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# On the centrally symmetric ovals circumscribing invariant maximal quadrilaterals

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**Abstract.** If in a centrally symmetric oval all the inscribed convex quadrilaterals of maximal area and/or perimeter have the property that for any point on the oval there is exactly one such quadrilateral having that point as a vertex, does the oval have to be an ellipse? In this paper we will give an answer ranging from partial to complete to this multiple question.

**Keywords:** Oval, ellipse, quadrilateral, area, perimeter

**MSC 2000 classification:** primary 52A40 52A38, secondary 52A10

## Introduction

It is well-known [2, 3, 5] that the inscribed convex $n$-gons which best approximate the area/perimeter of an ellipse exhibit the following invariance property: For any point on the ellipse, among all the inscribed $n$-gons having that point as a vertex there is only one of largest area/perimeter (unique pointwise maximality), and all these pointwise maximal $n$-gons have actually the same area/perimeter (global invariance). The area problem [5] is relatively simple compared to the perimeter one [3], a fact that parallels the area vs. perimeter difficulty contest for the ellipse itself. It is also worth noting that the same maximal $n$-gons answer both the area and the perimeter problems only when the ellipse is a circle.

In this paper, inspired by [2], we explore the converse situation: If a centrally symmetric smooth strictly convex closed plane curve (oval with a center) and its inscribed convex $n$-gons obey the invariant maximal area/perimeter property described above, does it have to be an ellipse? Actually, we tackle the simplest of cases, that of inscribed quadrilaterals ($n = 4$), that already unveils the complexity of the problem. The general case, $n$ arbitrary, will be pursued elsewhere.

Roughly, the answer to our question is no, if posed to the area or perimeter cases separately, and yes, when posed jointly. We will show that in both separate cases one quarter of any oval-solution can be prescribed relatively arbitrarily, after which the invariant maximal property locks in a neighboring quarter based on a concrete (area/perimeter dependent) recipe applied to the initial selection, and then the symmetry completes the oval. It turns out that the area problem admits a solution for any quarter-oval initial selection, if the product of the radii of curvature at the end-points of the quarter-oval equals the product of quarter-oval semi-axes. This is a transmission restriction which guarantees the global smoothness of the oval. The perimeter problem requires a more severe global curvature-growth restriction on the quarter-oval selection, in addition to a transmission restriction at the end-points. In fact, what we show is that the solutions in the perimeter case are exactly those ovals whose orthoptic curves are circles (For a similar result when quadrilaterals are replaced by parallelograms, see [1, 4]).

The joint case, when an oval simultaneously satisfies the invariant maximal area property and the invariant maximal perimeter property, leads to a differential-equation-look-alike for the initial quarter oval, whose solution most probably must be unique, therefore the ellipse. We are able to show this for an *analytic* quarter oval and conjecture it in general.

Throughout, we will try to use geometric arguments whenever feasible. When relying on analytic findings we will also strive to provide geometric interpretation.

## 1 The area problem

Let $\mathcal{C}$ be a smooth (class $C^2$) oval with center of symmetry $O$. Fixing a point $A$ on $\mathcal{C}$, consider the area functional associated to all possible convex quadrilaterals $ABCD$ inscribed in $\mathcal{C}$ and sharing the vertex $A$. The points $A, B, C$, and $D$, will always be positioned *clockwise* around $\mathcal{C}$. By the continuity of the area functional there is then at least one such quadrilateral of maximal area. It will be assumed that this pointwise maximal area quadrilateral is unique and also that the associated area is a constant, independent of $A$ (The invariant maximal area assumption).

**Proposition 1.** *Let $\mathcal{C}$ be a smooth oval with center of symmetry $O$, satisfying the invariant maximal area assumption. Then for any point $A$ on $\mathcal{C}$ the unique quadrilateral $ABCD$ of largest area is in fact a parallelogram. The vertex $C$ is the symmetric with respect to $O$ of the vertex $A$, and the vertices $B$ and $D$ are the unique points on $\mathcal{C}$ such that the two lines through them parallel to $\overleftrightarrow{AC}$ are tangent to $\mathcal{C}$. Moreover, the vertices $B$ and $D$ are also opposite each other* *with respect to* $O$ *and the two lines through* $A$ *and* $C$ *parallel to* $\overleftrightarrow{BD}$ *are tangent to* $\mathcal{C}$. (*Figure 1*).

*Therefore, for a quadrilateral* $ABCD$ *of largest area the tangent line to the oval at any one of its vertices is parallel to the diagonal of the quadrilateral not sharing that vertex* (*The Euler-Lagrange condition for area maximality*).

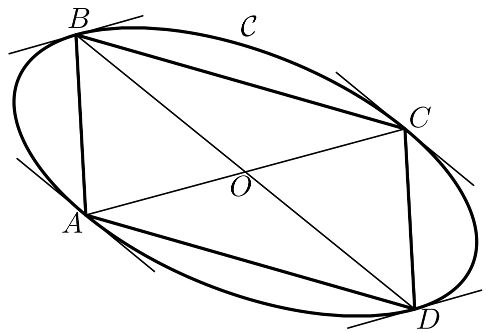


Figure 1. Maximal area inscribed quadrilateral based at $A$, a parallelogram

*Proof.* If $ABCD$ is a quadrilateral inscribed in $\mathcal{C}$ then $Area(ABCD) = \frac{1}{2}AC \cdot d$, where $d$ is the distance between the two lines through $B$ and $D$ parallel to $\overleftrightarrow{AC}$. $Area(ABCD)$ can be increased if $\overline{AC}$ is not the longest chord in $\mathcal{C}$ parallel to itself, or the lines through $B$ and $D$ mentioned above are not tangent to $\mathcal{C}$ (Figure 2).

If now vertex $A$ is fixed and $Area(ABCD)$ is maximal, then the invariant maximal area assumption guarantees that its area cannot be increased by either one of the two ways described above. By the strict convexity of $\mathcal{C}$, the longest chord of $\mathcal{C}$ in any fixed direction passes through the center $O$. This proves the first part of the Proposition. The second part follows from the the first part and the invariant maximal area assumption since a maximal quadrilateral $ABCD$ based at $A$ will also be maximal (and unique) when viewed as based at $B$. We point out that without the invariant maximal area assumption the conclusion of Proposition 1 might not hold in its entirety. It might then be possible for a maximal quadrilateral based at $A$ that the endpoints of the diagonal $\overline{AC}$ not be opposite each other or the line through $A$ parallel to $\overleftrightarrow{BD}$ not be tangent to $\mathcal{C}$. In such a case even the pointwise uniqueness of a maximal quadrilateral might be compromised. $\boxed{QED}$

A chord passing through the center of a centrally symmetric oval will be called *principal* if the tangent lines to the oval at the end-points of the chord

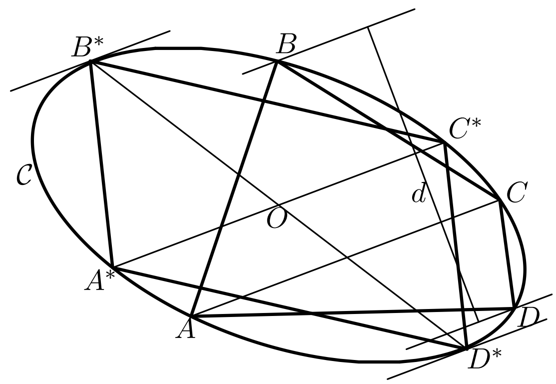


Figure 2. $Area(A^*B^*C^*D^*) > Area(ABCD)$

are perpendicular to the chord. Clearly, a diameter (longest chord) or a shortest chord among those passing through the center of the oval are principal chords. We will be interested in ovals admitting pairs of perpendicular principal chords. In such case a suitable coordinate system can be imposed on the oval, the coordinate axes precisely being supported by the principal chords. It is then natural to call *quarter-oval* the portion of the oval contained in any one of the four quadrants of such a coordinate system. Also, the *semi-axes* of the oval will be the distances from the center to the oval along the coordinate axes. While not every centrally symmetric oval admits pairs of perpendicular principal chords it is clear from Proposition 1 that if the invariant maximal area assumption is met by an oval then any chord perpendicular on a principal chord at the center of the oval is also principal. It is also obvious in such a case that principal perpendicular chords must be diagonals of inscribed parallelograms of maximal area which are *rhombi*.

We are now in a position to state the main result of this section.

**Theorem 1** (Classification of Centrally Symmetric Ovals Satisfying the Quadrilateral Invariant Maximal Area Property)**.** *The following two statements are equivalent:*

*a) $\mathcal{C}$, a centrally symmetric smooth (class $C^2$) oval with center $O$ satisfies the quadrilateral invariant maximal area property, that is for any point of $\mathcal{C}$, among all the quadrilaterals inscribed in $\mathcal{C}$ and having that point as a vertex there is only one of maximal area, and all these maximal quadrilaterals have the same area, independent of the point chosen.*

*b) A centrally symmetric smooth (class $C^2$) oval $\mathcal{C}$ with center $O$ is completely determined by its top left quarter, in the sense that starting with a smooth (class $C^2$) concave down arc $\widehat{WN}$ in the third quadrant of a coordinate system $XOY$, $W \in \overleftrightarrow{OX}$, $N \in \overleftrightarrow{OY}$, such that the half-tangents to the arc at the end-*

*points $W$ and $N$ are parallel to the coordinate axes, and such that the product of the radii of curvature of the arc at $W$ and $N$ equals $OW \cdot ON$, the mapping $A \to T(A)$, $A \in \widehat{WN}$, described below is a bijective, clockwise increasing transformation from the top left quarter of a centrally symmetric (center $O$) smooth oval $\mathcal{C}$ to the top right quarter of $\mathcal{C}$.*

*To the end of describing the transformation $A \longmapsto T(A)$, let $A$ be an arbitrary point on the arc $\widehat{WN}$ and let $P$ be the intersection point of the tangent line to the arc at $A$ and $\overleftrightarrow{ON}$. Let now the parallel line through $N$ to $\overleftrightarrow{PE}$, where $E$ denotes the symmetric point of $W$ with respect to $O$, intersect $\overleftrightarrow{OW}$ at a point $F$. Then $T(A)$ is the intersection point of the perpendicular line to $\overleftrightarrow{OW}$ at $F$ and the parallel line to $\overleftrightarrow{AP}$ through $O$ (Figure 3). Alternatively, $T(A)$ can be*

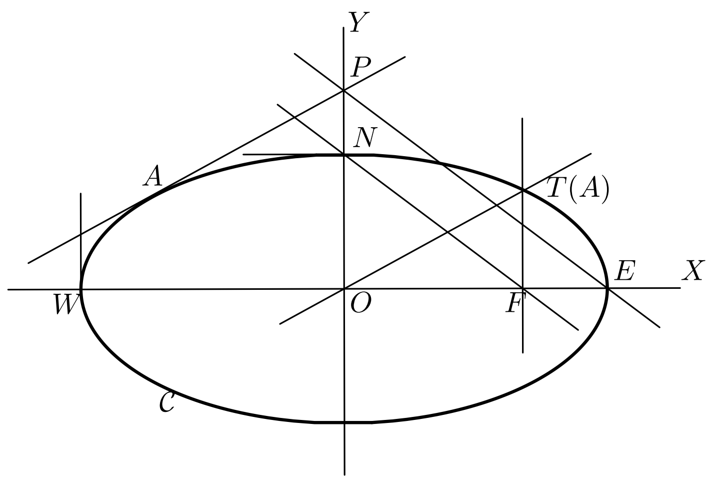


Figure 3. First construction of $T(A)$

*constructed in the following way: Let $M$ be the foot of the perpendicular dropped from $A$ to $\overleftrightarrow{OW}$ and let $Q$ be the intersection point of $\overleftrightarrow{ON}$ and the parallel line to $\overleftrightarrow{MN}$ through $W$. Then $T(A)$ is the intersection point of the parallel line to $\overleftrightarrow{AP}$ through $O$ and the parallel line to $\overleftrightarrow{OA}$ through $Q$ (Figure 4). In this description of $T(A)$ the line $\overleftrightarrow{QT(A)}$ is the tangent line to $\mathcal{C}$ at the point $T(A)$.*

*When the oval $\mathcal{C}$ satisfies the two equivalent characterizations above, $T(A)$ is the second vertex, vertex $B$, of the maximal area quadrilateral $ABCD$ based at $A$. In fact, by Proposition 1 this quadrilateral is a parallelogram, so the vertex $C$ is the symmetric with respect to $O$ of the vertex $A$ and the vertex $D$ is the symmetric with respect to $O$ of $T(A)$. Moreover, the maximal area value of inscribed quadrilaterals in $\mathcal{C}$ equals twice the product of the semi-axes of $\mathcal{C}$.*

*Proof.* $a) \Longrightarrow b)$ Assume that the centrally symmetric oval $\mathcal{C}$ satisfies the in-

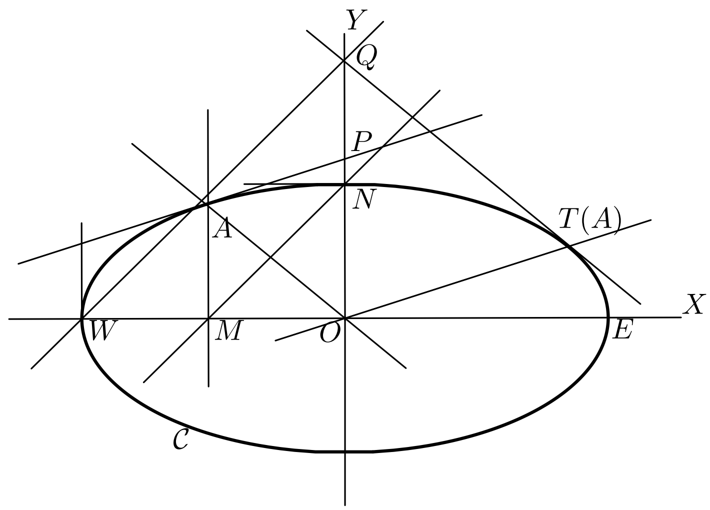


Figure 4. Second construction of $T(A)$

variant maximal area property. As noted before the oval admits then a pair of perpendicular principal axes which support a coordinate system $XOY$, $O$ being the center of the oval. With respect to this coordinate system there is a defining function for the top half of the oval, say $f : [-a, a] \to [0, b]$, $a > 0$, $b > 0$, of class $C^2$ on $(-a, a)$, such that

$$\begin{gathered}
f(-a) = f(a) = 0, \quad f(0) = b, \\
f'(x) > 0 \ \text{ on } (-a, 0), f'(0) = 0, f'(x) < 0 \ \text{ on } (0, a), \\
\lim_{x \to -a^+} f'(x) = +\infty, \ \lim_{x \to a^-} f'(x) = -\infty, \\
f''(x) < 0 \quad \text{on } (-a, a)
\end{gathered}$$

All these properties of $f$ follow simply from the strict convexity of the $C^2$-class oval and the particular choice of the coordinate system. Actually, the above equations do not capture the $C^2$-smoothness and strict convexity of the oval at its two points along the $X$-axis, $W(-a, 0)$ and $E(a, 0)$, due to the non-existence of the derivatives of $f$ at $\pm a$. This can be easily redressed by inverting $f$ near $x = -a^+$ or $x = a^-$. For instance, denoting by $f^{-1}$ the inverse of $f : [-a, 0] \to [0, b]$, the one-sided $C^2$-smoothness and strict convexity of the oval at $W$ implies the $C^2$-smoothness and strict convexity of $f^{-1}$ at $y = 0^+$, or equivalently

$$
\begin{aligned}
0 < (f^{-1})''(0) &= \lim_{y\to 0^+} \frac{(f^{-1})'(y)}{y} = \lim_{x\to -a^+} \frac{1}{f'(x)f(x)} \\
&= \lim_{y\to 0^+} (f^{-1})''(y) = -\lim_{x\to -a^+} \frac{f''(x)}{(f'(x))^3}
\end{aligned} \tag{1}
$$

The arc $\widehat{WN}$ mentioned in $b$) is simply the graph of $f(x)$ for $x \in [-a, 0]$, so $W$ and $N$ have coordinates $(-a, 0)$ and $(0, b)$, respectively. Therefore, $a$ and $b$ are the semi-axes of the oval.

Now, if the vertex $A$ of a maximal area parallelogram $ABCD$ is chosen to have arbitrary coordinates $(x, f(x))$, $x \in [-a, 0]$, by Proposition 1 there is an unique increasing function $\alpha : [-a, 0] \to [0, a]$, of class $C^2$ on $(-a, 0)$, such that the coordinates of $B$ are $(\alpha(x), f(\alpha(x)))$, the coordinates of $C$ are $(-x, -f(x))$, and those of $D$ are $(-\alpha(x), -f(\alpha(x)))$. In terms of the $\alpha$ function the conclusions of Proposition 1 simply become equivalent to

$$
f'(\alpha(x)) = \frac{f(x)}{x}, \quad f'(x) = \frac{f(\alpha(x))}{\alpha(x)}, \quad x \in (-a, 0) \tag{2}
$$

Since by Equation (2), $f(\alpha(x)) = \alpha(x)f'(x)$, $x \in (-a, 0]$, after differentiation we get

$$
f'(\alpha(x))\alpha'(x) = \alpha'(x)f'(x) + \alpha(x)f''(x) \tag{3}
$$

However, by the first part of Equation (2) we may substitute $\frac{f(x)}{x}$ for $f'(\alpha(x))$ in Equation (3) to get

$$
(f(x) - xf'(x))\alpha'(x) = xf''(x)\alpha(x) \tag{4}
$$

Equation (4) is now a first order differential equation in $\alpha(x)$, with initial condition $\alpha(0) = a$. Its solution on $(-a, 0]$ clearly is $\alpha(x) = \frac{ab}{f(x) - xf'(x)}$. From the second part of Equation (2) we conclude that $f(\alpha(x)) = \frac{abf'(x)}{f(x) - xf'(x)}$, $x \in (-a, 0]$.

Notice now that the curvatures of the quarter-oval $\widehat{WN}$ at the end-points $W$ and $N$ are $(f^{-1})''(0) = \lim_{x\to -a^+} \frac{1}{f'(x)f(x)}$ and $-f''(0)$, respectively. However, $f'(\alpha(x)) = \frac{f(x)}{x}$, $x \in (-a, 0)$, implies $\frac{f'(\alpha(x))}{\alpha(x)} = \frac{f(x)(f(x) - xf'(x))}{xab}$ and so, by Equation (1)

$$
f''(0) = \lim_{x\to -a^+} \frac{f'(\alpha(x))}{\alpha(x)} = \lim_{x\to -a^+} \frac{f(x)(f(x) - xf'(x))}{xab} = -\frac{1}{ab(f^{-1})''(0)}
$$

Therefore, the product of the radii of curvature at $W$ and $N$, $-\dfrac{1}{f''(0)(f^{-1})''(0)}$, equals $ab$, which is $OW \cdot ON$, or equivalently the product of the semi-axes of $\mathcal{C}$.

We want to show now that the point $T(A)$ obtained by the first construction recipe given in $b)$ is exactly vertex $B$ of the maximal quadrilateral/parallelogram $ABCD$, and so has coordinates $\left(\dfrac{ab}{f(x)-xf'(x)}, \dfrac{abf'(x)}{f(x)-xf'(x)}\right)$. By the first construction of $T(A)$, (Figure 3), $\triangle OFN \sim \triangle OEP$. Therefore, $\dfrac{OF}{OE} = \dfrac{ON}{OP}$. Now, $OE = a$, $ON = b$, and $OP = f(x) - xf'(x)$. To see that $OP = f(x) - xf'(x)$, simply write down the equation of the tangent line to the oval at $A$ and intersect it with the $Y$-axis. Thus, $OT(A) = \dfrac{ab}{f(x)-xf'(x)}$, and then $FT(A) = \dfrac{abf'(x)}{f(x)-xf'(x)}$, as needed.

For the alternative construction of $T(A) \equiv B$, (Figure 4), notice that since $\triangle OMN \sim \triangle OWQ$, $\dfrac{OM}{OW} = \dfrac{ON}{OQ}$, and so $OQ = -\dfrac{ab}{x}$. Since the parallel line to $\overleftrightarrow{AP}$ through $O$ has slope $f'(x)$ and the parallel line to $\overleftrightarrow{OA}$ through $Q$ has slope $\dfrac{f(x)}{x}$, the result follows by writing down the equations of these two lines, and then solving for their intersection point, which as expected has the coordinates $(\alpha(x), f(\alpha(x))$ of $B$.

Finally, in coordinates the area of a maximal parallelogram based at $A(x, f(x))$, for some $x \in [-a, 0]$, equals

$$\left|\det \begin{bmatrix} \alpha(x) - x & f(\alpha(x)) - f(x) \\ -\alpha(x) - x & -f(\alpha(x)) - f(x) \end{bmatrix}\right|$$
$$= 2(\alpha(x)f(x) - xf(\alpha(x))) = 2ab = OW \cdot ON.$$

$b) \implies a)$ Assume now that the smooth quarter oval $\widehat{WN}$ is given in the third quadrant of a coordinate system $XOY$ and that its end-points $W$ and $N$ satisfy the curvature condition given in $b)$. $\widehat{WN}$ can be viewed as the graph of a function $f : [-a, 0] \to [0, b]$, $f(-a) = 0$, i.e., $W$ has coordinates $(-a, 0)$, $f(0) = b$, i.e., $N$ has coordinates $(0, b)$, $f$ is differentiable on $(-a, 0]$ and $\lim\limits_{x\to -a^+} f'(x) = \infty$, $f'(x) > 0$ for $x \in (-a, 0)$, $f'(0) = 0$, and $f$ admits a continuous second derivative such that $f''(x) < 0$ for $x \in (-a, 0]$. Now, the curvature condition on $W$ and $N$ is equivalent to

$$-\frac{\lim_{x\to -a^+} f(x)f'(x)}{f''(0)} = ab \tag{5}$$

The task is to extend the quarter oval $\widehat{WN}$ (or equivalently $f$) to the first quadrant and then to conclude that symmetry with respect to $O$ completes the oval in such a way that the invariant maximal area property is satisfied. If $A(x, f(x))$, $x \in [-a, 0]$, is an arbitrary point on $\widehat{WN}$ then $T(A)$ as constructed in $b)$ will have coordinates $\left(\frac{ab}{f(x) - xf'(x)}, \frac{abf'(x)}{f(x) - xf'(x)}\right)$ (same proof as in $a) \Longrightarrow b)$). This allows us to double the domain and extend $f$ from $[-a, 0]$ to $[-a, a]$. Noticing that $\alpha : [-a, 0] \to [0, a]$, $\alpha(x) := \frac{ab}{f(x) - xf'(x)}$, is a smooth, increasing function, $f$ can simply be extended to $[0, a]$ by $f\left(\frac{ab}{f(x) - xf'(x)}\right) = \frac{abf'(x)}{f(x) - xf'(x)}$. Indeed, $\alpha'(x) = \frac{abxf''(x)}{(f(x) - xf'(x))^2} > 0$, for $x \in (-a, 0)$, shows that $\alpha$ is strictly increasing. Also, since $f(\alpha(x)) = \alpha(x)f'(x)$,

$$f'(\alpha(x)) = \frac{\alpha'(x)f'(x) + \alpha(x)f''(x)}{\alpha'(x)} = \frac{f(x)}{x} < 0, \quad x \in (-a, 0),$$

proves that $f$ is strictly decreasing on $[0, a]$.

Then, $f'(\alpha(x)) = \frac{f(x)}{x}$ yields

$$f''(\alpha(x)) = \frac{xf'(x) - f(x)}{x^2\alpha'(x)} = -\frac{(f(x) - xf'(x))^3}{abx^3 f''(x)} < 0, \quad x \in (-a, 0) \tag{6}$$

and so $f$ has class $C^2$ on $(0, a)$ and is concave down there.

Now Equations (6), (5), and (1) easily give $\lim_{x \to -a^+} f''(\alpha(x)) = f''(0)$, and this completes the proof of the $C^2$-smoothness and strict convexity of the upper-half of the oval, including the transitional point $N$. Symmetry with respect to $O$ completes the oval, and it is a routine exercise, similar to what was done so far, to check the full $C^2$-smoothness and strict convexity of the whole oval at $W$ and $E$. $\boxed{QED}$

## 2 The perimeter problem

Consider now the perimeter functional associated to all convex quadrilaterals $ABCD$ inscribed in $\mathcal{C}$ and sharing the vertex $A$. We will make the invariant maximal perimeter assumption, namely there is only one quadrilateral of maximal perimeter based at $A$, and all these maximal quadrilaterals have the same perimeter, irrespective of $A$. It is reasonable to expect that in any centrally symmetric oval an inscribed quadrilateral of maximal perimeter must be a parallelogram. This is not entirely obvious though, much more so given that the Euler-Lagrange condition required for perimeter maximality, namely that the trajectory $ABCD$ be a billiard of period 4 for the 'mirror' $\mathcal{C}$, may be satisfied by quadrilaterals other than parallelograms (e. g., trapezoids) for certain ovals. However, this will be the case if the oval satisfies the invariant maximal perimeter assumption, as the following proposition shows.

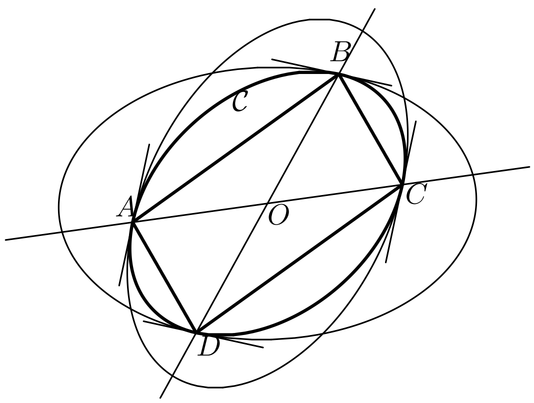


Figure 5. Maximal perimeter inscribed quadrilateral based at $A$, a parallelogram

**Proposition 2.** *Let $\mathcal{C}$ be a smooth oval with center of symmetry $O$, satisfying the invariant maximal perimeter assumption. Then for any point $A$ on $\mathcal{C}$ the unique quadrilateral $ABCD$ of largest perimeter is in fact a parallelogram. The vertex $C$ is the symmetric with respect to $O$ of the vertex $A$, and the vertices $B$ and $D$ are the unique points on $\mathcal{C}$ where the largest ellipse, in the family of confocal ellipses with foci $A$ and $C$ and intersecting $\mathcal{C}$, intersects $\mathcal{C}$. Moreover, $A$ and $C$ admit a similar ellipse characterization with respect to the family of confocal ellipses with foci $B$ and $D$, which are symmetric points with respect to $O$ (Figure 5).*

*It also follows that the trajectory $ABCD$ of any quadrilateral of largest perimeter is a billiard of period 4, that is any two consecutive sides of the quadrilateral form congruent angles with the tangent line to the oval at the point they share (The Euler-Lagrange condition for perimeter maximality).*

*Proof.* If $ABCD$ is a quadrilateral inscribed in $\mathcal{C}$ consider the family of confocal ellipses $\mathcal{E}$ with foci $A$ and $C$ and intersecting $\mathcal{C}$. There is an unique such ellipse containing $B$, say $\mathcal{E_B}$, and also one containing $D$, $\mathcal{E_D}$, and then the perimeter $Perim(ABCD) = (AB + BC) + (AD + DC) = MajorAxis(\mathcal{E_B}) + MajorAxis(\mathcal{E_D})$. $Perim(ABCD)$ can be increased if $\mathcal{E_B}$, $\mathcal{E_D}$ are not the largest intersecting the arc $\widehat{ABC}$, respectively the arc $\widehat{ADC}$, of the oval (Figure 6). Considering now the family of inscribed quadrilaterals of largest perimeter it is

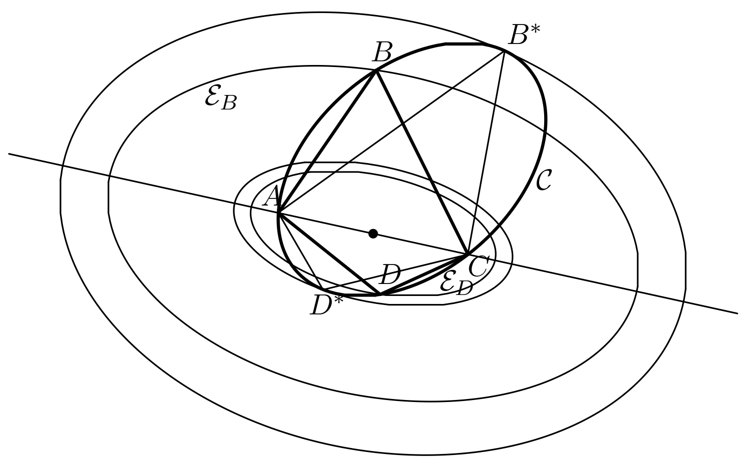


Figure 6. $Perim(AB^*CD^*) > Perim(ABCD)$

clear that the strict convexity and smoothness of $\mathcal{C}$ and the invariant maximal perimeter hypothesis of Proposition 2 imply that the second vertex $B$ (with respect to the clockwise orientation of $\mathcal{C}$), the third vertex $C$, and the forth vertex $D$ all vary smoothly with respect to the first vertex $A$. Also, as $A$ progresses increasingly clockwise, so will $B$, $C$, and $D$, in this order, or else the uniqueness part of the hypothesis will be contradicted, given that the lengths of all sides of quadrilaterals of largest perimeter must be bounded below by a strictly positive constant. In other words, fixing an inscribed quadrilateral of largest perimeter, $A_0B_0C_0D_0$, as $A$ increases steadily from $A_0$ to $B_0$, $B$ increases from $B_0$ to $C_0$, $C$ increases from $C_0$ to $D_0$, and $D$ increases from $D_0$ to $A_0$.

Now, if some maximal perimeter quadrilateral were not a parallelogram and the center $O$ of the oval were, say, inside the angle $\angle B_0O_0C_0$, where $O_0$ were the intersecting point of the diagonals of $A_0B_0C_0D_0$, then the quadrilateral $CDAB$, the symmetric of $A_0B_0C_0D_0$ with respect to $O$, which evidently had also maximal perimeter, would contradict the above discussion since consecutive vertices $D$ and $A$ could not both belong to the arc $\widehat{B_0C_0}$ (Figure 7). To conclude the proof, if $ABCD$ is a quadrilateral, in fact a parallelogram, of largest perimeter, then the confocal ellipse argument presented earlier cannot increase its perimeter, so the respective ellipses must completely contain the oval and be tangent to it at the appropriate vertices (Figure 5). Then the familiar reflective property of an ellipse also proves the billiard property of $ABCD$ with respect to $\mathcal{C}$.

$\boxed{QED}$

In order to state the main result of this section we need a certain global curvature-growth condition on a quarter-oval, dubbed below as *The k-condition.*

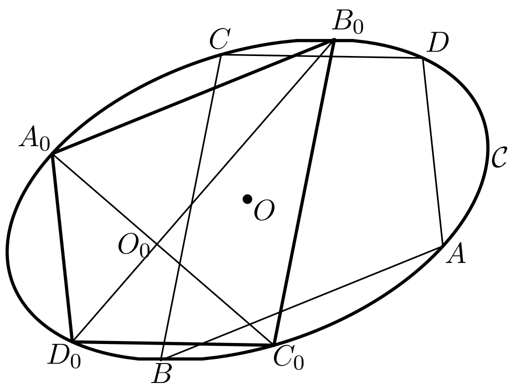


Figure 7. Inscribed quadrilateral of largest perimeter, $A_0B_0C_0D_0$, must be parallelogram

To this end, consider a coordinate system $XOY$ and in its third quadrant a smooth (class $C^2$), concave down arc $\widehat{WN}$ with defining function $f : [-a, 0] \longrightarrow [0, b]$, just as in the proof of the main result of the previous section. Then for every angle $\frac{\pi}{2} \leq t \leq \pi$ we define the *support function* $p(t)$ of $\widehat{WN}$ [4] as the distance $OT$ from $O$ to the unique tangent line to a suitable point of $\widehat{WN}$ perpendicular to the ray $\overrightarrow{OZ}$ making a (directed) angle $t$ with $\overrightarrow{OX}$ (Figure 8). Clearly, the support function $p(t)$ is smooth (class $C^2$) and elemen-

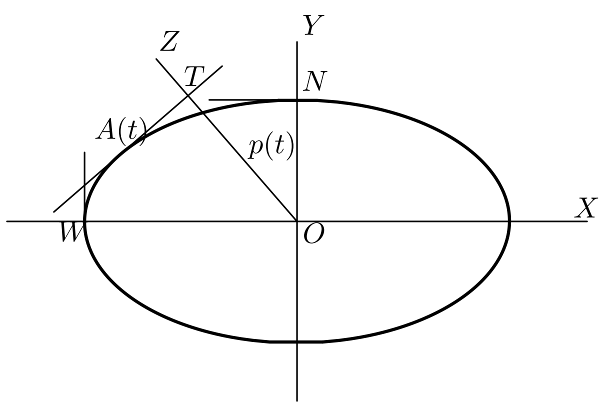


Figure 8. The support function $p(t)$

tary calculations show that the coordinates of the point of tangency $A(t)$ are

$(p(t)\cos t - p'(t)\sin t, p(t)\sin t + p'(t)\cos t)$. Consequently, for every $\frac{\pi}{2} \le t \le \pi$,

$$f(p(t)\cos t - p'(t)\sin t) = p(t)\sin t + p'(t)\cos t. \tag{7}$$

Conversely, if $(x, f(x))$, $x \in [-a, 0]$, is an arbitrary point of $\widehat{WN}$ then the unique angle $\frac{\pi}{2} \le t \le \pi$ such that $A(t)$ has coordinates $(x, f(x))$ satisfies

$$\tan t = -\frac{1}{f'(x)} \quad \text{and} \quad p(t) = \frac{f(x) - xf'(x)}{\sqrt{f'(x)^2 + 1}}, \tag{8}$$

a consequence of the fact that the point $T$ has coordinates

$$\left(\frac{-f'(x)\,(f(x) - xf'(x))}{f'(x)^2 + 1}, \frac{f(x) - xf'(x)}{f'(x)^2 + 1}\right).$$

When the arc $\widehat{WN}$ is parametrized via its support function $p(t)$ the radius of curvature at the point $A(t)$ equals [4]

$$R_{A(t)} = p(t) + p''(t), \tag{9}$$

and so the identification $t \equiv x$, $A(t) \equiv (x, f(x))$, yields

$$p(t) + p''(t) = -\frac{\left(f'(x)^2 + 1\right)^{3/2}}{f''(x)} \tag{10}$$

**Definition 1** (The $k$-condition)**.** It states that the function $\tilde{p}(t)$ given for $\frac{\pi}{2} \le t \le \pi$ by the formula $\tilde{p}(t) := \sqrt{a^2 + b^2 - p(t)^2}$ is well-defined and also satisfies

$$\tilde{p}(t) + \tilde{p}''(t) > 0, \quad \frac{\pi}{2} \le t \le \pi. \tag{11}$$

The geometric meaning of Equation (11), to be fully revealed below, is that by defining for $0 \le t \le \frac{\pi}{2}$, $p(t) := \tilde{p}\left(t + \frac{\pi}{2}\right)$, we obtain in $p(t)$, $0 \le t \le \pi$, the support function of a concave down half-oval $\widehat{WNE}$, an extension of $\widehat{WN}$ to the first quadrant of the coordinate system $XOY$.

We mention here for later use that under the identification $t \equiv x$ Equation (11) is equivalent to the fact that the transformation

$$[-a, 0] \ni x \longmapsto \pi(x) := \frac{a^2 + b^2 - f(x)^2 + xf(x)f'(x)}{\sqrt{(a^2 + b^2)(f'(x)^2 + 1) - (f(x) - xf'(x))^2}} \in [0, a] \tag{12}$$

is well-defined and strictly increasing ($\pi'(x) > 0$, for $-a < x \le 0$).

**Theorem 2** (Classification of Centrally Symmetric Ovals Satisfying the Quadrilateral Invariant Maximal Perimeter Property)**.** *The following two statements are equivalent:*

*a) $\mathcal{C}$, a centrally symmetric smooth (class $C^2$) oval with center $O$ satisfies the quadrilateral invariant maximal perimeter assumption, that is for any point of $\mathcal{C}$, among all the convex quadrilaterals inscribed in $\mathcal{C}$ and having that point as a vertex there is only one of maximal perimeter, and all these maximal quadrilaterals have the same perimeter, independent of the point selected.*

*b) A centrally symmetric smooth (class $C^2$) oval $\mathcal{C}$ with center $O$ is completely determined by its top left quarter, in the sense that starting with a smooth (class $C^2$) concave down arc $\widehat{WN}$ in the third quadrant of a coordinate system $XOY$, $W \in \overleftrightarrow{OX}$, $N \in \overleftrightarrow{OY}$, such that the half-tangents to the arc at the end-points $W$ and $N$ are parallel to the coordinate axes, such that the radii of curvature of the arc at $W$ and $N$, $R_W$ and $R_N$, satisfy the relation*

$$OW \cdot R_W + ON \cdot R_N = OW^2 + ON^2, \tag{13}$$

*and such that the support function $p(t)$ of the arc with respect to the point $O$ satisfies The $k$-condition (11), where $a = OW$ and $b = ON$, the mapping $A \to S(A)$, $A \in \widehat{WN}$, described below is a bijective, increasing transformation from the top left quarter of a centrally symmetric (center $O$) smooth oval $\mathcal{C}$ to the top right quarter of $\mathcal{C}$.*

*In order to describe the transformation $A \longmapsto S(A)$, let $A$ be an arbitrary point on the arc $\widehat{WN}$ and let $U$ be the intersection point of the ascending half-tangent to the arc at $A$ and the circle with center at $O$ and radius $\sqrt{OW^2 + ON^2}$. Further, let $V$ be the symmetric point with respect to the line $\lambda$, the perpendicular line to $\overrightarrow{AU}$ through the point $U$, of the point $C$ opposite to $A$ with respect to $O$. Then $S(A)$ is the intersection point of the line segment $\overline{AV}$ and the line $\lambda$ (Figure 9). Moreover, the line $\lambda = \overleftrightarrow{US(A)}$ is tangent to $\mathcal{C}$ at the point $S(A)$.*

*When the oval $\mathcal{C}$ satisfies the two equivalent characterizations above, $S(A)$ is the second vertex, vertex $B$, of the maximal perimeter quadrilateral $ABCD$ based at $A$. In fact, by Proposition 2 this quadrilateral is a parallelogram, so the vertex $C$ is the symmetric with respect to $O$ of the vertex $A$ and the vertex $D$ is the symmetric with respect to $O$ of $S(A)$. Moreover, the maximal perimeter value of inscribed quadrilaterals in $\mathcal{C}$ equals $4\sqrt{OW^2 + ON^2}$.*

*Proof.* If the invariant maximal perimeter assumption is restricted to the class of inscribed parallelograms rather than general quadrilaterals then the Theorem holds, compare it with the Corollary in [1]. In fact, the main result of [1] shows that a centrally symmetric oval $\mathcal{C}$ satisfies the parallelogram invariant maximal perimeter assumption if and only if its orthoptic curve is a circle, or equivalently

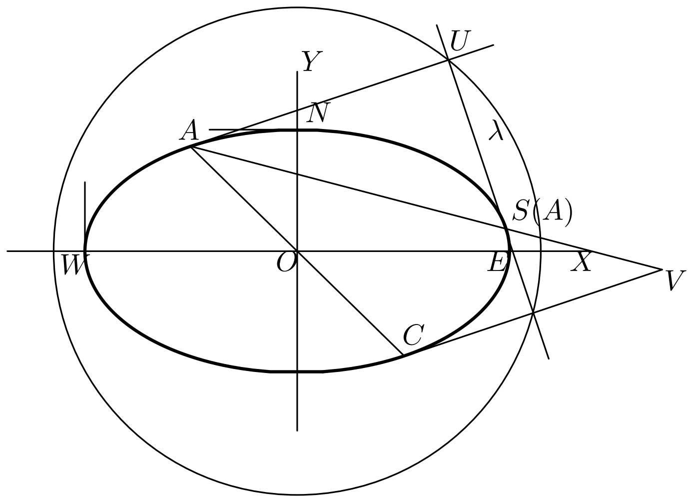


Figure 9. Construction of S(A)

[4], with respect to a coordinate system centered at the origin $O$ of the oval the support function $p(t)$, $t \in \mathbf{R}$, of $\mathcal{C}$ satisfies the property

$$p^2(t) + p^2\left(t + \frac{\pi}{2}\right) = \text{positive constant}, \quad t \in \mathbf{R} \tag{14}$$

Recall that the orthoptic curve of an oval is the locus of all the points from where the oval can be seen at a right angle.

It is now a routine exercise to see that the characterization $b)$ in our Theorem is a precise analytic transcription of the comment following the Corollary in [1].

For instance, referring to the points $W$ and $N$ of $b)$ we see that $p\left(\frac{\pi}{2}\right) = b$ and $p(\pi) = a$, so the constant in Equation (14) is $a^2 + b^2$. Moreover, if $A(t)$ is a point on the oval which is also an end-point to a principal chord in $\mathcal{C}$, then $p'(t) = 0$. Differentiating then twice Equation (14) with respect to $t$ and making suitable use of Equation (9) for $t = \frac{\pi}{2}$ and $t = \pi$, we get the transmission condition (13) at the end-points $W$ and $N$ of the arc $\widehat{WN}$.

Also, if the point $A$ on the arc $\widehat{WN}$ has coordinates $(x, f(x))$ then the point $S(A)$ described in $b)$ has coordinates

$$\left(\frac{a^2 + b^2 - f(x)^2 + xf(x)f'(x)}{\sqrt{E(f)(x)}}, \frac{(a^2 + b^2 - x^2)f'(x) + xf(x)}{\sqrt{E(f)(x)}}\right), \tag{15}$$

where $E(f)(x) := (a^2+b^2)(f'(x)^2+1)-(f(x)-xf'(x))^2$. Then The $k$-condition (11) is equivalent, via Equations (12) and (15), to the fact that the function

$$[-a, 0] \ni \text{abscissa of } A \longmapsto \text{abscissa of } S(A) \in [0, a]$$

is strictly increasing.

So all it remains to be shown is that the invariant maximal perimeter assumptions for quadrilaterals and parallelograms coincide. On one hand, Proposition 2 shows that the quadrilateral invariant maximal perimeter assumption implies the parallelogram invariant maximal perimeter assumption. On the other hand, assuming the parallelogram invariant maximal perimeter condition, or equivalently [1] assuming that the orthoptic curve of the oval is a circle, it suffices to show that no other inscribed quadrilaterals besides parallelograms can attain a maximal perimeter. By Proposition 2, it further suffices to show that no point $A$ of the oval can be vertex to a billiard trajectory of period 4, $ABCD$, which is not a parallelogram.

By contradiction, assume that $ABCD$ is a non-parallelogram billiard trajectory of period 4 in $\mathcal{C}$. To the point $A$ we associate the unique maximal perimeter inscribed parallelogram $A_0B_0C_0D_0$, where $A_0 = A$, which we know to be a billiard trajectory. Without loss of generality we can assume that the point $B$ is located inside the oval arc $\widehat{A_0B_0}$, traversed clockwise. Then the tangent lines to $\mathcal{C}$ at the points $A_0$, $B_0$, $C_0$, and $D_0$, meet at the points $P_0$, $Q_0$, $R_0$, and $S_0$, situated on the orthoptic circle $\Gamma$ of $\mathcal{C}$ (Figure 10). By symmetry, $P_0Q_0R_0S_0$ is a rectangle. To the point $B$ we can also associate the unique maximal perimeter inscribed parallelogram $B_1C_1D_1A_1$, where $B_1 = B$, with orthoptic rectangle $Q_1R_1S_1P_1$. Due to the assumption $B \in int(\widehat{A_0B_0})$ a simple slope argument shows that the distinct points $D_0, A_1, A_0 = A, B_1 = B, B_0, C_1$, and $C_0$ are situated clockwise exactly in this order around $\mathcal{C}$. Consequently, the interior of the arc $\widehat{A_1B_1}$ is completely contained inside the right triangle $\triangle A_1P_1B$, and so $meas(\angle ABP_1) < meas(\angle A_1BP_1) = meas(\angle C_1BQ_1)$. However, since $ABCD$ is a billiard trajectory, $meas(\angle CBQ_1) = meas(\angle ABP_1)$. It follows that $meas(\angle CBQ_1) < meas(\angle C_1BQ_1)$, which implies that the point $C$ belongs to the interior of the arc $\widehat{ABC_0}$. At last, if follows that the point $C$ belongs to the interior of the arc $\widehat{ABC_0}$.

Similar arguments, applied to the point $D$ instead of the point $B$, which must clearly belong to the interior of the arc $\widehat{D_0A_1A}$, show that the vertex $C$ must also belong to the interior of the arc $\widehat{C_0DA}$ (Figure 10). However, this is a contradiction since the interior of the arcs $\widehat{ABC_0}$ and $\widehat{C_0DA}$ are disjoint. $\boxed{QED}$

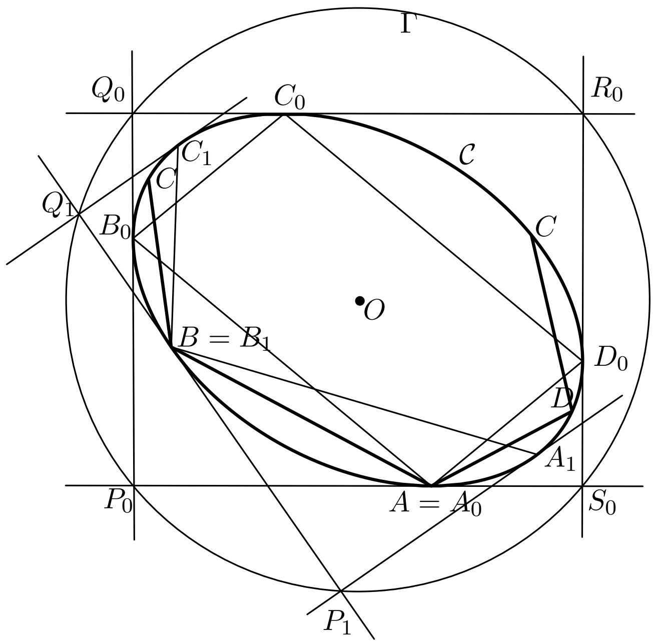


Figure 10. In oval with orthoptic curve a circle there are no non-parallelogram billiard trajectories $ABCD$ of period 4.

## 3 The mixed area-perimeter problem

We saw in the previous two sections that imposing on a centrally symmetric smooth oval either the inscribed quadrilateral invariant maximal area assumption or the maximal perimeter assumption still leaves one quarter of it pretty flexible. But what about imposing the area and perimeter assumptions simultaneously? Will the end-result be just the ellipse?

**Proposition 3.** *If a centrally symmetric smooth oval $\mathcal{C}$ with center $O$ obeys both the inscribed quadrilateral invariant maximal area assumption and the inscribed quadrilateral invariant maximal perimeter assumption then for any pair of perpendicular principal chords if in the associated coordinate system the oval has semi-axes $a$ and $b$, support function $p(t)$ and radial distance function $\rho(t)$, $t \in \mathbf{R}$, then*

$$\rho(t) = \frac{ab}{\sqrt{a^2 + b^2 - p^2(t)}}, \quad t \in \mathbf{R}. \tag{16}$$

*Furthermore, in terms of the defining function $f : [-a, 0] \longrightarrow [0, b]$ of the top* *left quarter of the oval $\mathcal{C}$ we have for any $x \in (-a, 0]$,*

$$f\left(-\frac{abf'(x)}{\sqrt{E(f)(x)}}\right) = \frac{ab}{\sqrt{E(f)(x)}}, \tag{17}$$

*where* $E(f)(x) = (a^2 + b^2)(f'(x)^2 + 1) - (f(x) - xf'(x))^2$.

*Proof.* The proof is a simple consequence of the main results of the previous two sections. Indeed, from Section 2 in a coordinate system adapted to a pair of perpendicular principal chords in which the oval has semi-axes $a$ and $b$ we know that the support function $p(t)$ satisfies Equation (14) for the constant $a^2 + b^2$. At the same time, for fixed $t \in \mathbf{R}$ the parallelogram of largest area based at the point with coordinates $(\rho(t)\cos t, \rho(t)\sin t)$ has area, by Proposition 1 in Section 1, equal to $2\rho(t)p\left(t + \frac{\pi}{2}\right)$, which also equals $2ab$ by maximal area invariance. Now, the two equations

$$p(t)^2 + p\left(t + \frac{\pi}{2}\right)^2 = a^2 + b^2 \quad \text{and} \quad \rho(t)p\left(t + \frac{\pi}{2}\right) = ab$$

yield the conclusion of Equation (16). Also, Equation (17) follows from the fact that for $\frac{\pi}{2} \leq t \leq \pi$, $f(\rho(t)\cos t) = \rho(t)\sin t$, via Equations (16) and (8). $\boxed{QED}$

The geometric condition (16) or the analytic condition (17) appear to be strong enough to force an oval $\mathcal{C}$ as in Proposition 3 to be the ellipse centered at $O$ and with semi-axes $a$ and $b$. It is not entirely obvious though how to show, for instance, that the differential-equation-look-alike (17) has the unique solution $f(x) = \frac{b}{a}\sqrt{a^2 - x^2}$, $-a \leq x \leq 0$. However, under the stronger hypothesis that the oval $\mathcal{C}$ be analytic this is indeed the case.

**Theorem 3** (Classification of Centrally Symmetric Analytic Ovals Simultaneously Satisfying the Quadrilateral Invariant Maximal Area and Perimeter Properties)**.** *If a centrally symmetric analytic oval $\mathcal{C}$ with center $O$ obeys both the inscribed quadrilateral invariant maximal area assumption and the inscribed quadrilateral invariant maximal perimeter assumption then for any pair of perpendicular principal chords if in the associated coordinate system the oval has semi-axes $a$ and $b$ it must be the ellipse with center $O$ and semi-axes $a$ and $b$.*

*Proof.* By invoking Proposition 3 if suffices to show that an analytic function $f : (-a, 0] \longrightarrow (0, b]$, $f(0) = b$, $\lim_{x \to -a^+} f(x) = 0$, $f'(0) = 0$, $f'(x) > 0$, $x \in (-a, 0)$,

$f''(x) < 0$, $x \in (-a, 0]$ and satisfying Equation (17) must be the quarter-ellipse $f(x) = \dfrac{b}{a}\sqrt{a^2 - x^2}$, $-a < x \leq 0$.

Without loss of generality we can assume $a \geq b$. It is now a routine exercise to show that $f(x) := \dfrac{b}{a}\sqrt{a^2 - x^2}$, $-a < x \leq 0$ is a solution of (17), as expected, but also that for $x$ negative and sufficiently close to 0, $f(x) := \sqrt{b^2 - x^2}$ is again solution of Equation (17), near $x = 0$. This is surprising, given that (17) involves $a$, which does not appear in the second solution $f(x)$. Notice that the two solutions are distinct if $a > b$ and that in this case the second solution cannot represent a valid analytic quarter-oval as it does not extend through analyticity past $x = -b$.

It then suffices, in the case $a > b$, to prove that there are only two solutions of Equation (17) which are analytic at $x = 0$. In fact, what we will show is that for an analytic solution of (17) the derivatives $f^{(n)}(0)$, $n \geq 3$, can be recurrently expressed in terms of the lower order derivatives, and that $f''(0)$ can take only one of two values, $-\dfrac{b}{a^2}$ or $-\dfrac{1}{b}$, which indeed correspond to the second order derivatives for the solutions exhibited in the previous paragraph.

If $f(x)$ is an analytic solution of Equation (17) on the interval $(-a, 0]$, denote by $u(x)$, $v(x)$, and $\gamma(x)$ the associated functions

$$\begin{aligned} u(x) &:= (a^2 + b^2 - x^2)f'(x) + xf(x) \\ v(x) &:= a^2 + b^2 - f(x)^2 + xf(x)f'(x) \\ \gamma(x) &:= -\frac{abf'(x)}{\sqrt{E(f)(x)}} = -\frac{abf'(x)}{\sqrt{u(x)f'(x) + v(x)}} \end{aligned} \tag{18}$$

Clearly, Equation (17) is equivalent to

$$f'(x)f(\gamma(x)) + \gamma(x) = 0, \quad x \in (-a, 0]. \tag{19}$$

Differentiating once with respect to $x$ the last equation in (18), and (19), yields

$$\begin{aligned} \gamma'(x) &= -\frac{abf''(x)v(x)}{(u(x)f'(x) + v(x))^{3/2}} \\ f'(\gamma(x)) &= \frac{u(x)}{v(x)} \end{aligned} \tag{20}$$

Further,

$$f''(\gamma(x))\gamma'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)},$$

which upon setting $x = 0$ becomes

$$-bf''(0) = \frac{(a^2 + b^2)f''(0) + b}{a^2}. \tag{21}$$

Equation (21) gives $f''(0) = -\dfrac{b}{a^2}$ or $f''(0) = -\dfrac{1}{b}$, as inferred earlier. Let us mention that the two curvature transmission conditions at the end-points of the arc $\widehat{WN}$ in the main theorems of the previous two sections give precisely the same values for $f''(0)$ as Equation (21) does.

In order to conclude that the higher order derivatives $f^{(n)}(0)$, $n \geq 3$, depend recursively on $f(0)$, $f'(0)$, $f''(0)$, ..., $f^{(n-1)}(0)$, we make use now of two classical formulas giving the $n^{\text{th}}$-order derivatives of products of functions and composition of functions, the Leibniz product formula and the Faà di Bruno composition formula [6]: If $g(x)$ and $h(x)$ are two $C^\infty$ functions on some interval and their composite $g(h(x))$ is well-defined, then

$$\begin{aligned} (gh)^{(n)}(x) &= \sum_{p=0}^{n} \binom{n}{p} g^{(n-p)}(x) h^{(p)}(x) \\ (g(h))^{(n)}(x) &= \sum \frac{n!}{k_1! k_2! \dots k_n!} g^{(k)}(h(x)) \left(h'(x)\right)^{k_1} \left(h''(x)\right)^{k_2} \dots \left(h^{(n)}(x)\right)^{k_n}, \end{aligned} \tag{22}$$

where in the second sum $k = k_1 + k_2 + \cdots + k_n$, and the sum is taken over all the non-negative integers $k_1$, $k_2$, ..., $k_n$ such that $k_1 + 2k_2 + \cdots + nk_n = n$. Using now Equation (18) it is straightforward to conclude that for $n \geq 1$ and $x \in (-a, 0]$,

$$\begin{aligned} u^{(n)}(x) &= (a^2 + b^2 - x^2) f^{(n+1)}(x) + (1 - 2n) x f^{(n)}(x) + n(2 - n) f^{(n-1)}(x) \\ v^{(n)}(x) &= \sum_{p=0}^{n} \binom{n}{p} \left(x f^{(p+1)}(x) + (p - 1) f^{(p)}(x)\right) f^{(n-p)}(x) \end{aligned}$$

which, when specialized to $x = 0$, yield

$$\begin{aligned} u^{(n)}(0) &= (a^2 + b^2) f^{(n+1)}(0) + n(2 - n) f^{(n-1)}(0) \\ v^{(n)}(0) &= \sum_{p=0}^{n} \binom{n}{p} (p - 1) f^{(p)}(0) f^{(n-p)}(0) \end{aligned} \tag{23}$$

We claim now that for $n \geq 1$,

$$\gamma^{(n)}(0) = -bf^{(n+1)}(0) + E_n\left(a, f(0), f'(0), \dots, f^{(n)}(0)\right), \tag{24}$$

where $E_n$ is an expression depending only on $a$ and the derivatives $f^{(i)}(0)$, $0 \leq i \leq n$. This follows by induction on $n$, taking into account that since

$$\sqrt{u(x)f'(x) + v(x)}\gamma(x) = -abf'(x),$$

the Leibniz product formula gives for any $k$,

$$\sum_{p=0}^{k} \binom{k}{p} \left(\sqrt{uf' + v}\right)^{(k-p)} (0)\gamma^{(p)}(0) = -abf^{(k+1)}(0),$$

and also that by the Faà di Bruno composition formula, the Leibniz product formula, and Equation (23), $\left(\sqrt{uf' + v}\right)^{(i)} (0)$ depends only on $a$, $f(0)$, $f'(0)$, $\dots$, $f^{(i)}(0)$, and $f^{(i+1)}(0)$.

Similarly, starting with the formula given by the second equation (20), the Faà di Bruno composition formula yields, for $n \geq 2$,

$$\begin{aligned}\left(\frac{u}{v}\right)^{(n)} (0) =& f^{(n+1)}(\gamma(0)) \left(\gamma'(0)\right)^n + f''(\gamma(0))\gamma^{(n)}(0) + \\ & F_n \left(\gamma'(0), \dots, \gamma^{(n-1)}(0), f''(0), \dots f^{(n)}(0)\right),\end{aligned} \tag{25}$$

where $F_n$ depends on the quantities specified. Equations (25) and (24) give then

$$\begin{aligned}\left(\frac{u}{v}\right)^{(n)} (0) = {}& f^{(n+1)}(0) \left(-bf''(0)\right)^n \\ & + f''(0) \left(-bf^{(n+1)}(0)\right) + \tilde{F}_n \left(a, f(0), f'(0), \dots f^{(n)}(0)\right).\end{aligned} \tag{26}$$

Finally, the Leibniz product formula applied to the quotient $\dfrac{u}{v}$ gives

$$\left(\frac{u}{v}\right)^{(n)} (0) = \frac{a^2 + b^2}{a^2} f^{(n+1)}(0) + G_n \left(a, f(0), f'(0), \dots f^{(n)}(0)\right). \tag{27}$$

Putting together Equations (26) and (27) we conclude that for $n \geq 2$

$$\begin{aligned}&\left((-1)^{n+1} \left(f''(0)\right)^n b^n + bf''(0) + \frac{a^2 + b^2}{a^2}\right) f^{(n+1)}(0) \\ &\qquad = H_n \left(a, f(0), f'(0), \dots f^{(n)}(0)\right).\end{aligned} \tag{28}$$

Equation (28) shows now that for $n \geq 2$, $f^{(n+1)}(0)$ can be expressed recursively in terms of lower order derivatives if and only if the coefficient

$$(-1)^{n+1} \left(f''(0)\right)^n b^n + bf''(0) + \frac{a^2 + b^2}{a^2} \tag{29}$$

does not vanish. This is indeed the case for the two admissible values of $f''(0)$, if $a > b$.

The case $a = b$ when the ellipse becomes a circle cannot be settled by exactly the same argument as above since the coefficient (29) vanishes now. However, it can be seen that $f''(0)$ admits only one value, $-\dfrac{b}{a^2} = -\dfrac{1}{b}$, so an argument along similar lines as above can be pursued. The proof of the theorem is complete. $\boxed{QED}$

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