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Two notes on the axiomatics of structures with parallelism

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Abstract. We provide: (1) an independent quantifier-free axiomatization for André's central translation structures and state a conjecture, which, if true, would show a very strong connection between central translation structures and translation planes; (2) a first-order axiomatization of Everett's and Permutti's affine geometries over rings without zero divisors in which any two non-zero elements have a right greatest common divisor.

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1 What are central translation structures?

Central translation structures were introduced by J. André [2] as a generalization of translation planes. A central translation structure was defined in [2] as an incidence structure — in which every line is incident with at least two points, and which contains three non-collinear points — endowed with a parallelism relation that satisfies the Euclidean parallel postulate (existence and uniqueness of a parallel through a given point to a given line) and all the universal statements valid in arbitrary affine planes, such that all of its translations (fixpoint-free collineations which map lines into parallel lines, and the identity) are central (if τ is a proper translation and p, q are two points, then the lines $\langle p, \tau(p)\rangle$ and $\langle q, \tau(q)\rangle$ are parallel), form a group, and act transitively on the set of points. These are axioms (a0)–(a3), (t1), (t2) and 'All translations are central' in [1] and [2]. It is easily seen that these may be restated without using two sorts of individuals, for 'points' and 'lines', by rephrasing 'incidence' in terms of the ternary predicate of 'collinearity' (to be denoted by L, with $L(abc)$ to be read as 'points a, b, c are collinear' or 'point c belongs to the line determined by a and b, or $a = b$'), and by rephrasing the binary predicate of 'line parallelism' as a quaternary predicate among points, (to be denoted by $\parallel$, with '$ab \parallel cd$' to be read as ' $a \neq b$, $c \neq d$, and the line determined by a and b is parallel to the line determined by c and d').

Our first order language $\mathcal{S}$ thus contains one sort of variables, for 'points', the relation symbols L, $\|$, and the ternary operation symbol τ, with $\tau(abc) = d$ to be read as 'd is the image of c under the translation that maps a into b'. A more suggestive notation, which we shall use instead of $\tau(abx)$, is $\tau_{ab}(x)$. We shall also use $\sigma_a(b)$ as an abbreviation for $\tau_{ba}(a)$.

André's axiom system can be rephrased as:

A 1. $L(aba)$,

A 2. $L(abc) \to L(cba) \wedge L(bac)$,

A 3. $a \neq b \wedge L(abc) \wedge L(abd) \to L(acd)$,

A 4. $ab \parallel cd \to a \neq b$,

A 5. $c \neq e \wedge ab \parallel cd \wedge L(cde) \to ab \parallel ce$,

A 6. $a \neq b \to ab \parallel ba$,

A 7. $a \neq b \to ab \parallel ab$,

A 8. $ab \parallel cd \wedge ab \parallel ef \to cd \parallel ef$,

A 9. $(\forall pab)(\exists q)[a \neq b \to ab \parallel pq]$,

A 10. $ab \parallel cd \wedge ab \parallel ce \to L(cde)$,

A 11. $(\exists abc)\, \neg L(abc)$,

A 12. $\tau_{ab}(a) = b$,

A 13. $\tau_{ab}(x) = x \leftrightarrow a = b$,

A 14. $p \neq q \to pq \parallel \tau_{ab}(p)\tau_{ab}(q)$,

A 15. $a \neq b \to p\tau_{ab}(p) \parallel q\tau_{ab}(q)$,

A 16. $\tau_{cd}(\tau_{ab}(x)) = \tau_{a\tau_{cd}(b)}(x)$.

Axioms A1–A11 are axioms stating the properties of the parallelism and collinearity relations, which may be seen as reformulations of André's [2] axioms (a0)–(a3) and (t2). The remaining axioms express in our language the axiom (t1) and the axiom "All translations are central". To be precise: A12 states that translations act transitively, A13 that proper translations don't have fixed points, A14 that the image of a line under a translation is a parallel line; A15 that translations are central, and A16 that the composition of two translations is a translation (thus translations form a group under composition, with τ_{aa} as unit, and τ_{ba} the inverse of τ_{ab}).

The first-order language $\mathcal{L}$ in which the quantifier-free axioms will be formulated has only one sort of variables (to be interpreted as 'points'), three individual constants a_0, a_1, a_2, the ternary relation L for collinearity, and the ternary operation τ. The axioms are A3 and

B 1. $L(abc) \to L(bac)$,

B 2. $\tau_{ab}(c) = \tau_{ac}(b)$,

B 3. $L(ab\sigma_a(b))$,

B 4. $L(abc) \to L(x\tau_{ab}(x)\tau_{ac}(x))$,

B 5. $\tau_{ab}(x) = x \to a = b$,

B 6. $\tau_{ab}(x) = \tau_{c\tau_{ab}(c)}(x)$,

B 7. $\neg L(a_0a_1a_2)$.

Let $\Gamma = \{\mathrm{A1}-\mathrm{A16}\}$ and $\Sigma = \{\mathrm{B1}-\mathrm{B7}, \mathrm{A3}\}$. To show that Σ is equivalent to André's axiom system for central translation structures, we need to show that the axioms in Γ are provable from $\Sigma \cup \{(1)\}$, where

$$ab \parallel cd \leftrightarrow a \neq b \wedge c \neq d \wedge L(cd\tau(bac)). \tag{1}$$

That, vice-versa, all axioms of Σ hold in all central translation structures is trivial to check, given the representation theorem for central translation planes (see below).

Theorem 1. $\Sigma \cup \{(1)\} \vdash \Gamma$.

Proof. Suppose $\tau_{ab}(x) = \tau_{ab}(y)$. We want to conclude that $x = y$. By B6 and B2 we have $\tau_{ab}(x) = \tau_{yx}(\tau_{ab}(y))$, whence $\tau_{yx}(\tau_{ab}(y)) = \tau_{ab}(y)$. Therefore, by B5,

$$\tau_{ab}(x) = \tau_{ab}(y) \to x = y. \tag{2}$$

By B6 we have $\tau_{ab}(b) = \tau_{a\tau_{ab}(a)}(b)$, so using B2 and (2) we get

$$\tau_{ab}(a) = \tau_{aa}(b) = b, \tag{3}$$

which shows that A12 holds. A13 follows from B5 and (3). As a special case of (3) we get

$$\sigma_a(a) = a. \tag{4}$$

We also notice that, from the definition of σ and B5 we deduce

$$\sigma_a(b) = a \to a = b. \tag{5}$$

With $c = b$ and $x = a$, B6 becomes $\tau_{b\tau_{ab}(b)}(a) = \tau_{ab}(a)$, which, by (3) and B2, implies

$$\tau_{ba}(\sigma_b(a)) = b. \tag{6}$$

Let $a' = \sigma_b(a)$. By B6 we have $\tau_{a'b}(a') = \tau_{b\tau_{a'b}(b)}(a')$, *i. e.* $\tau_{b\tau_{a'b}(b)}(a') = b$ (by (3)). By (6) this means that $\tau_{b\tau_{a'b}(b)}(a') = \tau_{ba}(a')$. Using B2 and (2) we conclude that $\tau_{a'b}(b) = a$, *i. e.* that

$$\sigma_b(\sigma_b(a)) = a. \tag{7}$$

Notice that B1 is a truncated version of A2, whose first conjunct of the consequent has been deleted. We now turn to the proof of the deleted part of A2, *i. e.* to $L(abc) \to L(cba)$. If $a = c$, then $L(abc) \to L(cba)$ is a tautology, so we may always assume that $a \neq c$. We shall first prove it for $a \neq b$. Suppose $L(abc)$ and $a \neq b$. By B3 we have $L(ab\sigma_a(b))$, whereas A3 gives $a \neq b \wedge L(ab\sigma_a(b)) \wedge L(abc) \to L(a\sigma_a(b)c)$. Since the antecedent of the above implication holds, its consequent, $L(a\sigma_a(b)c)$, holds as well. By B3 we have $L(a\sigma_a(b)\sigma_a(\sigma_a(b)))$, *i. e.*, since $\sigma_a(\sigma_a(b)) = b$ (by (7)), we have $L(a\sigma_a(b)b)$. Since $a \neq \sigma_a(b)$ (by (5)), the antecedent of $a \neq \sigma_a(b) \wedge L(a\sigma_a(b)c) \wedge L(a\sigma_a(b)b) \to L(acb)$ (which holds by A3) holds, and so its consequent, $L(acb)$, holds as well. This proves

$$a \neq b \wedge L(abc) \to L(acb), \tag{8}$$

and since the antecendent of (8) holds, so does the consequent, $L(acb)$. Applying B1 with antecedent $L(acb)$, we get $L(cab)$, and, given that we may assume $a \neq c$, applying (8) with $L(cab)$ as antecedent we get $L(cba)$. We have thus shown that

$$a \neq b \wedge L(abc) \to L(cba). \tag{9}$$

We now turn to the proof of A1. Let $a \neq b$. Then $\sigma_b(a) \neq b$ (by (5)). We have $L(ba\sigma_b(a))$ (by B3)), therefore $L(b\sigma_b(a)a)$ (by (8)). By A3, $a \neq b \wedge L(b\sigma_b(a)a) \wedge L(b\sigma_b(a)a) \to L(baa)$, therefore $L(baa)$, whence, by B1, $L(aba)$, which is A1 for $a \neq b$. For $a = b$, A1 is a consequence of B3 and of (4).

Apply now B1 with $L(aca)$ (which holds, by A1) as antecedent to get $L(caa)$. This shows that (9) holds without the condition $a \neq b$ as well, and proves that A2 holds in $\Sigma \cup \{(1)\}$.

Since $\tau_{ba}(\tau_{ab}(c)) = \tau_{b\tau_{ac}(b)}(a) = \tau_{ac}(a) = c$ (the first equation follows from B2, the second from B6 and the third from (3)) and

$$\tau_{ba}(c) = \tau(\tau(abc)\tau(ba\tau(abc))c)$$

(by B6), we get

$$\tau_{ba}(c) = \sigma_c(\tau_{ab}(c)). \tag{10}$$

Notice that A4 is an immediate consequence of (1), A5 follows from (1) and A3, and A7 is an immediate consequence of B3 and (1). To prove A6 notice that, by (1), $ab \parallel ba$ is equivalent to $a \neq b \wedge L(ba\tau_{ba}(b))$, i. e. to $a \neq b$ and $L(baa)$ (by (3)), which is a consequence of A1 and B1.

Let now $ab \parallel cd$ and $ab \parallel ef$. We want to show that $cd \parallel ef$, thus proving A8. By (1) our hypothesis amounts to $a \neq b$, $c \neq d$, $e \neq f$, $L(cd\tau_{ba}(c))$ and $L(ef\tau_{ba}(e))$. From B4 applied to $L(cd\tau_{ba}(c))$ we get $L(e\tau(cde)\tau(c\tau(bac)e))$, *i. e.*, since $\tau(cde) = \sigma_e(\tau(dce))$ (by (10)) and $\tau_{c\tau_{ba}(c)}(e) = \tau_{ba}(e)$ (by B6), we have $L(e\sigma_e(\tau_{dc}(e))\tau_{ba}(e))$. Since we also have $L(e\tau_{dc}(e)\sigma_e(\tau_{dc}(e))$ (by B3) and $\sigma_e(\tau_{dc}(e)) \neq e$ (by $c \neq d$, (5) and B5), using A2 and A3 we conclude that $L(e\tau_{dc}(e)\tau_{ba}(e))$. Since we also have $L(ef\tau_{ba}(e))$ from our hypothesis and $\tau_{ba}(e) \neq e$ (by B5), using A2 and A3 we get $L(ef\tau_{dc}(e))$, i. e. $cd \parallel ef$ (by (1)). This proves A8.

To see that A9 follows from $\Sigma \cup \{(1)\}$, notice that, with $q = \tau_{ba}(p)$, $ab \parallel pq$ becomes $L(p\tau_{ba}(p)\tau_{ba}(p))$, which holds by A1 and B1. Since $a \neq b$, we conclude from B5 that $p \neq q$.

A10 is a consequence of (1), A2 and A3, and A11 follows from B7.

Let $p \neq q$. Then, since $\tau_{pq}(\tau_{ab}(p)) = \tau_{ab}(q)$ (by B6, B2) and thus by A1 and A2 $L(\tau_{ab}(p)\tau_{ab}(q)\tau_{pq}(\tau_{ab}(p)))$, we also have $qp \| \tau_{ab}(p)\tau_{ab}(q)$ (by (1)), from which we derive, using A6 and A8, $pq \| \tau_{ab}(p)\tau_{ab}(q)$, which proves A14.

Let $a \neq b$. Then $p \neq \tau_{ab}(p)$ and $q \neq \tau_{ab}(q)$ (by B5) and $L(q\tau_{ab}(q)\tau_{p\tau_{ab}(p)}(q))$ (by A1 and A2, since $\tau_{p\tau_{ab}(p)}(q) = \tau_{ab}(q)$ (by B6)), i. e. $\tau_{ab}(p)p \| q\tau_{ab}(q)$ (by (1)), which proves A15 (by A6, A8).

By B6 we have both $\tau_{b\tau_{cd}(b)}(\tau_{ab}(x)) = \tau_{cd}(\tau_{ab}(x))$ and $\tau_{b\tau_{ax}(b)}(\tau_{cd}(b)) = \tau_{ax}(\tau_{cd}(b))$, and since the left hand sides of these equations are equal (which follows from applying B2 twice), we get $\tau_{ax}(\tau_{cd}(b)) = \tau_{cd}(\tau_{ab}(x))$, i. e. $\tau_{a\tau_{cd}(b)}(x) = \tau_{cd}(\tau_{ab}(x))$ (by B2), which is A16. $\boxed{QED}$

For Fano translation structures (diagonals of parallelograms are not parallel) we need the additional axiom

B 8. $\sigma_a(b) = b \to a = b$.

Let $\Sigma' = \Sigma \cup \{\text{B8}\}$. We shall say that $\langle G, \mathfrak{P}\rangle$ is an Abelian group with partition if G is an Abelian group and $\mathfrak{P}$ consists of proper subgroups of G, different from $\{0\}$, which cover G and whose pairwise intersection is $\{0\}$, 0 being the neutral element of G. The representation theorem proved by André [2, Satz 1.3] translates into

Theorem 2 (Representation Theorem).

$\mathfrak{M} \in Mod(\Sigma)$ *iff* $\mathfrak{M} \simeq \langle G, \mathbf{L}, \tau\rangle$*, where* G *is the underlying Abelian group of an Abelian group with partition* $\langle G, \mathfrak{P}\rangle$*,*

$$\mathbf{L} = \{(\mathbf{a}, \mathbf{b}, \mathbf{c}) \in G^3 \mid \mathbf{b} - \mathbf{a}, \mathbf{c} - \mathbf{a} \textit{ belong to the same member of } \mathfrak{P}\},$$

$\tau_{\mathbf{ab}}(\mathbf{c}) = \mathbf{b} + \mathbf{c} - \mathbf{a}$.

$\mathfrak{M} \in Mod(\Sigma')$ *iff* $\mathfrak{M} \simeq \langle G, \mathbf{L}, \tau\rangle$*, with all notations as above, and* G *having no element of order two.*

Theorem 3. *The axiom system* Σ' *is independent.*

Proof. Independence models:

(i) for B1:[1] $u(\mathfrak{M})=\mathbb{Z}\times\mathbb{Z}$, $\tau(\mathbf{abc}) = \mathbf{b}+\mathbf{c}-\mathbf{a}$,

$$\mathbf{L} = \{(\mathbf{a},\mathbf{b},\mathbf{c}) \mid \mathbf{a},\ \mathbf{b},\ \mathbf{c} \text{ are three different collinear points in } \mathbb{Z}\times\mathbb{Z}, \text{ or else } \mathbf{a}=\mathbf{b} \text{ or } \mathbf{b}=\mathbf{c}\},$$

$\mathbf{a}_0 = (0,0), \mathbf{a}_1 = (1,0), \mathbf{a}_2 = (0,1)$;
(ii) for A3: same $u(\mathfrak{M})$, τ, and $\mathbf{a}_0,\mathbf{a}_1,\mathbf{a}_2$ as above, $\mathbf{L} = \{(\mathbf{a},\mathbf{b},\mathbf{c}) \mid \mathbf{c} = 2\mathbf{a}-\mathbf{b} \vee \mathbf{c} = 2\mathbf{b}-\mathbf{a}\}$;
(iii) for B2: (communicated in [4]) $u(\mathfrak{M})$ a Frobenius group with its natural partition $\mathfrak{P}$ (see [8, 12.6.1, p. 348]), $\mathbf{L} = \{(\mathbf{a},\mathbf{b},\mathbf{c}) \mid (\exists U \in \mathfrak{P})\ \mathbf{a}^{-1}\mathbf{b}, \mathbf{a}^{-1}\mathbf{c} \in U\}$, $\tau(\mathbf{abc}) = \mathbf{ba}^{-1}\mathbf{c}$, with a convenient choice of $\mathbf{a}_0,\mathbf{a}_1,\mathbf{a}_2$;
(iv) for B3: same as in (i), except that $\mathbf{L}_{\mathfrak{M}}$ is empty;
(v) for B6: an affine plane that is not a translation plane, with the standard interpretations of the primitive notions; for B7: $u(\mathfrak{M})=\{\mathbf{e}\}$, $\mathbf{L}_{\mathfrak{M}}=\{(\mathbf{e},\mathbf{e},\mathbf{e})\}$, and $\tau_{\mathfrak{M}}(\mathbf{eee}) = \mathbf{e}$; for B8: a translation plane over a field of characteristic 2.

Since for B4 and B5 I could not find independence models, the proof of independence will be syntactic, following a method presented in [6]. Axioms are turned into rules of inference, in which from a certain *sequent* another sequent may be inferred, and a certain sequent in which no logical symbols appear is provable only if it can be derived as the endsequent of a chain of inferences, where only the inferences corresponding to the axioms and two logical axioms are allowed to appear, and the succedent has to be the same throughout (see [6] for details). In our case, the axioms B1, A3, B2, B3, B4, B5, B6, B7, B8 turn into the following rules of inference (Γ and Δ are arbitrary multisets of formulas (repetitions are allowed)):

$$\frac{L(abc), L(bac), \Gamma \Rightarrow \Delta}{L(abc), \Gamma \Rightarrow \Delta} \tag{11}$$

$$\frac{L(abc), L(abd), L(acd), \Gamma \Rightarrow \Delta \quad L(abc), L(abd), a=b, \Gamma \Rightarrow \Delta}{L(abc), L(abd), \Gamma \Rightarrow \Delta} \tag{12}$$

$$\frac{\tau_{ab}(c) = \tau_{ac}(b), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \tag{13}$$

$$\frac{L(ab\tau_{ba}(b)), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \tag{14}$$

$$\frac{L(abc), L(x\tau_{ab}(x)\tau_{ac}(x)), \Gamma \Rightarrow \Delta}{L(abc), \Gamma \Rightarrow \Delta} \tag{15}$$

[1] $u(\mathfrak{M})$ denotes the universe of the model.

$$\frac{\tau_{ab}(x) = x, a = b, \Gamma \Rightarrow \Delta}{\tau_{ab}(x) = x, \Gamma \Rightarrow \Delta} \tag{16}$$

$$\frac{\tau_{ab}(x) = \tau_{c\tau_{ab}(c)}(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \tag{17}$$

$$\frac{}{L(a_0a_1a_2), \Gamma \Rightarrow \Delta} \tag{18}$$

$$\frac{\tau_{ba}(a) = b, a = b, \Gamma \Rightarrow \Delta}{\tau_{ba}(a) = b, \Gamma \Rightarrow \Delta} \tag{19}$$

To these inference rules we must add two inference rules corresponding to the equality axioms, namely

$$\frac{a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \tag{20}$$

$$\frac{s = t, Q(s), Q(t), \Gamma \Rightarrow \Delta}{s = t, Q(s), \Gamma \Rightarrow \Delta} \tag{21}$$

where s and t are terms, and $Q(a)$ is an atomic predicate from our language, in which the variable a occurs, *i. e.* a predicate of the form $t_1 = t_2$ or $t_1t_2 \parallel t_3t_4$, where the t_i are terms, and the variable a occurs in at least one of the terms involved.

To show that the sequent $\tau_{ab}(x) = x \Rightarrow a = b$ is not derivable from the above rules of inference, except for the rule (16), by using only the two logical axioms $P, \Gamma \Rightarrow \Delta, P$ and $\bot, \Gamma \Rightarrow \Delta$, (where P is an atomic formula, and $\bot$ is the logical symbol for falsity), we inspect the possible rules of inference starting with the last one that could have been used in a deduction of $\tau_{ab}(x) = x \Rightarrow a = b$, and prove that one cannot eventually arrive, in a bottom-up search, regardless of the combination of inference rules used in the presumed deduction tree, at a logical axiom. Notice that rules (18) and (19) could never be used, because their antecedents cannot be obtained as the premise by using any other rule. We can never arrive at a sequent of the form $\bot, \Gamma \Rightarrow \Delta$, because $\bot$ is not part of our sequent, and $\bot$ cannot be added to the antecedent by any successive use of the rules of inference. So the only axiom where a deduction tree, followed bottom-up, of $\tau_{ab}(x) = x \Rightarrow a = b$ could possibly end is a logical axiom of the type $P, \Gamma \Rightarrow \Delta, P$. Since the succedent must be the same throughout the derivation, that logical axiom would have to be of the form $a = b, \tau_{ab}(x) = x, \Gamma \Rightarrow a = b$ (since none of the formulas of the antecedent can ever be lost). Now the only inference rule (besides (19) which cannot be used) in which equality between two variables appears in the premise and not necessarily in the conclusion is (12).

The problem with that rule is that it appears only in *one* of the two premises, whereas both need to become axioms, leading to an infinite loop.

To show that the sequent $L(abc) \Rightarrow L(x\tau_{ab}(x)\tau_{ac}(x))$ is not derivable from the above inference rules except (15) from logical axioms (notice again that $\perp, \Gamma \Rightarrow \Delta$ cannot appear in this derivation tree), we notice again that we ought to arrive at an axiom of the type $L(abc), L(x\tau_{ab}(x)\tau_{ac}(x)), \Gamma \Rightarrow L(x\tau_{ab}(x)\tau_{ac}(x))$. But the only inference rule from which an atomic formula containing L and τ could appear in the antecedent of a premise, without having to appear in the antecedent of its conclusion as well, is (14), but $L(uv\tau_{vu}(u))$ cannot become $L(x\tau_{ab}(x)\tau_{ac}(x))$, for the terms u and v will have to contain only the variables a, b, x, and $\tau_{vu}(u)$ will have to contain the variable c. $\boxed{QED}$

If we add the axiom

A 17. $\neg L(c\tau_{ab}(c)d) \to (\exists p)\, L(pab) \wedge L(pcd)$

to Σ (or Σ') we get an axiom system for translation planes (or Fano translation planes). Let $\Delta = \Sigma \cup \{A17\}$ and $\Delta' = \Sigma' \cup \{A17\}$.

We conjecture that the theory of central translation structures is the universal part of the theory of translation planes, i. e. that every universal sentence formulated in the language $\mathcal{L}$ that is valid in all translation planes, is a consequence of Σ, and analogously for the Fano case.

Denoting by $Cn(A)$ the set of all logical consequences of A and by $\mathcal{T}_\forall$ the set of all universal sentences of a theory $\mathcal{T}$, we can formulate our conjecture as

Conjecture. . Σ is an axiom system for $(Cn(\Delta))_\forall$ and Σ' is an axiom system for $(Cn(\Delta'))_\forall$.

A natural question to ask would be what the universal theory of Desarguesian and Pappian affine planes is. Since none of the known proofs for the implications ‘Pappus implies Desargues’ or for the many equivalent configuration theorems to Desargues or Pappus can be carried through without using the existence of any other points but those produced by τ, they are not valid in $Cn(\Sigma)$ or $Cn(\Sigma')$. This leads us to believe that the universal theory of Desarguesian or Pappian affine planes is not finitely axiomatizable.

The representation theorem for translation planes is usually phrased in terms of Veblen-Wedderburn systems (also called strong left quasi-fields in [9]). However, they can also be algebraically described as follows (cf. [9, Th. 3.4.5]):

Their point-sets are commutative groups $(G, 0, +, -, \sim)$, endowed with an equivalence relation $\sim$ on $G \setminus \{0\}$, satisfying the following axioms: $-a \sim a$, $a \sim b \wedge a+b \neq 0 \to a+b \sim b$, $c-a \not\sim b-a \wedge a-b \sim c-d \wedge a-c \sim b-d \to a+d = b+c$ and $a \neq 0 \wedge b \neq 0 \wedge x \neq 0 \wedge a \not\sim b \wedge x \not\sim a \wedge x \not\sim b \to (\exists yz) y \sim a \wedge z \sim b \wedge y+z = x$. The notions τ and L are interpreted as follows: $\tau_{ab}(c) = b + c - a$ and $L(abc)$ iff $a = b$ or $b = c$ or $a = c$ or $c - a \sim c - b$.

Central translation structures can be described in the same way, with the only difference that the equivalence relation $\sim$ need not satisfy the last axiom. Our conjecture can now be reformulated to say that the theory (in a language with $0, +, -, \sim$) of commutative groups with $\sim$ (satisfying all but the last axiom) is the universal theory of the theory of commutative groups with $\sim$ satisfying all of the above axioms.

2 A first-order axiomatization of affine spaces over rings with right GCDs

An axiom system for a plane affine geometry over rings with unit element, without zero divisors, for which there exists a right greatest common divisor for any two non-zero elements, was first presented, modeled on E. Artin's approach, in [3]. An axiom system for 'n-dimensional' affine spaces coordinatized over the same class of rings was given in [7]. Both axiom systems are formulated in a bi-sorted language, with 'points' and 'lines' as individuals, and 'incidence' and 'parallelism' as primitive notions. In both papers there is an axiom which is not expressed in this language, but is rather a property of models of the axiom system, formulated in terms of translations and trace-preserving endomorphisms of the group of translations. The aim of this section is to provide a purely geometric axiomatization of 'n-dimensional' affine spaces over rings with a right gcd. For $n = 2$ the axiomatization is quantifier-free.

Permutti's [7] axiom system can be easily seen to be an axiom system for central translation structures, to which the following axioms are added

P 1. For $n = 2$: There are two lines l_1 and l_2, such that every line parallel to l_1 cuts every line parallel to l_2; for $n > 2$: There are n lines $l_1, \ldots, l_n$ such that for any lines g_1 and g_n with $l_1 \parallel g_1$ and $l_n \parallel g_n$, there exists exactly one $(n-2)$-tuple $(g_2, \ldots g_{n-1})$ such that $g_i \parallel l_i$ for $i \in \{2, \ldots, n-1\}$ and such that g_i intersects g_{i+1} for $i \in \{1, \ldots, n-1\}$.

P 2.. Given two lines l_1 and l_2 meeting in T, there is a point $P \neq T$ on l_1 such that for all P' on l_1 and Q on l_2, the parallel through P' to PQ intersects l_2.

P 3.. For any line l there exists a translation $\tau_0 \neq 1$, with traces parallel to l, such that any translation τ with traces parallel to l can be written as $\tau = \tau_0^\alpha$, where α is an endomorphism of the group of translations such that, for any translation θ, θ^α is either the identity or has the same traces as θ.

It is this last axiom, P3, which ought to be replaced by purely geometric statements, that do not refer to elements of groups of translations or to special endomorphisms thereof.

Our axiom system will be expressed in an extension $\mathcal{L}'_n$ of $\mathcal{L}$ by two operation symbols, a quaternary one, χ, with $\chi(abcd)$ to be read as 'the intersection point of lines ab and cd, in case $a \neq b, c \neq d$, and ab and cd are two different intersecting lines, an arbitrary point, otherwise', and a binary one φ, with $\varphi(ab)$ to be read as 'a point at unit distance from a on the line ab, provided that $a \neq b$'. For $n > 2$ we also add the individual constants a_i with $i \in \{3, \ldots, n\}$ to the language $\mathcal{L}'_n$. For improved readability we shall also use, besides the notion of parallelism, as defined in (1), the following abbreviation (only for a, b, c, d satisfying $(\neg L(abc) \vee \neg L(abd)) \wedge c \neq d$):

$$I(abcd) :\leftrightarrow L(ab\chi(abcd)) \wedge L(cd\chi(abcd)).$$

The axiom system consists of the axioms A3, B1–B6, as well as:

C 1. $(\neg L(abc) \vee \neg L(abd)) \wedge c \neq d \wedge L(abx) \wedge L(cdx) \rightarrow \chi(abcd) = x$,

C 2. For $n = 2$: $\neg L(a_0a_1a_2) \wedge \chi(x\tau_{a_0a_1}(x)y\tau_{a_0a_2}(y))$.
For $n > 2$: $(\forall x_1 x_{n+1})(\exists x_2 \ldots x_n)(\forall z_2 \ldots z_n) \bigwedge_{1\leq i<j\leq n} \neg L(a_0a_ia_j)$
$\bigwedge_{i=1}^{n} L(x_i\tau_{a_0a_i}(x_i)x_{i+1}) \wedge [\bigwedge_{i=2}^{n-1} L(z_i\tau_{a_0a_i}(z_i)z_{i+1}) \wedge L(x_1\tau_{a_0a_1}(x_1)z_2)$
$\wedge L(z_n\tau_{a_0a_1}(z_n)x_{n+1}) \rightarrow \bigwedge_{i=2}^{n} x_i = z_i]$,

C 3. $o \neq a \wedge o \neq b \wedge L(oab) \rightarrow \varphi(oa) = \varphi(ob)$,

C 4. $o \neq a \rightarrow L(oa\varphi(oa)) \wedge \varphi(oa) \neq o$,

C 5. $\neg L(oab) \rightarrow I(a\tau_{\varphi(oa)b}(a)ob)$,

C 6. $\neg L(oab) \wedge \neg L(oac) \wedge \neg L(obc) \rightarrow bc \parallel \chi(oba\tau_{\varphi(oa)b}(a))\chi(oca\tau_{\varphi(oa)c}(a))$,

C 7. $\neg L(oab) \wedge L(oac) \rightarrow$
$ab \parallel \chi(obc\tau_{\varphi(oa)b}(c))\chi(oa\chi(obc\tau_{\varphi(oa)\varphi(ob)}(c))\tau_{\varphi(ob)a}(\chi(obc\tau_{\varphi(oa)\varphi(ob)}(c)))$.

C1 states that, if two different lines ab and cd have a point x in common, then x is $\chi(abcd)$; C2_2 states that there are two lines, a_0a_1 and a_0a_2, such that any parallel to the former intersects any parallel to the latter: C2_n states that there exist n distinct lines, a_0a_i with $i = 1, \ldots n$, such that, given any two lines, $x_1\tau_{a_0a_1}(x_1)$ and $x_{n+1}\tau_{a_0a_n}(x_{n+1})$, parallel to a_0a_1 and a_0a_n, there is a unique $(n-1)$-tuple of points $(x_2, \ldots, x_n)$, such that the lines x_ix_{i+1} are parallel to a_0a_i for $i = 1, \ldots, n$; C3 and C4 state that $\varphi(o, a)$ is in fact a function of o and the *line* oa, not the *point* a, and that the value of this function is a point on the line oa that is different from o (a point we may think of as at *unit distance* from o); C5 is a rephrasing of P2 in our language; C6 is a weak form of the Desargues axiom, in which one of the two triangle has as vertex the point at unit distance from 'point of perspectivity' (the intersection of the three lines on which the vertices of the two triangles lie), $\varphi(o, a)$; C7 is a weak version of what is commonly called the 'scissors theorem' (*Scherensatz*, see [5, p. 19f]), in which one of the 'scissors' has the points at unit distance from the intersection point of the lines on which the two scissors lie as its ‘handles’. The scissors theorem is a consequence of the major Desargues axiom, but the weak versions of each seem to be independent of each other.

Let $\Xi_n = \{\mathrm{A3}, \mathrm{B1}-\mathrm{B6}, \mathrm{C1}-\mathrm{C7}\}$. To prove that Permutti’s axiom system follows from Ξ_n, all we have to do is to prove that P3 can be deduced from it. Let $l := oa$ be a line, and let τ_0 be the translation $\tau_{o\varphi(oa)}$. Let τ be a translation with traces parallel to l. Then $\tau = \tau_{op}$ for some point p on l. We shall define a dilatation σ, such that $\sigma\tau_0\sigma^{-1} = \tau$, thus proving P3, since $\theta \mapsto \sigma\theta\sigma^{-1}$ is a trace-preserving endomorphism of the group of translations. We define σ by letting $\sigma(o) = o$, $\sigma(\varphi(op)) = p$, as well as:
if x is not on l, then $\sigma(x) = \chi(oxp\tau_{\varphi(op)x}(p))$, *i. e.* the point of intersection of the lines ox and the parallel from p to $\varphi(op)x$, which exists by C5;
if x is on l, and if different from $\varphi(op)$, then we pick a point q that is not on l and let $\sigma_q(x) = \chi(ox\sigma(\varphi(oq))\tau_{\varphi(oq)x}(\sigma(\varphi(oq))))$.

Since this definition depends, when $x \neq \varphi(op)$ lies on l, on the arbitrary point q not on l, we have to show that $\sigma_q(x) = \sigma_r(x)$ whenever q and r are not on l, and that we may define by $\sigma(x) := \sigma_q(x)$ the value of σ in points $x \neq \varphi(op)$ on l. Suppose that p, q, r are not collinear. From a first application of the weak Desargues axiom C6 to the triangles $\varphi(op)\varphi(oq)\varphi(or)$ and $p\sigma(\varphi(oq))\sigma(\varphi(or))$ we deduce that $\varphi(oq)\varphi(or)$ is parallel to $\sigma(\varphi(oq))\sigma(\varphi(or))$, and a second application of C6, this time to the triangles $x\varphi(oq)\varphi(or)$ and $\sigma_q(x)\sigma(\varphi(oq))\sigma(\varphi(or))$, allows us to conclude that $x\varphi(or)$ is parallel to $\sigma_q(x)\sigma(\varphi(or))$, thus $\sigma_q(x) = \sigma_r(x)$. If x and y are two points, both not on l, and such that x, y, o are not collinear, then we deduce that xy is parallel to $\sigma(x)\sigma(y)$ by using C6; if x is on l and y is not on l, then the same conclusion follows from C7. Thus, σ is a dilatation. To see that $\sigma\tau_0 = \tau\sigma$, let x be a point that is no on l, and let q be the intersection of the parallel from p to $\varphi(oa)\tau_0(x)$ (*i. e.* $q = \chi(o\tau_0(x)p\tau_{\varphi(oa)\tau_0(x)}(p))$). Applying C6 to the two triangles, perspective from o, $\varphi(oa)x\tau_0(x)$ and $p\sigma(x)q$, we infer that $x\tau_0(x)$ is parallel to $\sigma(x)q$. Since both q and $\tau(\sigma(x))$ are on the parallel from p to $\varphi(oa)\tau_0(x)$, and both $\sigma(x)q$ and $\sigma(x)\tau(\sigma(x))$ are parallel to $x\tau_0(x)$, we must have $q = \tau(\sigma(x))$. This implies that $\sigma(\tau_0(x)) = \tau(\sigma(x))$. That the equality $\sigma(\tau_0(\sigma^{-1}(x))) = \tau(x)$ holds for points x on l as well follows from the fact that the left hand side is a dilatation, and from the fact that we know the equality holds for x not on l.

Permutti also considers the following two axioms, the first ensuring that the dimension of the space can indeed be said to exist and be n (*i. e.* that $\mathrm{C2}_n$ holds only for one value of n), the second, a weak form of Pappus’s axiom, that ring multiplication is commutative:

C 8. $(\forall x_1 x_2)(\exists u)\ \bigwedge_{i=1}^{2}[L(a_0a_1x_i) \wedge x_i \neq a_0] \rightarrow L(a_0a_1u) \wedge u \neq a_0$
$\bigwedge_{i=1}^{2} I(a_0\tau_{a_0x_i}(\varphi(oa_2))u\tau_{a_0a_2}(u))$,

C 9. $\neg L(oab) \to \varphi(oa)\varphi(ob) \parallel \chi(oab\tau_{\varphi(ob)a}(b))\chi(oba\tau_{\varphi(oa)b}(a))$.

From the representation theorem proved in [7] we get

Theorem 4 (Representation theorem).

$\mathfrak{M} \in Mod(\Xi_n)$ iff $\mathfrak{M} \simeq \langle R^n, \tau, \varphi, \chi\rangle$, where R is a ring with unit, such that any two non-zero elements have a right greatest common divisor (i. e. if a, b are two non-zero elements, then there exists a d such that $a = a'd$ and $b = b'd$, for some $a', b' \in R$, and for any e with $a = ue$ and $b = ve$, we have $d = d'e$, for some $d' \in R$); $\tau_{\mathbf{ab}}(\mathbf{c}) = \mathbf{b} + \mathbf{c} - \mathbf{a}$; $\varphi(\mathbf{oa})$ is $(\alpha_1 + \lambda_1 u, \ldots, \alpha_n + \lambda_n u)$, where the u is a unit in R, $\mathbf{o} = (\alpha_1, \ldots, \alpha_n)$, and the points on the line $\mathbf{oa}$ are all of the form $(\alpha_1 + \lambda_1 k, \ldots \alpha_n + \lambda_n k)$, with $k \in R$, and with the right GCD of $(\lambda_1, \ldots, \lambda_n)$ being 1; $\chi(\mathbf{abcd})$ is the intersection point of lines $\mathbf{ab}$ and $\mathbf{cd}$, i. e. a point that is both of the form $(\beta_1 + \mu_1 k, \ldots, \beta_n + \mu_n k)$ and of the form $(\gamma_1 + \nu_1 k, \ldots, \gamma_n + \nu_n k)$, with $k \in R$ and with the μ_i as well as the ν_i having right GCD 1, where both $\mathbf{a}$ and $\mathbf{b}$ are of the former form and both $\mathbf{c}$ and $\mathbf{d}$ are of the latter form.

$\mathfrak{M} \in Mod(\Xi_n \cup \{C8\})$ iff $\mathfrak{M}$ is as above, with R having the additional property that, for any non-zero a and b in R, there are non-zero a' and b' such that $aa' = bb'$.

$\mathfrak{M} \in Mod(\Xi_n \cup \{C9\})$ iff $\mathfrak{M}$ is as above, with R being commutative as well.

Note that C8 is independent from Ξ_n, given that the free $\mathbb{Z}$-algebra $\mathbb{Z}\langle x, y\rangle$ is an integral domain in which any two elements have a right GCD, but the elements x and y do not have a right common multiple. I thank Prof. P. M. Cohn for having pointed out this example to me.

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