}\prod_{i=1}^{r}\left(\frac{\left(\gamma_i^{(j)}\right)_{k_il_i^{(j)}}}{(k_i)!}\left(\omega_i^{(j)}\right)^{k_i}\right)\right]\frac{x^{\alpha+\mu_j+\sum_{i=1}^{r}\rho_i^{(j)}k_i}}{\Gamma\left(\mu+\alpha+\sum_{i=1}^{r}\rho_i^{(j)}k_i+1\right)}$$

Now on interpreting with the help of definition of $E^{(\gamma_r),(l_r)}_{(\rho_r),\mu}[.]$ given in (9) we at once arrive at the desired result in (10).

3 Special Cases

(1) If in Theorems-1 to 4, we take $r = 1$ then these reduce to the following results involving generalized Mittag – Leffler function due to Srivastava and Tomovski [10] as follows:

(i) Let $a \in R_+$; $\alpha,\ \mu,\ \gamma,\ \rho,\ l,\ \omega \in C$; $Re(\alpha) > 0$; $Re(\rho) > 0$; $Re(\mu) > 0$; $Re(l) > 0$. Then for $x > a$, there hold the relations

$$D_{a+}^{\alpha}\left[(t-a)^{\mu-1}\,E_{\rho,\mu}^{\gamma,l}[\omega(t-a)^{\rho}]\right](x) = (x-a)^{\mu-\alpha-1}\,E_{\rho,\mu-\alpha}^{\gamma,l}\left[\omega(x-a)^{\rho}\right] \tag{1}$$

and

$$I_{a+}^{\alpha}\left[(t-a)^{\mu-1}\,E_{\rho,\mu}^{\gamma,l}[\omega(t-a)^{\rho}]\right](x) = (x-a)^{\mu+\alpha-1}E_{\rho,\mu+\alpha}^{\gamma,l}\left[\omega(x-a)^{\rho}\right] \tag{2}$$

Corollary 1. *The following fractional differential equation*

$$\left(D_{0+}^{\alpha}y\right)(x) = \lambda\left(E_{0+,\rho,\mu}^{\omega;\gamma,l}\right)(x) + f(x) \tag{3}$$

($0 < \alpha < 1$*;* $\omega \in C$*;* $Re(\rho) > max\{0, Re(l)-1\}$*;* $min\{Re(\mu), Re(\gamma), Re(l)\} > 0$ *)*

With the initial condition
$\left(I_{0+}^{1-\alpha}y\right)(0+) = c$ *has its solution in the space* $L(0,\infty)$ *given by*

$$y(x) = c\frac{x^{\alpha-1}}{\Gamma(\alpha)} + \lambda x^{\mu+\alpha}\left(E_{\rho,\mu+\alpha+1}^{\gamma,l}\right)(\omega x^{\rho}) + \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)\,dt \tag{4}$$

where c is an arbitrary constant and $\left(E_{0+,\rho,\mu}^{\omega,\gamma,l}\right)(x)$ *is the integral operator studied by Srivastava and Tomovski [10, p.202, Eq. (12)]:*

$$\left(E_{a+;\alpha,\beta}^{\omega,\gamma,K}\phi\right)(x) = \int_a^x (x-t)^{\beta-1}E_{\alpha,\beta}^{\gamma,K}\left[\omega(x-t)^{\alpha}\right]\phi(t)dt(x>a) \tag{5}$$

$$(\gamma,\omega \in C; Re(\alpha) > \max\{0, Re(K)-1\}; \min\{Re(\beta), Re(K)\} > 0)\,.$$

Corollary 2. *The following fractional differential equation*

$$\left(D_{0+}^{\alpha}y\right)(x) = \lambda\left(E_{0+,\rho,\mu}^{\omega;\gamma,l}\right)(x) + px^{\mu}E_{\rho,\mu+1}^{\gamma,l}\left[\omega x^{\rho}\right] \tag{6}$$

($0 < \alpha < 1$*;* $\omega \in C$*;* $Re(\rho) > max\{0, Re(l)-1\}$*;* $min\{Re(\mu), Re(\gamma), Re(l)\} > 0$*)*

with the initial condition

$\left(I_{0+}^{1-\alpha}y\right)(0+) = c$

has its solution in the space $L(0,\infty)$ *given by*

$$y(x) = c\frac{x^{\alpha-1}}{\Gamma(\alpha)} + (\lambda + p)\, x^{\mu+\alpha} E_{\rho,\mu+\alpha+1}^{\gamma,l}\left[\omega x^{\rho}\right] \tag{7}$$

where c is an arbitrary constant and $\left(E_{0+,\rho,\mu}^{\omega,\gamma,l}\right)(x)$ *is the integral operator defined in* (5) .

Corollary 3. *The following fractional differential equation*

$$x\left(D_{0+}^{\alpha}y\right)(x) = \lambda\left(E_{0+,\rho,\mu}^{\omega,\gamma,l}\right)(x) \tag{8}$$

($0<\alpha<1;\ \omega\in C;\ Re(\rho) > max\{0, Re(l)-1\};\ min\{Re(\mu), Re(\gamma), Re(l)\} > 0$*) with the initial condition* $\left(I_{0+}^{1-\alpha}y\right)(0+) = c$ *has its solution in the space* $L(0,\infty)$ *given by*

$$y(x) = c\frac{x^{\alpha-1}}{\Gamma(\alpha)} - \frac{\lambda}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}(x-t)^{\mu-1} E_{\rho,\mu+1}^{\gamma,l}\left[\omega(x-t)^{\rho}\right]dt \tag{9}$$

where c is an arbitrary constant and $\left(E_{0+,\rho,\mu}^{\omega,\gamma,l}\right)(x)$*is the integral operator defined in* (5) .

Remark: The results in (1) to (9) may be obtained from the results of Srivastava and Tomovski [10] by taking $\nu = 0$therein.

(1) If in Theorems-1 to 4, we take $l_1 = l_2 = ... = l_r = 1$ then these reduce to the following results involving generalized Mittag–Leffler function due to Gautam [1] and Saxena et al. [7].

(i) Let $a \in R_+;\ \alpha,\ \mu,\ \rho_j,\ \gamma_j,\ \omega_j \in C;\ Re(\alpha) > 0;\ Re(\rho_j) > 0;\ Re(\mu) > 0;\ j = 1,2,...,r$. Then for $x > a$, there hold the relations

$$D_{a+}^{\alpha}\left[(t-a)^{\mu-1} E_{(\rho_r),\mu}^{(\gamma_r)}[\omega_1(t-a)^{\rho_1},...,\omega_r(t-a)^{\rho_r}]\right](x)$$

$$= (x-a)^{\mu-\alpha-1} E_{(\rho_r),\mu-\alpha}^{(\gamma_r)}[\omega_1(x-a)^{\rho_1},...,\omega_r(x-a)^{\rho_r}] \tag{10}$$

and

$$I_{a+}^{\alpha}\left[(t-a)^{\mu-1} E_{(\rho_r),\mu}^{(\gamma_r)}[\omega_1(t-a)^{\rho_1},...,\omega_r(t-a)^{\rho_r}]\right](x)$$

$$= (x-a)^{\mu+\alpha-1} E_{(\rho_r),\mu+\alpha}^{(\gamma_r)}[\omega_1(x-a)^{\rho_1},...,\omega_r(x-a)^{\rho_r}]] \tag{11}$$

Remark: The above results in (10) and (11) are known results due to Gautam [1] and Saxena et al. [7].

Corollary 4. *The following fractional differential equation*

$$\left(D_{0+}^{\alpha}y\right)(x)=\lambda\left(E_{(\rho_r),\mu,(\omega_r);0+}^{(\gamma_r)}\right)(x)+f\left(x\right) \tag{12}$$

($\alpha,\mu,\rho_j,\gamma_j,\omega_j\in C$; $Re\left(\alpha\right)>0$; $Re\left(\rho_j\right)>0$; $Re\left(\mu\right)>0$; $j=1,2,...,r$) with the initial condition $\left(I_{0+}^{1-\alpha}y\right)(0+)=c$ has its solution in the space $L\left(0,\infty\right)$ given by

$$y\left(x\right)=c\frac{x^{\alpha-1}}{\Gamma\left(\alpha\right)}+\lambda x^{\mu+\alpha}E_{(\rho_r),\mu+\alpha+1}^{(\gamma_r)}\left[\omega_1x^{\rho_1},...,\omega_rx^{\rho_r}\right]+\frac{1}{\Gamma\left(\alpha\right)}\int_0^x\left(x-t\right)^{\alpha-1}f\left(t\right)dt \tag{13}$$

where c is an arbitrary constant and $\left(E_{(\rho_r),\mu,(\omega_r);0+}^{(\gamma_r)}\right)(x)$ is the integral operator defined by Gautam [1] and Saxena et al. [8, p, 540, Eq. 4.1]:

$$\left(E_{(\rho_r),\mu,(\omega_r);a+}^{(\gamma_r)}\phi\right)(x)\phi(t)dt\int_a^x(x-a)^{\mu-1}E_{(\rho_r),\mu}^{(\gamma_r)}[\omega_1(x-t)^{\rho_1},...,\omega_r(x-t)^{\rho_r}]\phi(t)dt \tag{14}$$

$$(x>a)$$

with μ, ρ_j, γ_j, $\omega_j\in C$; $Re\left(\rho_j\right)>0$; $Re\left(\mu\right)>0$; $j=1,2,...,r$.

Corollary 5. *The following fractional differential equation*

$$\left(D_{0+}^{\alpha}y\right)(x)=\lambda\left(E_{(\rho_r),\mu,(\omega_r);0+}^{(\gamma_r)}\right)(x)+px^{\mu}E_{(\rho_r),\mu+1}^{(\gamma_r)}\left[\omega_1x^{\rho_1},...,\omega_rx^{\rho_r}\right] \tag{15}$$

($\alpha,\mu,\rho_j,\gamma_j,\omega_j\in C$; $Re\left(\alpha\right)>0$; $Re\left(\rho_j\right)>0$; $Re\left(\mu\right)>0$; $j=1,2,...,r$) with the initial condition $\left(I_{0+}^{1-\alpha}y\right)(0+)=c$ has its solution in the space $L\left(0,\infty\right)$ given by

$$y\left(x\right)=c\frac{x^{\alpha-1}}{\Gamma\left(\alpha\right)}+\left(\lambda+p\right)x^{\mu+\alpha}E_{(\rho_r),\mu+\alpha+1}^{(\gamma_r)}\left[\omega_1x^{\rho_1},...,\omega_rx^{\rho_r}\right] \tag{16}$$

where c is an arbitrary constant and $\left(E_{(\rho_r),\mu,(\omega_r);0+}^{(\gamma_r)}\right)(x)$ is the integral operator defined in (14).

Corollary 6. *The following fractional differential equation*

$$x\left(D_{0+}^{\alpha}y\right)(x)=\lambda\left(E_{(\rho_r),\mu,(\omega_r);0+}^{(\gamma_r)}\right)(x) \tag{17}$$

($\alpha,\mu,\rho_j,\gamma_j,\omega_j\in C$; $Re\left(\alpha\right)>0$; $Re\left(\rho_j\right)>0$; $Re\left(\mu\right)>0$; $j=1,2,...,r$) with the initial condition $\left(I_{0+}^{1-\alpha}y\right)(0+)=c$ has its solution in the space $L\left(0,\infty\right)$ given by

$$y\left(x\right)=c\frac{x^{\alpha-1}}{\Gamma\left(\alpha\right)}-\frac{\lambda}{\Gamma\left(\alpha\right)}\int_0^x t^{\alpha-1}\left(x-t\right)^{\mu-1}E_{(\rho_r),\mu+1}^{(\gamma_r)}\left[\omega_1\left(x-t\right)^{\rho_1},...,\omega_r\left(x-t\right)^{\rho_r}\right]dt \tag{18}$$

where c is an arbitrary constant and $\left(E^{(\gamma_r)}_{(\rho_r),\mu,(\omega_r);0+}\right)(x)$*is the integral operator defined in* (14).

(1) If in Theorems-1 to 4 we take $l_1 = l_2 = ... = l_r = 1$, $\rho_1 = \rho_2 = ... = \rho_r = 1$ *then these reduce to the results involving multivariable confluent hypergeometric function as follows:*

(i) Let $a \in R_+$; $\alpha,\ \mu,\ \gamma_j,\ \omega_j \in C$; $Re\,(\alpha) > 0$; $Re\,(\mu) > 0$; $j = 1, 2, ..., r$. *Then for* $x > a$, *there hold the relations*

$$D^{\alpha}_{a+}\left[(t-a)^{\mu-1}\,\Phi^{(r)}_2[\gamma_1, ..., \gamma_r; \mu; \omega_1\,(t-a)\,, ..., \omega_r\,(t-a)]\right](x)$$

$$= \frac{\Gamma(\mu)}{\Gamma(\mu-\alpha)}\,(x-a)^{\mu-\alpha-1}\,\Phi^{(r)}_2[\gamma_1, ..., \gamma_r; (\mu-\alpha); \omega_1\,(x-a)\,, ..., \omega_r\,(x-a)] \quad (19)$$

and

$$I^{\alpha}_{a+}\left[(t-a)^{\mu-1}\,\Phi^{(r)}_2[\gamma_1, ..., \gamma_r; \mu; \omega_1\,(t-a)\,, ..., \omega_r\,(t-a)]\right](x)$$

$$= \frac{\Gamma(\mu)}{\Gamma(\mu+\alpha)}\,(x-a)^{\mu+\alpha-1}\,\Phi^{(r)}_2[\gamma_1, ..., \gamma_r; (\mu+\alpha); \omega_1\,(x-a)\,, ..., \omega_r\,(x-a)] \quad (20)$$

Remark: The above results in (19) and (20) are also obtained by Gautam [1] and Saxena et al. [7, p.539, Eqs. (6) and (4)] as special cases of their main results.

Corollary 7. *The following fractional differential equation*

$$\left(D^{\alpha}_{0+}y\right)(x) = \lambda\left(\Phi^{(\gamma_j)}_{\mu,(\omega_j);0+}\right)(x) + f\,(x) \quad (21)$$

$(\alpha, \mu, \gamma_j, \omega_j \in C$; $Re\,(\alpha) > 0$; $Re\,(\mu) > 0$; $j = 1, 2, ..., r)$ *with the initial condition* $\left(I^{1-\alpha}_{0+}y\right)(0+) = c$ *has its solution in the space* $L\,(0, \infty)$ *given by*

$$y\,(x) = c\frac{x^{\alpha-1}}{\Gamma\,(\alpha)} + \frac{\lambda x^{\mu+\alpha}}{\Gamma(\mu+\alpha+1)}\Phi^{(r)}_2[\gamma_1, ..., \gamma_r; \mu+\alpha+1; \omega_1 x^{\rho_1}, ..., \omega_r x^{\rho_r}]$$

$$+\frac{1}{\Gamma\,(\alpha)}\int_0^x (x-t)^{\alpha-1}\,f\,(t)\,dt \quad (22)$$

where c is an arbitrary constant and $\left(\Phi^{(\gamma_j)}_{\mu,(\omega_j);0+}\right)(x)$ *is the integral operator defined by Saxena and Kalla [7] , Srivastava and Saxena [11] and Gautam [1] defined as follows:*

$$\left(\Phi^{(\gamma_j)}_{\mu,(\omega_j);a+}y\right)(x)=\frac{1}{\Gamma(\mu)}\int_a^x(x-t)^{\mu-1}\Phi_2^{(r)}[\gamma_1,...,\gamma_r;\mu;\omega_1(x-t),...,\omega_r(x-t)]y(t)dt, \tag{23}$$

$(x>a)$ $(\mu,\gamma_j\omega_j\in C;$ $Re(\mu)>0;$ $j=1,...,r)$.

Corollary 8. *The following fractional differential equation*

$$\left(D^{\alpha}_{0+}y\right)(x)=\lambda\left(\Phi^{(\gamma_j)}_{\mu,(\omega_j);0+}\right)(x)+\frac{px^{\mu}}{\Gamma(\mu+1)}\Phi_2^{(r)}[\gamma_1,...,\gamma_r;(\mu+1);\omega_1x,...,\omega_rx] \tag{24}$$

$$(\alpha,\mu,\gamma_j,\omega_j\in C;Re\,(\alpha)>0;Re\,(\mu)>0;j=1,2,...,r)$$

with the initial condition

$$\left(I^{1-\alpha}_{0+}y\right)(0+)=c$$

has its solution in the space $L(0,\infty)$ given by

$$y\,(x)=c\frac{x^{\alpha-1}}{\Gamma\,(\alpha)}+(\lambda+p)\,\frac{x^{\mu+\alpha}}{\Gamma(\mu+\alpha+1)}\Phi_2^{(r)}[\gamma_1,...,\gamma_r;\mu+\alpha+1;\omega_1x,...,\omega_rx] \tag{25}$$

where c is an arbitrary constant and $\left(\Phi^{(\gamma_j)}_{\mu,(\omega_j);0+}\right)(x)$ is the integral operator defined in (23) *.*

Corollary 9. *The following fractional differential equation*

$$x\left(D^{\alpha}_{0+}y\right)(x)=\lambda\left(\Phi^{(\gamma_j)}_{\mu,(\omega_j);0+}\right)x \tag{26}$$

$$(\alpha,\mu,\gamma_j,\omega_j\in C;Re\,(\alpha)>0;Re\,(\mu)>0;j=1,2,...,r)$$

with the initial condition

$$\left(I^{1-\alpha}_{0+}y\right)(0+)=c$$

has its solution in the space $L(0,\infty)$ given by

$$y\,(x)=-\frac{\lambda}{\Gamma\,(\alpha)\,\Gamma(\mu+1)}\int_0^x t^{\alpha-1}\,(x-t)^{\mu-1}\,\Phi_2^{(r)}[\gamma_1,...,\gamma_r;\mu+1;\omega_1\,(x-t)\,,...,\omega_r\,(x-t)]dt+c\frac{x^{\alpha-1}}{\Gamma\,(\alpha)} \tag{27}$$

where c is an arbitrary constant and $\left(\Phi^{(\gamma_j)}_{\mu,(\omega_j);0+}\right)(x)$ *is the integral operator defined in* (23).

Remark: The Theorem-5 is further extension of Theorem-3, one can obtain similar special cases of Theorem-5 as discussed above, and therefore we omit the details here.

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