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Real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ whose structure Jacobi operator is Lie $\mathbb{D}$-parallel

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Abstract. In [3], [7] and [8] results concerning the parallelness of the Lie derivative of the structure Jacobi operator of a real hypersurface with respect to ξ and to any vector field X were obtained in both complex projective space and complex hyperbolic space. In the present paper, we study the parallelness of the Lie derivative of the structure Jacobi operator of a real hypersurface with respect to vector field $X \in \mathbb{D}$ in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. More precisely, we prove that such real hypersurfaces do not exist.

Keywords: Real hypersurface, Lie $\mathbb{D}$-parallelness, Structure Jacobi operator, Complex projective space, Complex hyperbolic space.

MSC 2000 classification: primary 53C40, secondary 53C15, 53D15

1 Introduction

A complex n-dimensional Kaehler manifold of constant holomorphic sectional curvature c is called a complex space form and it is denoted by $M_n(c)$. A complete and simply connected complex space form is complex analytically isometric to a complex projective space $\mathbb{C}P^n$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{C}H^n$ if $c > 0$, $c = 0$ or $c < 0$ respectively.

Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Then an almost contact metric structure (φ, ξ, η, g) can be defined on M induced from the Kaehler metric and complex structure J on $M_n(c)$. The structure vector field ξ is called principal if $A\xi = \alpha\xi$, where A is the shape operator of M and $\alpha = \eta(A\xi)$ is a smooth function. A real hypersurface is said to be a *Hopf hypersurface* if ξ is principal.

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The classification problem of real hypersurfaces in complex space form is of great importance in Differential Geometry. The study of this was initiated by Takagi (see [10]), who classified homogeneous real hypersurfaces in $\mathbb{C}P^n$ and showed that they could be divided into six types, which are said to be of type A_1, A_2, B, C, D and E. Berndt (see [1]) classified homogeneous real hypersurfaces in $\mathbb{C}H^n$ with constant principal curvatures.

The Jacobi operator with respect to X on M is defined by $R(\cdot, X)X$, where R is the Riemannian curvature of M. For $X = \xi$ the Jacobi operator is called structure Jacobi operator and is denoted by $l = R_\xi = R(\cdot, \xi)\xi$. It has a fundamental role in almost contact manifolds. Many differential geometers have studied real hypersurfaces in terms of the structure Jacobi operator.

The study of real hypersurfaces whose structure Jacobi operator satisfies conditions concerned to the parallelness of it, is a problem of great importance. In [6] the non-existence of real hypersurfaces in nonflat complex space form with parallel structure Jacobi operator ($\nabla l = 0$) was proved. In [9] a weaker condition $\mathbb{D}$-parallelness, where $\mathbb{D} = \ker(\eta)$, that is $\nabla_X l = 0$ for any vector field X orthogonal to ξ, was studied and it was proved the non-existence of such hypersurfaces in case of $\mathbb{C}P^n$, $(n \geq 3)$. The ξ-parallelness of structure Jacobi operator in combination with other conditions was another problem that was studied by many authors such as Ki, Perez, Santos, Suh ([4]).

The Lie derivative of the structure Jacobi operator is another condition that has been studied extensively. More precisely, in [7] it was proved the non-existence of real hypersurfaces in $\mathbb{C}P^n$, $(n \geq 3)$, whose Lie derivative of the structure Jacobi operator with respect to any vector field X vanishes (i.e. $\mathcal{L}_X l = 0$). On the other hand, real hypersurfaces in $\mathbb{C}P^n$, $(n \geq 3)$, whose Lie derivative of the structure Jacobi operator with respect to ξ vanishes (i.e. $\mathcal{L}_\xi l = 0$, Lie ξ-parallel) are classified (see [8]). Ivey and Ryan in [3] extend some of the above results in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. More precisely, they proved that in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ there exist no real hypersurfaces satisfying condition $\mathcal{L}_X l = 0$, for any vector field X, but real hypersurfaces satisfying condition $\mathcal{L}_\xi l = 0$ exist and they classified them. Additional, they proved that there exist no real hypersurfaces in $\mathbb{C}P^n$ or $\mathbb{C}H^n$, $(n \geq 3)$, satisfying condition $\mathcal{L}_X l = 0$, for any vector field X.

Following the notion of [8], the structure Jacobi operator is said to be *Lie $\mathbb{D}$-parallel*, when the Lie derivative of it with respect to any vector field $X \in \mathbb{D}$ vanishes. So the following question raises naturally

"*Do there exist real hypersurfaces in non-flat $M_2(c)$ with Lie $\mathbb{D}$-parallel structure Jacobi operator?*"

In this paper, we study the above question in $\mathbb{C}P^2$ and $\mathbb{C}H^2$. The condition of *Lie $\mathbb{D}$-parallel structure Jacobi operator*, i.e. $\mathcal{L}_X l = 0$ with $X \in \mathbb{D}$, implies

$$\nabla_X(lY) + l\nabla_Y X = \nabla_{lY} X + l\nabla_X Y, \quad Y \in TM. \tag{1}$$

We prove the following

Theorem 1. *There exist no real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ equipped with Lie $\mathbb{D}$-parallel structure Jacobi operator.*

2 Preliminaries

Throughout this paper all manifolds, vector fields etc are assumed to be of class C^∞ and all manifolds are assumed to be connected. Let M be a connected real hypersurface immersed in a nonflat complex space form $(M_n(c), G)$ with complex structure J of constant holomorphic sectional curvature c. Let N be a locally defined unit normal vector field on M and $\xi = -JN$. For a vector field X tangent to M we can write $JX = \varphi(X) + \eta(X)N$, where φX and $\eta(X)N$ are the tangential and the normal component of JX respectively. The Riemannian connection $\overline{\nabla}$ in $M_n(c)$ and ∇ in M are related for any vector fields X, Y on M

$$\overline{\nabla}_Y X = \nabla_Y X + g(AY, X)N, \quad \overline{\nabla}_X N = -AX,$$

where g is the Riemannian metric on M induced from G of $M_n(c)$ and A is the shape operator of M in $M_n(c)$. M has an almost contact metric structure (φ,ξ,η,g) induced from J on $M_n(c)$ where φ is a (1,1) tensor field and η a 1-form on M such that (see [2])

$$g(\varphi X, Y) = G(JX, Y), \qquad \eta(X) = g(X, \xi) = G(JX, N).$$

Then we have

$$\varphi^2 X = -X + \eta(X)\xi, \qquad \eta \circ \varphi = 0, \qquad \varphi\xi = 0, \qquad \eta(\xi) = 1 \tag{2}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \qquad g(X, \varphi Y) = -g(\varphi X, Y) \tag{3}$$

$$\nabla_X \xi = \varphi AX, \qquad (\nabla_X \varphi)Y = \eta(Y)AX - g(AX, Y)\xi \tag{4}$$

Since the ambient space is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi for any vector fields X, Y, Z on M are respectively given by

$$\begin{aligned} R(X,Y)Z =& \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y \\ & - 2g(\varphi X, Y)\varphi Z] + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{5}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\varphi Y - \eta(Y)\varphi X - 2g(\varphi X, Y)\xi] \tag{6}$$

where R denotes the Riemannian curvature tensor on M.

Relation (5) implies that the structure Jacobi operator l is given by

$$lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - \eta(AX)A\xi \tag{7}$$

For every point $P \in M$, the tangent space T_PM can be decomposed as following

$$T_PM = <\xi> \oplus \ker(\eta),$$

where $\ker(\eta) = \{X \in T_PM \mid \eta(X) = 0\}$. Due to the above decomposition,the vector field $A\xi$ can be written

$$A\xi = \alpha\xi + \beta U, \tag{8}$$

where $\beta = |\varphi\nabla_\xi\xi|$ and $U = -\frac{1}{\beta}\varphi\nabla_\xi\xi \ \in \ \ker(\eta)$, provided that $\beta \neq 0$.

3 Some Previous Results

Let M be a non-Hopf hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, i.e. $M_2(c)$, $c \neq 0$. Then the following relations holds on every three-dimensional real hypersurface in $M_2(c)$.

Lemma 1. *Let M be a real hypersurface in $M_2(c)$. Then the following relations hold on M*

$$AU = \gamma U + \delta\varphi U + \beta\xi, \qquad A\varphi U = \delta U + \mu\varphi U, \tag{9}$$

$$\nabla_U\xi = -\delta U + \gamma\varphi U, \qquad \nabla_{\varphi U}\xi = -\mu U + \delta\varphi U, \qquad \nabla_\xi\xi = \beta\varphi U, \tag{10}$$

$$\nabla_U U = \kappa_1\varphi U + \delta\xi, \qquad \nabla_{\varphi U}U = \kappa_2\varphi U + \mu\xi, \qquad \nabla_\xi U = \kappa_3\varphi U, \tag{11}$$

$$\nabla_U\varphi U = -\kappa_1 U - \gamma\xi, \qquad \nabla_{\varphi U}\varphi U = -\kappa_2 U - \delta\xi, \qquad \nabla_\xi\varphi U = -\kappa_3 U - \beta\xi, \tag{12}$$

where γ ,δ, μ, κ_1, κ_2, κ_3 are smooth functions on M.

Proof. Let $\{U, \varphi U, \xi\}$ be a local orthonormal basis of M. Then we have

$$AU = \gamma U + \delta\varphi U + \beta\xi, \qquad A\varphi U = \delta U + \mu\varphi U,$$

where γ, δ, μ are smooth functions, since $g(AU, \xi) = g(U, A\xi) = \beta$ and $g(A\varphi U, \xi) = g(\varphi U, A\xi) = 0$. The first relation of (4), because of (8) and (9), for $X = U$, $X = \varphi U$ and $X = \xi$ implies (10). From the well known relation $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ for $X, Y, Z \in \{\xi, U, \varphi U\}$ we obtain (11) and (12), where κ_1, κ_2 and κ_3 are smooth functions. QED

The Codazzi equation for $X \in \{U, \varphi U\}$ and $Y = \xi$, because of Lemma 1 implies

$$U\beta - \xi\gamma = \alpha\delta - 2\delta\kappa_3, \tag{13}$$
$$\xi\delta = \alpha\gamma + \beta\kappa_1 + \delta^2 + \mu\kappa_3 + \frac{c}{4} - \gamma\mu - \gamma\kappa_3 - \beta^2, \tag{14}$$
$$U\alpha - \xi\beta = -3\beta\delta, \tag{15}$$
$$\xi\mu = \alpha\delta + \beta\kappa_2 - 2\delta\kappa_3, \tag{16}$$
$$(\varphi U)\alpha = \alpha\beta + \beta\kappa_3 - 3\beta\mu \tag{17}$$
$$(\varphi U)\beta = \alpha\gamma + \beta\kappa_1 + 2\delta^2 + \frac{c}{2} - 2\gamma\mu + \alpha\mu. \tag{18}$$

Furthermore, the Codazzi equation for $X = U$ and $Y = \varphi U$ implies

$$U\delta - (\varphi U)\gamma = \mu\kappa_1 - \kappa_1\gamma - \beta\gamma - 2\delta\kappa_2 - 2\beta\mu, \tag{19}$$
$$U\mu - (\varphi U)\delta = \gamma\kappa_2 + \beta\delta - \kappa_2\mu - 2\delta\kappa_1. \tag{20}$$

We recall the following Proposition ([3])

Proposition 1. *There does not exist real non-flat hypersurface in $M_2(c)$, whose structure Jacobi operator vanishes.*

4 Auxiliary Relations

If M is a real non-flat hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, i.e. $M_2(c)$, $c \neq 0$, we consider the open subset $\mathcal{W}$ of points $P \in M$, such that there exists a neighborhood of every P, where $\beta = 0$ and $\mathcal{N}$ the open subset of points $Q \in M$, such that there exists a neighborhood of every Q, where $\beta \neq 0$. Since, β is a smooth function on M, then $\mathcal{W} \cup \mathcal{N}$ is an open and dense subset of M. In $\mathcal{W}$ ξ is principal. Furthermore, we consider $\mathcal{V}$, Ω open subsets of $\mathcal{N}$

$$\mathcal{V} = \{Q \in \mathcal{N} \mid \alpha = 0 \ \textit{in a neighborhood of } Q\},$$
$$\Omega = \{Q \in \mathcal{N} \mid \alpha \neq 0 \ \textit{in a neighborhood of } Q\},$$

where $\mathcal{V} \cup \Omega$ is open and dense in the closure of $\mathcal{N}$.

Lemma 2. *Let M be a real hypersurface in $M_2(c)$, equipped with Lie $\mathbb{D}$-parallel structure Jacobi operator. Then $\mathcal{V}$ is empty.*

Proof. Let $\{U, \varphi U, \xi\}$ be a local orthonormal basis on $\mathcal{V}$. The following relations hold, because of Lemma 1

$$AU = \gamma' U + \delta' \varphi U + \beta\xi, \quad A\varphi U = \delta' U + \mu' \varphi U, \quad A\xi = \beta U \tag{21}$$

$$\nabla_U \xi = -\delta' U + \gamma' \varphi U, \quad \nabla_{\varphi U}\xi = -\mu' U + \delta' \varphi U, \quad \nabla_\xi \xi = \beta\varphi U, \tag{22}$$

$$\nabla_U U = \kappa_1' \varphi U + \delta' \xi, \quad \nabla_{\varphi U} U = \kappa_2' \varphi U + \mu' \xi, \quad \nabla_\xi U = \kappa_3' \varphi U, \tag{23}$$

$$\nabla_U \varphi U = -\kappa_1' U - \gamma' \xi, \quad \nabla_{\varphi U}\varphi U = -\kappa_2' U - \delta' \xi, \quad \nabla_\xi \varphi U = -\kappa_3' U - \beta\xi, \tag{24}$$

where γ', δ', μ', κ_1', κ_2', κ_3' are smooth functions on $\mathcal{V}$.

From (7) for $X = U$ and $X = \varphi U$, taking into account (21), we obtain

$$l\varphi U = \frac{c}{4}\varphi U, \quad lU = (\frac{c}{4} - \beta^2)U. \tag{25}$$

Relation (1) , because of (22), (23) (24) and (25), implies

$$\delta' = 0, \text{ for } \quad X = \varphi U \text{ and } Y = \xi \tag{26}$$

$$(\mu' - \kappa_3')(\frac{c}{4} - \beta^2) = 0, \text{ for } X = \varphi U \text{ and } Y = \xi \tag{27}$$

$$\kappa_3' = \gamma', \text{ for } X = U \text{ and } Y = \xi. \tag{28}$$

In $\mathcal{V}$ relations (14), (17), (18) and (19), taking into account (26), become:

$$\beta\kappa_1' + \mu'\kappa_3' + \frac{c}{4} = \gamma'\mu' + \gamma'\kappa_3' + \beta^2, \tag{29}$$

$$\kappa_3' = 3\mu', \tag{30}$$

$$(\varphi U)\beta = \beta\kappa_1' + \frac{c}{2} - 2\gamma'\mu', \tag{31}$$

$$(\varphi U)\gamma' = \kappa_1'\gamma' + \beta\gamma' + 2\beta\mu' - \mu'\kappa_1'. \tag{32}$$

Due to (27), suppose that there is a point $Q \in \mathcal{V}$, such that $\beta^2 \neq \frac{c}{4}$ in a neighborhood of Q. Then we obtain $\mu' = \kappa_3'$. Because of relations (28) and (30), we obtain $\mu' = \kappa_3' = \gamma' = 0$. Relation (1), for $X = U$ and $Y = \varphi U$, due to (23), (24) and (25), implies $\kappa_1' = 0$. Substituting in (29) $\mu' = \kappa_3' = \gamma' = \kappa_1' = 0$, leads to $\beta^2 = \frac{c}{4}$, which is impossible. So the relation $\beta^2 = \frac{c}{4}$ holds on $\mathcal{V}$. In $\mathcal{V}$, because of (28) and (30), we have $\gamma' = \kappa_3' = 3\mu'$. Substituting the last two relations in (29), implies $\beta\kappa_1' = 9\mu'^2$. Differentiation of $\beta^2 = \frac{c}{4}$ with respect to φU and taking into account (31), $\gamma' = 3\mu'$ and $\beta\kappa_1' = 9\mu'^2$, yields $c = -6\mu'^2$, which is a contradiction because $\beta^2 = \frac{c}{4}$. Hence, $\mathcal{V} = \emptyset$. QED

In what follows we work in Ω. By using (7), because of (9), we obtain

$$lU = (\frac{c}{4} + \alpha\gamma - \beta^2)U + \alpha\delta\varphi U, \quad l\varphi U = \alpha\delta U + (\alpha\mu + \frac{c}{4})\varphi U. \tag{33}$$

Relation (1) because of (10), (11), (12) and (33) implies

$$\delta(\alpha\kappa_3 + \frac{c}{4} - \beta^2) = 0, \text{ for } X = U \text{ and } Y = \xi \tag{34}$$

$$(\frac{c}{4} + \alpha\mu)(\kappa_3 - \gamma) + \alpha\delta^2 = 0, \text{ for } X = U \text{ and } Y = \xi \tag{35}$$

$$(\frac{c}{4} + \alpha\gamma - \beta^2)(\mu - \kappa_3) - \alpha\delta^2 = 0, \text{ for } X = \varphi U \text{ and } Y = \xi \tag{36}$$

$$\delta(\alpha\kappa_3 + \frac{c}{4}) = 0, \text{ for } X = \varphi U \text{ and } Y = \xi. \tag{37}$$

Due to (37), we consider the open subsets Ω_1 and Ω_1' of Ω

$$\Omega_1 = \{Q \in \Omega \mid \delta \neq 0 \textit{ in a neighborhood of } Q\},$$
$$\Omega_1' = \{Q \in \Omega \mid \delta = 0 \textit{ in a neighborhood of } Q\},$$

where $\Omega_1 \cup \Omega_1'$ is open and dense in the closure of Ω.

In Ω_1, from (34) and (37), we have $\beta = 0$, which is a contradiction, therefore $\Omega_1 = \emptyset$. Thus we have $\delta = 0$ in Ω and relations from Lemma 1, (33), (35) and (36) become respectively

$$AU = \gamma U + \beta\xi, \quad A\varphi U = \mu\varphi U, \quad A\xi = \alpha\xi + \beta U \tag{38}$$

$$\nabla_U\xi = \gamma\varphi U, \quad \nabla_{\varphi U}\xi = -\mu U, \quad \nabla_\xi\xi = \beta\varphi U \tag{39}$$

$$\nabla_U U = \kappa_1\varphi U, \quad \nabla_{\varphi U}U = \kappa_2\varphi U + \mu\xi, \quad \nabla_\xi U = \kappa_3\varphi U, \tag{40}$$

$$\nabla_U\varphi U = -\kappa_1 U - \gamma\xi, \; \nabla_{\varphi U}\varphi U = -\kappa_2 U, \; \nabla_\xi\varphi U = -\kappa_3 U - \beta\xi, \tag{41}$$

$$lU = (\frac{c}{4} + \alpha\gamma - \beta^2)U, \quad l\varphi U = (\alpha\mu + \frac{c}{4})\varphi U, \tag{42}$$

$$(\frac{c}{4} + \alpha\gamma - \beta^2)(\mu - \kappa_3) = 0, \tag{43}$$

$$(\frac{c}{4} + \alpha\mu)(\kappa_3 - \gamma) = 0. \tag{44}$$

Owing to (43), we consider the open subsets Ω_2 and Ω_2' of Ω

$$\Omega_2 = \{Q \in \Omega \mid \mu \neq \kappa_3 \textit{ in a neighborhood of } Q\},$$
$$\Omega_2' = \{Q \in \Omega \mid \mu = \kappa_3 \textit{ in a neighborhood of } Q\},$$

where $\Omega_2 \cup \Omega_2'$ is open and dense in the closure of Ω. So in Ω_2, we have

$$\gamma = \frac{\beta^2}{\alpha} - \frac{c}{4\alpha} \text{ and (42) implies } lU = 0. \tag{45}$$

Lemma 3. *Let M be a real hypersurface in $M_2(c)$, equipped with Lie $\mathbb{D}$-parallel structure Jacobi operator. Then Ω_2 is empty.*

Proof. Due to (44) we consider Ω_{21} and Ω'_{21} the open subsets of Ω_2

$$\Omega_{21}=\{Q\ \in\ \Omega_2\mid\ \mu\neq-\frac{c}{4\alpha}\ \textit{in a neighborhood of } Q\},$$
$$\Omega'_{21}=\{Q\ \in\ \Omega_2\mid\ \mu=-\frac{c}{4\alpha}\ \textit{in a neighborhood of } Q\},$$

where $\Omega_{21}\cup\Omega'_{21}$ is open and dense in the closure of Ω_2. In Ω_{21} since $\mu\neq-\frac{c}{4\alpha}$, due to (42), we obtain $l\varphi U\neq 0$. Furthermore, because of (44) $\kappa_3=\gamma$.

Relation (1) for $X=\varphi U$ and $Y=U$ owing to (40), (41) and (45) implies $\kappa_2 l\varphi U=0$ and so $\kappa_2=0$. Due to the last, relations (16) and (20) imply

$$U\mu=\xi\mu=0. \tag{46}$$

Relation (1) for $X=U$ and $Y=\varphi U$ taking into account (40), (41), (42) and (45) implies $\mu=-\gamma$ and $\kappa_1=0$. Substitution in (14) of the relations which hold on Ω_{21} implies $\gamma=0$ and this results in $\beta^2=\frac{c}{4}$. Differentiation of the last with respect to φU, because of (18) leads to $c=0$, which is a contradiction. So Ω_{21} is empty and $\mu=-\frac{c}{4\alpha}$ holds on Ω_2. So in Ω_2, (42) implies $l\varphi U=0$ and because of (45) we obtain that the structure Jacobi operator vanishes in Ω_2. So because of Proposition 1 we have that the subset $\Omega_2=\emptyset$ and this completes the proof of the present Lemma. QED

Summarizing, in Ω we have $\delta=0$ and $\mu=\kappa_3$. Due to (44), we consider Ω_3 and Ω'_3 the open subsets of Ω

$$\Omega_3=\{Q\ \in\ \Omega\mid\ \mu\neq-\frac{c}{4\alpha}\ \textit{in a neighborhood of } Q\},$$
$$\Omega'_3=\{Q\ \in\ \Omega\mid\ \mu=-\frac{c}{4\alpha}\ \textit{in a neighborhood of } Q\},$$

where $\Omega_3\cup\Omega'_3$ is open and dense in the closure of Ω. Since $\mu\neq-\frac{c}{4\alpha}$, due to (42) and (44) we obtain $l\varphi U\neq 0$ and $\gamma=\mu=\kappa_3$, in Ω_3.

Lemma 4. *Let M be a real hypersurface in $M_2(c)$, equipped with Lie $\mathbb{D}$-parallel structure Jacobi operator. Then Ω_3 is empty.*

Proof. In Ω_3 relations (14), (17), (18) and (19) become respectively

$$\alpha\gamma+\beta\kappa_1+\frac{c}{4}=\beta^2+\gamma^2 \tag{47}$$
$$(\varphi U)\alpha=\alpha\beta-2\beta\gamma \tag{48}$$
$$(\varphi U)\beta=2\alpha\gamma+\beta\kappa_1+\frac{c}{2}-2\gamma^2 \tag{49}$$
$$(\varphi U)\mu=(\varphi U)\gamma=3\beta\gamma. \tag{50}$$

Relation (1) for $X = Y = \varphi U$, because of (42), (48) and (50) implies $\gamma(2\alpha - \gamma) = 0$. Suppose that there is a point $Q \in \Omega_3$, such that $\gamma \neq 0$ in a neighborhood of Q. Then we obtain $\gamma = 2\alpha$. Differentiation of the latter with respect to φU and taking into account (48) and (50) leads to $\alpha\beta = 0$, which is impossible. So in Ω_3, $\gamma = 0$.

Resuming in Ω_3 we have $\gamma = \mu = \kappa_3 = 0$ and relation (16) implies $\kappa_2 = 0$. Relation (1) for $X = U$ and $Y = \varphi U$, because of (40), (41) and (42) yields $\kappa_1 = 0$ and so relation (47) implies $\beta^2 = \frac{c}{4}$. Differentiation of the last along φU and because of (49) leads to $c = 0$, which is a contradiction and this completes the proof of Lemma 4. $\boxed{QED}$

So in Ω the following relations hold

$$\delta = 0, \quad \mu = \kappa_3 = -\frac{c}{4\alpha},$$

and relation (42), because of the last one implies $l\varphi U = 0$.

Relation (1), for $X = U$ and $Y = \varphi U$, because of (40) and (41) implies $\kappa_1 lU = 0$. Suppose that there is a point $Q \in \Omega$, such that $\kappa_1 \neq 0$ in a neighborhood of Q, so we obtain $lU = 0$. Since $l\varphi U = 0$, we have that the structure Jacobi operator vanishes. Owing to Proposition 1 we derive a contradiction and so in Ω $\kappa_1 = 0$ holds.

In Ω relations (17), (18) and (19) become respectively

$$(\varphi U)\alpha = \alpha\beta + \frac{c\beta}{2\alpha}, \tag{51}$$

$$(\varphi U)\beta = \alpha\gamma + \frac{c\gamma}{2\alpha} + \frac{c}{4}, \tag{52}$$

$$(\varphi U)\gamma = \beta\gamma - \frac{c\beta}{2\alpha}. \tag{53}$$

Relation (1) for $X = \varphi U$ and $Y = U$ taking into account (40), (41) and (42) yields

$$(\varphi U)(\frac{c}{4} + \alpha\gamma - \beta^2) = 0 \tag{54}$$

$$\kappa_2(\frac{c}{4} + \alpha\gamma - \beta^2) = 0 \tag{55}$$

$$(\mu + \gamma)(\frac{c}{4} + \alpha\gamma - \beta^2) = 0. \tag{56}$$

Lemma 5. *Let M be a real hypersurface in $M_2(c)$, equipped with Lie $\mathbb{D}$-parallel structure Jacobi operator. Then Ω is empty.*

Proof. Due to (55), suppose that there is a point $Q \in \Omega$, such that $\frac{c}{4} + \alpha\gamma - \beta^2 \neq 0$ in a neighborhood of Q. Then from (55) we obtain $\kappa_2 = 0$. Relation

(56) implies $\mu = -\gamma$ and since $\mu = -\frac{c}{4\alpha}$ we have that $\gamma = \frac{c}{4\alpha}$ and this results in $\alpha\gamma = \frac{c}{4}$. Relation (54), owing to the last relation, implies that $(\phi U)\beta = 0$ and so relation (52) taking into consideration the relation $\alpha\gamma = \frac{c}{4}$ implies $4\gamma^2 + c = 0$. Differentiation of the last one with respect to ϕU and because of (53) we lead to the following $c = 0$, which is a contradiction. So in Ω relation $\gamma = \frac{\beta^2}{\alpha} - \frac{c}{4\alpha}$ holds. Because of (42) $lU = 0$ and since $l\varphi U = 0$, due to Proposition 1 we obtain that $\Omega = \emptyset$ and this completes the proof of Lemma 5. $\boxed{QED}$

We lead to the following due to Lemmas 2 and 5

Proposition 2. *Every real hypersurface in $M_2(c)$, equipped with Lie $\mathbb{D}$-parallel structure Jacobi operator is a Hopf hypersurface.*

5 Proof of Main Theorem

Since M is a Hopf hypersurface, due to Theorem 2.1, ([5]) , we have that α is a constant. We consider a point $P \in M$ and we choose principal vector field $Z \in \ker(\eta)$ at P, such that $AZ = \lambda Z$ and $A\varphi Z = \nu\varphi Z$. The following relation holds on M, (Corollary 2.3, [5])

$$\lambda\nu = \frac{\alpha}{2}(\lambda + \nu) + \frac{c}{4}. \tag{57}$$

Relation (7) implies

$$lZ = (\frac{c}{4} + \alpha\lambda)Z \quad l\varphi Z = (\frac{c}{4} + \alpha\nu)\varphi Z. \tag{58}$$

Relation (1) for $X = Z$ and $Y = \varphi Z$ and for $X = \varphi Z$ and $Y = Z$, because of (58) and taking the inner product of them with ξ implies respectively

$$(\lambda + \nu)(\frac{c}{4} + \alpha\nu) = 0, \tag{59}$$

$$(\nu + \lambda)(\frac{c}{4} + \alpha\lambda) = 0. \tag{60}$$

Suppose that λ, ν are distinct and unequal at point P. Then from (59) we obtain $\lambda = -\nu$ and because of (57) we have $c = -4\lambda^2$. From the last relation we conclude that $c < 0$ and $\lambda =$constant. The only hypersurface that we have in this case is of type B in $\mathbb{C}H^2$. Substituting the eigenvalues of this hypersurface in relation $\lambda = -\nu$ leads to a contradiction (see [1]).

So the remaining case is that of $\lambda = \nu$ at all points. Then from (59), we obtain that either $\lambda = 0$ or $\frac{c}{4} + \alpha\lambda = 0$. In both cases, substitution of these relations in (57) leads us to a contradiction. This completes the proof of Theorem 1.

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