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Generalized digital (k_0, k_1)-homeomorphism

Sang-Eon Han
Department of Computer and Applied Mathematics
College of Natural Science, Honam University
Gwangju, 506-714, Korea
sehan@honam.ac.kr

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Abstract. The aim of this paper is to introduce a generalized digital (k_0, k_1)-homeomorphism of the digital curve and the digital surface in $\mathbb{Z}^n$. The generalized digital (k_0, k_1)-continuity is studied with the n kinds of k-adjacency relations in $\mathbb{Z}^n$. The k-type digital fundamental group of the digital image comes from the generalized digital (k_0, k_1)-homotopy, $i \in \{0, 1\}$. Furthermore, we show how a digital (k_0, k_1)-homeomophism induces a digital fundamental group (k_0, k_1)-isomorphism.

Keywords: digital (k_0, k_1)-continuity, digital (k_0, k_1)-homeomorphism, digital curve, digital surface.

MSC 2000 classification: primary: 55P10; secondary 55P15.

Introduction

The digital k-adjacency on digital curves and digital surfaces in $\mathbb{Z}^3$ are investigated in [7, 8, 9]. The digital continuity was introduced in [1, 2, 10] and further an advanced concept of the digital continuity was also introduced [1].

Recently, the digital (k_0, k_1)-continuity was investigated with relation to the digital (k_0, k_1)-homeomorphism, and further it is a generalization of the concepts from [1, 2, 10] relative to the dimension and the adjacency.

By virtue of a generalization of the k-adjacency relations, we consider the generalized digital (k_0, k_1)-continuity and the generalized digital (k_0, k_1)-homeomorphism in $\mathbb{Z}^4$ and $\mathbb{Z}^5$.

We work in the category of finite digital images and digitally (k_0, k_1)-continuous maps.

1 Notation and basic terminology

In the set $\mathbb{Z}^n$ of points in the Euclidean n-dimensional space, $n = 4, 5$, that have integer coordinates, two metric spaces $(\mathbb{Z}^n, d_n)$ and $(\mathbb{Z}^n, d_*)$ are considered with the following metric functions:

$d_n, d_* : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{N} \cup \{0\}$ are defined by

$(M1)$ $d_n(p, q) = \sum_{i=1}^{n} |p_i - q_i|$ and

$(M2)$ $d_*(p,q) = max\{|p_i - q_i|\}_{i\in M}$, $M = \{1, 2, \cdots, n\}$, respectively for two points $p, q \in \mathbb{Z}^n$, $\mathbb{N}$ is the set of natural numbers.

By use of the above two metric functions we get the k-adjacency relations of a digital image in $\mathbb{Z}^4$ and $\mathbb{Z}^5$.

Basically, two pixels $(p_1, p_2), (q_1, q_2) \in \mathbb{Z}^2$ are called 4-adjacent if $|p_1 - q_1| + |p_2 - q_2| = 1$. And they are called 8-adjacent if max $\{|p_1 - q_1|, |p_2 - q_2|\} = 1$[7, 8].

Two voxels $(p_1, p_2, p_3), (q_1, q_2, q_3)(\in \mathbb{Z}^3)$ are called 6-adjacent if

$$|p_1 - q_1| + |p_2 - q_2| + |p_3 - q_3| = 1.$$

They are called 26-adjacent if $max\{|p_1 - q_1|, |p_2 - q_2|, |p_3 - q_3|\} = 1$ [8, 9].

Furthermore, two points are 18-adjacent if they are 26-adjacent and differ in at most two of their coordinates [8].

Concretely, a digital picture is considered as a quadruple $P = (V, k, \bar{k}, X)$ with black points set $X \subset V$ and white points set $V - X$. If $V = \mathbb{Z}^2$, $(k, \bar{k}) = (4, 8)$ or $(8, 4)$, and if $V = \mathbb{Z}^3$, $(k, \bar{k}) = (6, 26), (26, 6), (6, 18)$ or $(18, 6)$[11, 8, 9].

The point $p = (p_1, p_2, p_3, p_4, p_5) \in \mathbb{Z}^5$ is considered as a 5-cube $\{(p_1 \pm 1/2, p_2 \pm 1/2, p_3 \pm 1/2, p_4 \pm 1/2, p_5 \pm 1/2)\}$ with a center p, whose edges are parallel to each axes.

Now in $\mathbb{Z}^5$, we consider the following equations which are relevant for the k-neighborhood and the k-adjacency relations.

For two 5-xels $p = (p_1, p_2, p_3, p_4, p_5), q = (q_1, q_2, q_3, q_4, q_5) \in \mathbb{Z}^5$

(1) $d_5(p, q) = 5, d_*(p, q) = 1 \Rightarrow$ then p shares a point with q,

(2) $d_5(p, q) = 4, d_*(p, q) = 1 \Rightarrow$ then p shares an edge with q,

(3) $d_5(p, q) = 3, d_*(p, q) = 1 \Rightarrow$ then p shares a face with q,

(4) $d_5(p, q) = 2, d_*(p, q) = 1 \Rightarrow$ then p shares a cube with q,

(5) $d_5(p, q) = 1, d_*(p, q) = 1 \Rightarrow$ then p shares a 4-cube with q.

Consequently, in $\mathbb{Z}^5$, the 5 kinds of digital k-neighborhoods are obtained from (1) $\sim$ (5) above and by the properties of the combination as follows:

$(1)'$ $N_{242}(p) = \{q \in \mathbb{Z}^5 | d_5(p, q) \leq 5, d_*(p, q) = 1\}$ from the above formula (1) such that $\sharp\{q \in \mathbb{Z}^5 | d_5(p, q) \leq 5, d_*(p, q) = 1\} = 242$, where $\sharp$ means the cardinality of the set.

$(2)'$ $N_{210}(p) = \{q \in \mathbb{Z}^5 | d_5(p, q) \leq 4, d_*(p, q) = 1\}$ from the above formula (2).

Namely, for $p = (p_1, p_2, p_3, p_4, p_5) \in \mathbb{Z}^5$, $N_{210}(p) = N_{242}(p) - X_5(p)$, where $X_5(p) = \{q \in \mathbb{Z}^5 | d_5(p, q) = 5, d_*(p, q) = 1\}$. In fact, $X_5(p) = \{(p_1 \pm 1, p_2 \pm 1, p_3 \pm 1, p_4 \pm 1, p_5 \pm 1)\}$. Now we use the notation, $X_5(p) = \cup_{i=0}^{5} X_5(p)^i$ in terms of the following notations:

$X_5(p)^0 = \{(p_1 + 1, p_2 + 1, p_3 + 1, p_4 + 1, p_5 + 1)\}$ with $\sharp X_5(p)^0 = C_0^5 = 1$, where C_i^5 stands for the combination of 5 objects taken i.

$X_5(p)^1 = \{(p_1+1, p_{i-1}+1, p_i-1, p_{i+1}+1, p_5+1) | i \in [1,5]_{\mathbb{Z}}\}$ and $\sharp X_5(p)^1 = C_1^5$, i.e., $X_5(p)^1$ consists of the elements which have the coordinates with only one element $p_i - 1 (1 \leq i \leq 5)$ and the others are $p_j + 1 (i \neq j)$.

$X_5(p)^2 = \{(p_{i-1}+1, p_i-1, p_j-1, p_{j+1}+1, p_{j+2}+1)\}$, where $i \neq j \in [1,5]_{\mathbb{Z}}$ and $\sharp X_5(p)^2 = C_2^5$, i.e., $X_5(p)^2$ consists of the elements which have the coordinates with only two elements, $p_i - 1, p_j - 1, 1 \leq i, j \leq 5$, and the others $p_k + 1 (k \neq i, j)$.

$X_5(p)^3 = \{(p_i - 1, p_{i+1} + 1, p_j - 1, p_k - 1, p_{k+1} + 1)\}$, where $i \neq j \neq k \in [1,5]_{\mathbb{Z}}$ with $\sharp X_5(p)^3 = C_3^5$, i.e., $X_5(p)^3$ consists of the elements which have the coordinates with only three elements, $p_i - 1, p_j - 1, p_k - 1, 1 \leq i, j, k \leq 5$, and the others are $p_l + 1, 1 \leq l \leq 5, l \neq i, j, k$.

$X_5(p)^4 = \{(p_1 + 1, p_2 - 1, p_3 + 1, p_4 + 1, p_5 - 1), (p_1 - 1, p_2 + 1, p_3 - 1, p_4 - 1, p_5 - 1), \cdots, (p_1 - 1, p_2 - 1, p_3 - 1, p_4 - 1, p_5 + 1)\}$ with $\sharp X_5(p)^4 = C_4^5$.

Finally,

$X_5(p)^5 = \{(p_1 - 1, p_2 - 1, p_3 - 1, p_4 - 1, p_5 - 1)\}$ with $\sharp X_5(p)^5 = C_5^5$.

Then $X_5(p)^i$ and $X_5(p)^j$ are disjoint for $i \neq j \in \{0, 1, 2, 3, 4, 5\}$.

Thus we get $\sharp X_5(p) = \sum_{i=0}^{5} C_i^5$.

Consequently, $\sharp\{q \in \mathbb{Z}^5 | d_5(p,q) \leq 4, d_*(p,q) = 1\}$

$= \sharp(N_{242}(p) - X_5(p)) = 242 - (C_0^5 + C_1^5 + C_2^5 + \cdots + C_5^5) = 210$.

$(3)'$ $N_{130}(p) = \{q \in \mathbb{Z}^5 | d_5(p,q) \leq 3, d_*(p,q) = 1\}$

$= N_{210}(p) - X_4(p)$ from (3) above, where $X_4(p) = \{q \in \mathbb{Z}^5 | d_5(p,q) = 4, d_*(p,q)1\}$.

Actually, $X_4(p) = \{(p_{i-2} \pm 1, p_{i-1} \pm 1, p_i, p_{i+1} \pm 1, p_{i+2} \pm 1)\}$ $(i \in \{1, 2, 3, 4, 5\})$ $= \cup_{i=0}^{4} X_4(p)^i$ via the following notations:

$X_4(p)^0 = \{(p_{j-1} + 1, p_j + 1, p_{j+1} + 1, p_i, p_{i+1} + 1) | i \neq j \in [1,5]_{\mathbb{Z}}\}$ with $\sharp X_4(p)^0 = C_0^4$.

$X_4(p)^1 = \{(p_{j-1}+1, p_j-1, p_{i-1}+1, p_i, p_{i+1}+1) | i \neq j \in [1,5]_{\mathbb{Z}}\}$, i.e., $X_4(p)^1$ consists of the elements which have the coordinates with only one $p_j - 1, 1 \leq j (\neq i) \leq 5$, and the others are $p_k + 1, 1 \leq k \leq 5, i \neq j \neq k$, except p_i with $\sharp X_4(p)^1 = C_1^4$.

$\cdots$,

Finally,

$X_4(p)^4 = \{(p_{i-2} - 1, p_{i-1} - 1, p_i, p_{i+1} - 1, p_{i+2} - 1)\}$ with $\sharp X_4(p)^4 = C_4^4$.

Then $X_4(p)^i$ and $X_4(p)^j$ are disjoint for $i \neq j \in \{0, 1, 2, 3, 4\}$.

Therefore $\sharp X_4(p) = C_1^5(C_0^4 + C_1^4 + C_2^4 + C_3^4 + C_4^4)$.

Thus we get the following:

$\sharp\{q \in \mathbb{Z}^5 | d_5(p,q) \leq 3, d_*(p,q) = 1\} = \sharp(N_{210}(p) - X_4(p))$

$= 210 - C_1^5(C_0^4 + C_1^4 + C_2^4 + C_3^4 + C_4^4) = 130$. Similarly,

$(4)'$ $N_{50}(p) = N_{130}(p) - X_3(p) = \{q \in \mathbb{Z}^5 | d_5(p,q) \leq 2, d_*(p,q) = 1\}$ from (4) above,

where $X_3(p) = \{q \in \mathbb{Z}^5 | d_5(p,q) = 3, d_*(p,q) = 1\}$

$= \{(p_{i-1} \pm 1, p_i, p_{i+1} \pm 1, p_j, p_{j+1} \pm 1)\} = \cup_{i=0}^{3} X_3(p)^i$.

By the same method above, we get

$\sharp X_3(p) = C_2^5(C_0^3 + C_1^3 + C_2^3 + C_3^3) = 50$.

$(5)'$ $N_{10}(p) = \{q \in \mathbb{Z}^5 | d_5(p,q) \leq 1\}$ from (5) above such that $\sharp N_{10}(p) = 10$.

At last, 5 kinds of k-adjacency relations in $\mathbb{Z}^5$ are obtained from the above formulas $(1)' \sim (5)'$:

We now say that p and q are called k-adjacent if $q \in N_k(p)$ in $\mathbb{Z}^5$, where $k \in \{242, 210, 130, 50, 10\}$.

Similarly, by the same method above, we get 4 kinds of k- adjacency relations in $\mathbb{Z}^4$ are followed. Namely, for two 4-xels $p = (p_1, p_2, p_3, p_4), q = (q_1, q_2, q_3, q_4) \in \mathbb{Z}^4$, the following equations are considered,

(6) $d_4(p,q) = 4, d_*(p,q) = 1 \Rightarrow$ then p shares a point with q,

(7) $d_4(p,q) = 3, d_*(p,q) = 1 \Rightarrow$ then p shares an edge with q,

(8) $d_4(p,q) = 2, d_*(p,q) = 1 \Rightarrow$ then p shares a face with q,

(9) $d_4(p,q) = 1, d_*(p,q) = 1 \Rightarrow$ then p shares a cube with q.

From (6) $\sim$ (9) above, the following equations are taken by the same method as $\mathbb{Z}^5$.

We now say that p and q are called k-adjacent if $q \in N_k(p)$ in $\mathbb{Z}^4$, where $k \in \{80, 64, 32, 8\}$.

Consequently, we get the following:

1 Proposition. *There are* 4 *kinds of k-adjacency relations in $\mathbb{Z}^4$, $k \in \{80, 64, 32, 8\}$ and* 5 *kinds of k-adjacency relations in $\mathbb{Z}^5$, $k \in \{242, 210, 130, 50, 10\}$.*

Thus in $\mathbb{Z}^4$, the digital pictures $(\mathbb{Z}^4, k, \bar{k}, X)$ are considered for the following cases: $(k, \bar{k}) \in \{(80, 8), (8, 80), (64, 8), (8, 64), (32, 8), (8, 32)\}$.

Furthermore, in $\mathbb{Z}^5$, the digital pictures $(\mathbb{Z}^5, k, \bar{k}, X)$ are obtained for the following cases as follows: $(k, \bar{k}) \in \{(242, 10), (10, 242), (210, 10), (10, 210), (130, 10), (10, 130), (50, 10), (10, 50)\}$.

For a digital image $X(\subset \mathbb{Z}^n)$, two points $x(\neq)y(\in X)$ are called k-connected [1, 6] if there is a k-path $f : [0, m]_{\mathbb{Z}} \to X$ where the image is a sequence $(x_0, x_1, \ldots, x_m)$ from the set of points $\{f(0) = x_0 = x, f(1) = x_1, \ldots, f(m) = x_m = y\}$ such that x_i and x_{i+1} are k-adjacent, $i \in \{0, 1, \ldots, m-1\}, m \geq 1$[1, 10].

And a simple closed k-curve is considered as a sequence $(x_0, x_1, \ldots, x_m)$ of the k-path where x_i and x_j are k-adjacent if and only if $j = i + 1(\mathrm{mod}\, m)$ or $i = j - 1(\mathrm{mod}\, m)$[1, 3].

2 Digital (k_0, k_1)-homotopy

On the basis of the digital continuity and the digital (k_0, k_1)-continuity [1, 10], the convenient digital (k_0, k_1)-continuity in terms of the digital k-connectedness was introduced in [2]. But for the study of pointed digital homotopy theory, we need some reformations. Furthermore, the former digital (k_0, k_1)-continuity with the standard k_i-adjacency relations will be generalized to the digital (k_0, k_1)-continuity with the n types of k_i-adjacency relations in $\mathbb{Z}^n$, $i \in \{0, 1\}, n \in \{4, 5\}$.

Now we define a digital (k_0, k_1)-continuity as a generalization of the digital (k_0, k_1)-continuity of [2]; such an approach is essential in studying the pointed digital (k_0, k_1)-homotopy theory [2].

2 Definition. For two digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$ and $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$, we say that a map $f : X \to Y$ is digitally (k_0, k_1)-continuous at the point $x \in X$ if for every k_0-connected subset $O_{k_0}(x)$ containing x, $f(O_{k_0}(x))$ is k_1-connected, where $k_i \in \{242, 210, 130, 50, 10\}$ in $\mathbb{Z}^5$, $k_i \in \{80, 64, 32, 8\}$ in $\mathbb{Z}^4$, $k_i \in \{26, 18, 6\}$ in $\mathbb{Z}^3$, $k_i \in \{8, 4\}$ in $\mathbb{Z}^2$ and so on.

If f is digitally (k_0, k_1)-continuous at any point $x \in X$ then f is called a digitally (k_0, k_1)-continuous map.

From now on, all spaces are considered under the following k_i-adjacency relations,

$k_i \in \{242, 210, 130, 50, 10\}$ in $\mathbb{Z}^5$, $k_i \in \{80, 64, 32, 8\}$ in $\mathbb{Z}^4$, $k_i \in \{26, 18, 6\}$ in $\mathbb{Z}^3$, $k_i \in \{8, 4\}$ in $\mathbb{Z}^2$ and so on.

For two digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, (X, A))$ and $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, (Y, B))$, we say that a map $f : (X, A) \to (Y, B)$ is digitally (k_0, k_1)-continuous if $f : X \to Y$ is digitally (k_0, k_1)-continuous and $f(A) \subset B$, respectively.

In [1, 2], the digital homotopy was introduced. Now we define the generalized digital (k_0, k_1)-homotopy.

For digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$ and $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$, let $f, g : X \to Y$ be digitally (k_0, k_1)-continuous functions. And suppose that there are a positive integer m and a function, $F : X \times [0, m]_{\mathbb{Z}} \to Y$ such that

- for all $x \in X, F(x, 0) = f(x)$ and $F(x, m) = g(x)$,
- for all $x \in X$, the induced map $F_x : [0, m]_{\mathbb{Z}} \to Y$ defined by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbb{Z}}$ is digitally $(2, k_1)$-continuous, and
- for all $t \in [0, m]_{\mathbb{Z}}$, the induced map F_t which is defined by $F_t(x) = F(x, t) :$ $X \to Y$ is digitally (k_0, k_1)-continuous for all $x \in X$.

If, further, $F(x_0, t) = y_0$ for some $(x_0, y_0) \in X \times Y$ and all $t \in [0, m]_{\mathbb{Z}}$, we say F is a pointed (k_0, k_1)-homotopy.

If $X = [0, m_X]_{\mathbb{Z}}$ and for all $t \in [0, m]_{\mathbb{Z}}$ we have $F(0, t) = F(0, 0)$ and $F(m_X, t) = F(m_X, 0)$, we say F holds the endpoints fixed.

We say an image X is k-contractible [2] if the identity map 1_X is (k, k)-homotopic in X to a constant map with image consisting of some $x_0 \in X$. If such a homotopy is a pointed homotopy, we say (X, x_0) is *pointed k-contractible.*

We say that f and g are digitally pointed homotopic and then we use a notation $f \simeq_{d\cdot(k_0,k_1)\cdot h} g$.

Especially, for the case of the digital pointed (k,k)-homotopy, we call it a digital pointed k-homotopy and use the notation: $f \simeq_{d\cdot k\cdot h} g$ instead of $f \simeq_{d\cdot(k,k)\cdot h} g$.

For the digital image X with a k-adjacency and its subimage A, we call (X,A) a digital image pair with a k-adjacency. Furthermore, if A is a singleton set $\{p\}$ then (X,p) is called a pointed digital image.

For a digital image (X,A) with a k-adjacency, we say that X is k-deformable into A if there is a digital pointed k-homotopy $D : X \times [0,m]_{\mathbb{Z}} \to X$ such that $D(x,0) = x$ and $D(x,m) \subset A$, $x \in X$. The above digital pointed k-homotopy is called a digital k-deformation. The current pointed k-homotopy means that $D(x_0,t) = x_0$ for $x_0 \in A$ and all $t \in [0,m]_{\mathbb{Z}}$.

Actually, the digital fundamental group was developed for the digital image in dimension at most three image in $\mathbb{Z}^3$ [6] and was derived from an approach to algebraic topology under the standard k-adjacency in $\mathbb{Z}^n$, where $k \in \{3^n - 1(n \geq 2), 2n(n \geq 1), 18(n = 3)\}$ [5].

Now we make a reformation in terms of the generalized pointed digital homotopy without any restriction to the dimension and the k-adjacency of the image. The k-type digital fundamental group is induced via the generalized pointed k-homotopy. Namely, we study the image in $\mathbb{Z}^n$ with the n-kinds of the k-adjacency in $\mathbb{Z}^n$, $k \in \{242, 210, 130, 50, 10\}$ in $\mathbb{Z}^5$, $k \in \{80, 64, 32, 8\}$ in $\mathbb{Z}^4$, $k \in \{26, 18, 6\}$ in $\mathbb{Z}^3$, $k \in \{8, 4\}$ in $\mathbb{Z}^2$ and $k \in \{3^n - 1(n \geq 2), 2n(n \geq 1)\}$ in $\mathbb{Z}^n, n \geq 6$.

Since the preservation of the base point is essential in studying the pointed digital (k_0,k_1)-homotopy theory, the digital (k_0,k_1)-continuity is very meaningful.

Thus the k-type digital fundamental group is a generalization of the digital fundamental group of [2, 5, 6] relative to the adjacency and the dimension of the image.

Concretely, for a pointed digital image (X,p), a k-loop f based at p is a k-path in X with $f(0) = p = f(m)$. And we put $F_1^k(X,p) = \{f | f$ is a k-loop based at $p\}$.

For maps $f, g (\in F_1^k(X,p))$, i.e., $f : [0,m_1]_{\mathbb{Z}} \to (X,p)$ with $f(0) = p = f(m_1)$ and $g : [0,m_2]_{\mathbb{Z}} \to (X,p)$ with $g(0) = p = g(m_2)$, we get a map $f * g : [0, m_1 + m_2]_{\mathbb{Z}} \to (X,p)$ as follows [5]:

$f * g : [0, m_1 + m_2]_{\mathbb{Z}} \to (X,p)$ is defined by $f * g(t) f(t), (0 \leq t \leq m_1)$ and $g(t - m_1), (m_1 \leq t \leq m_1 + m_2)$. Then $f * g \in F_1^k(X,p)$.

We denote the digital k-homotopy class of f by $[f]$. Obviously, the homotopy class $[f * g]$ depends on the homotopy classes $[f]$ and $[g]$.

Furthermore, for any $f_1, f_2, g_1, g_2 \in F_1^k(X,p)$ such that $f_1 \in [f_2], g_1 \in [g_2]$

we get the map $f_1 * g_1 \in [f_2 * g_2]$, i.e., $[f_1 * g_1] = [f_2 * g_2]$.

Consequently, we put $\pi_1^k(X,p) = \{[f] | f \in F_1^k(X,p)\}$. And we take an operation $\cdot$ on $\pi_1^k(X,p)$ as follows: $[f] \cdot [g] = [f * g]$.

The group structure on $\pi_1^k(X,p)$ is checked by the same method as in [1] with respect to the digital $(2,k)$-continuity.

For our emphasizing on the k-connectivity of the digital image X, we use the superscript k like $\pi_1^k(X,p)$.

Consequently, we get a group $\pi_1^k(X,p)$ with the above operation $\cdot$, which is called the k-type digital fundamental group of a pointed digital image (X,p).

Actually, if p and q belong to the same k-connected component of X, then $\pi_1^k(X,p)$ is isomorphic to $\pi_1^k(X,q)$ [1].

For digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$, $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$ and a digitally (k_0,k_1)-continuous based map $h : (X,p) \to (Y,q)$, the map h induces a digital fundamental group (k_0,k_1)-homomorphism as follows.

Define $\pi_1^{(k_0,k_1)}(h) = h_* : \pi_1^{k_0}(X,p) \to \pi_1^{k_1}(Y,q)$ by the equation $h_*([f_1]) = [h \circ f_1]$, where $[f_1] \in \pi_1^{k_0}(X,p)$, which is well defined. Particularly, if $k_0 = k_1$, we use the following notation, $\pi_1^{k_0}(h)$[1].

For digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$, $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$ and $(\mathbb{Z}^{n_2}, k_2, \bar{k}_2, Z)$, let $f : X \to Y$ be digitally (k_0,k_1)-continuous based map and $g : Y \to Z$ be digitally (k_1,k_2)-continuous function. Then obviously $\pi_1^{(k_0,k_2)}(g \circ f) = \pi_1^{(k_1,k_2)}(g) \circ \pi_1^{(k_0,k_1)}(f)$[1]. In particular, if $k_0 = k_1 = k_2$, $\pi_1^{k_0}(g \circ f) = \pi_1^{k_0}(g) \circ \pi_1^{k_0}(f)$. Actually, if a pointed image (X,p) is k-connected, for any point $q \in X$ there is an isomorphism $\phi : \pi_1^k(X,p) \cong \pi_1^k(X,q)$[1].

3 Theorem. *For a digital image picture $(\mathbb{Z}^n, k, \bar{k}, (X,A))$, if (X,p) is k-deformable into (A,p) then $\pi_1^k(X,p) \cong \pi_1^k(A,p)$.*

PROOF. First, from the digital k-deformation $D : X \times [0,m]_{\mathbb{Z}} \to X$ such that $D(X \times \{m\}) \subset A$, let $r : (X,p) \to (A,p)$ be defined as follows: $(i \circ r)(x) = D(x,m)$, $x \in X$ and $i : (A,p) \to (X,p)$ is the inclusion map. Then D makes $1_{(X,p)}$ be digitally pointed k-homotopic to $i \circ r$. And further, $D(x_0,t) = x_0$ for some $x_0 \in A$. Thus r is a right digital k-homotopy inverse of i. Namely, $i \circ r \simeq_{d \cdot k \cdot h} 1_{(X,p)}$. Therefore $\pi_1^k(i \circ r) = \pi_1^k(i) \circ \pi_1^k(r) = 1_{\pi_1^k(X,p)}$. Thus $\pi_1^k(r)$ is a monomorphism.

Second, for any $[g] \in \pi_1^k(A,p)$, there are a k-path $f \in F_1^k(X,p)$ and a set of k-paths $\{g_1, g_2, \cdots, g_c\} \subset F_1^k(X,p)$, such that $f \simeq_{d \cdot k \cdot h} g_1, g_i \simeq_{d \cdot k \cdot h} g_{i+1}$ for $i \in \{1, 2, \cdots, c-1\}$ and $g_c \simeq_{d \cdot k \cdot h} g$. Thus $\pi_1^k(r)([f]) = [g]$. Therefore $\pi_1^k(r)$ is an epimorphism. QED

4 Corollary. *[1] If X is pointed k-contractible then $\pi_1^k(X,p)$ is trivial.*

3 Digital (k_0, k_1)-homeomorphism

For our classification of digital images, we need special relations among digital images with k_i-adjacencies $i \in \{0,1\}$. One of them is the digital (k_0, k_1)-homeomorphism as follows:

5 Definition. [1, 3, 4] For digital pictures $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, X)$ and $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, Y)$, a map $h : X \to Y$ is called a digital (k_0, k_1)-homeomorphism if h is digitally (k_0, k_1)-continuous and bijective and further $h^{-1} : Y \to X$ is digitally (k_1, k_0)-continuous. Then we write it by $X \approx_{d\cdot(k_0,k_1)\cdot h} Y$. If $k_0 = k_1$, we say that h is a digital homeomorphism [1].

The minimal simple closed curves in $\mathbb{Z}^2$ with three types which are not digital homeomorphic to each other are MSC_8, MSC_4 and $MSC_8'(\subset \mathbb{Z}^2)$[3, 4].

Let MSC_8 be the set which is digitally 8-homeomorphic to the image [4],

$$\{(x_1, y_1), (x_1-1, y_1+1), (x_1-2, y_1), (x_1-2, y_1-1), (x_1-1, y_1-2), (x_1, y_1-1)\}.$$

Let MSC_4 be the set which is digitally 4-homeomorphic to the image,

$$\{(x_1, y_1), (x_1, y_1+1), (x_1-1, y_1+1), (x_1-2, y_1+1), (x_1-2, y_1), (x_1-2, y_1-1), (x_1-1, y_1-1), (x_1, y_1-1)\},$$

i.e., $MSC_4 \approx_{d\cdot 4\cdot h} N_8(p_3), p_3 \in \mathbb{Z}^2$ [3, 4].

Let MSC_8' be the set which is digitally 8-homeomorphic to the image,

$$\{(x_1, y_1), (x_1-1, y_1+1), (x_1-2, y_1), (x_1-1, y_1-1)\}$$

[1, 3].

We can classify digital images from the following induced digital fundamental group (k_0, k_1)- isomorphism.

6 Theorem. *Let $(\mathbb{Z}^{n_0}, k_0, \bar{k}_0, (X, x_0))$ and $(\mathbb{Z}^{n_1}, k_1, \bar{k}_1, (Y, y_0))$ be digital pictures, where $k_i \in \{242, 210, 130, 50, 10\}$ in $\mathbb{Z}^5$, $k_i \in \{80, 64, 32, 8\}$ in $\mathbb{Z}^4$, $k_i \in \{26, 18, 6\}$ in $\mathbb{Z}^3$,$k_i \in \{8, 4\}$ in $\mathbb{Z}^2$ and $k_i \in \{3^n-1, 2n\}$ in $\mathbb{Z}^n, n \geq 6$, $i \in \{0,1\}$. If $h : (X, x_0) \to (Y, y_0)$ is a digital (k_0, k_1)-homeomorphism then the induced map $h_* : \pi_1^{k_0}(X, p) \to \pi_1^{k_1}(Y, q)$ defined by $h_*([f]) = [h \circ f]$, $[f] \in \pi_1^{k_0}(X, p)$ is a digital fundamental group isomorphism.*

Proof. First, h_* is well-defined. If $f' \in [f] \in \pi_1^{k_0}(X, p)$, let $F : (X, p) \times [0, m]_{\mathbb{Z}} \to (X, p)$ be a digital k_0-homotopy between f and f'. Then $h \circ F$ is a digital k_1-homotopy between the k_1-loops $h \circ f$ and $h \circ f'$. Thus $h \circ f' \in [h \circ f]$.

Second, the induced map h_* is a homomorphism.

For any maps $f, g \in F_1^{k_0}(X, p)$, the digitally $(2, k_0)$-continuous maps $f : [0, m_1]_{\mathbb{Z}} \to (X, p)$ and $g : [0, m_2]_{\mathbb{Z}} \to (X, p)$, the map $h \circ (f * g) : [0, m_1 + m_2]_{\mathbb{Z}} \to$

(Y, q) is defined as follows:

$$h \circ (f * g) : [0, m_1 + m_2]_{\mathbb{Z}} \to (Y, q)$$
$$h \circ (f * g)(t) = \begin{cases} h(f(t)), & (0 \le t \le m_1), \\ h(g(t - m_1)), & (m_1 \le t \le m_1 + m_2) \end{cases}$$

Thus $h \circ (f * g) = (h \circ f) * (h \circ g)$ and $h_*([f] \cdot [g]) = h_*([f * g]) = [h \circ (f * g)][(h \circ f) * (h \circ g)] = [h \circ f] \cdot [h \circ g] = h_*([f]) \cdot h_*([g])$.

The induced map h_* depends not only on the digitally (k_0, k_1)-continuous map $h : (X, p) \to (Y, q)$ but also on the choice of the base points p and q.

Second, h_* is surjective: for any $[g] \in \pi_1^{k_1}(Y, q)$, we get $g : [0, m]_{\mathbb{Z}} \to (Y, q)$ is a digitally $(2, k_1)$-continuous map such that $g(0) = q = g(m)$. Because h is a digital (k_0, k_1)-homeomorphism, there is a digitally $(2, k_0)$-continuous map: $f_1 : [0, m]_{\mathbb{Z}} \to (X, p)$ such that $f_1(0) = p = f_1(m)$ and $h \circ f_1 = g$. Thus $h_*([f_1]) = [h \circ f_1] = [g]$.

Third, h_* is injective: if $h_*([f_1]) = [h \circ f_1] = c_{\{q\}} \in \pi_1^{k_1}(Y, q)$, we only prove that $f_1 \simeq_{d \cdot k_0 \cdot h} c_{\{p\}}$. Since $h \circ f_1 \simeq_{d \cdot k_1 \cdot h} c_{\{q\}}$, there is a digitally $(2, k_0)$-continuous map $f_1 : [0, m]_{\mathbb{Z}} \to (X, p)$ such that $f_1(0) = p = f_1(m)$ and $f_1 \simeq_{d \cdot k_0 \cdot h} c_{\{p\}}$.

Fourth, h_* is a homomorphism. For any $[f_1], [f_2] \in \pi_1^{k_0}(X, p)$, $h_*([f_1] \cdot [f_2]) = h_*[f_1 * f_2] = [h \circ ([f_1 * f_2])] = [(h \circ f_1 * h \circ f_2)] = [(h \circ f_1] \cdot [h \circ f_2] h_*[f_1] \cdot h_*[f_2]$. *QED*

A black point in a digital picture $P = (\mathbb{Z}^n, k, \bar{k}, X)$ is called a border point if it is k-adjacent to one or more white points. The border of X in the above digital picture P is the set of all border points and it is denoted by $\mathrm{Bd}(X)$.

7 Example. The group $\pi_1^4(MSC_4, x_0) \simeq \pi_1^8(Bd(B_2(p, 2)))$.

Proof. Since $Bd(B_2(p, 2))$ is $(8, 4)$-homeomorphic to MSC_4, the proof is completed. *QED*

8 Example. For the image $W_1 = B_2(p_1, 2) - \{p_1, (x_1 + 1, y_1)\} \cup N_8(p_3)$, where $p_1 = (x_1, y_1)$, $p_2 = (x_1 + 2, y_1)$ and $p_3 = (x_1 + 3, y_1)$, $\pi_1^8(W_1, p_2) \cong \pi_1^8(MSC_8)$. Assume that $N_8(p_3) = \{q_0 = (x_1 + 4, y_1), q_1 = (x_1 + 4, y_1 + 1), q_2 = (x_1 + 3, y_1 + 1), q_3 = (x_1 + 2, y_1 + 1), q_4 = (x_1 + 2, y_1), q_5 = (x_1 + 2, y_1 - 1), q_6 = (x_1 + 3, y_1 - 1), q_6 = (x_1 + 4, y_1 - 1)\}$.

Proof. (Step 1): Without loss of generality, assume that MSC_8 is a subset of $B_2(p_1, 2) - \{p_1, (x_1 + 1, y_1)\}$. We get easily $B_2(p_1, 2) - \{p_1, (x_1 + 1, y_1)\}$ is 8-deformable into $W_2 (\approx_{d \cdot 8 \cdot h} MSC_8)$, where $W_2 = \{(x_1 + 2, y_1), (x_1 + 1, y_1 + 1), (x_1, y_1 + 1), (x_1 - 1, y_1), (x_1, y_1 - 1), (x_1 + 1, y_1 - 1)\}$.

(Step 2): We prove that $N_8(p_3)$ is pointed 8-contractible into $\{p_2\}$. Namely, there is a digital 8-homotopy $H : N_8(p_3) \times [0, 3]_{\mathbb{Z}} \to N_8(p_3)$ as follows:

First, $H(q_i, 0) = q_i$, for any $q_i \in N_8(p_3)$.

Second, $H(q_{2i+1}, 1) = q_{2i}, H(q_{2i}, 1) = q_{2i}, i \in [0, 3]_{\mathbb{Z}}$,

Third, $H(q_i, 2) = q_4, i \in \{2, 3, 4, 5\}$ and $H(q_j, 2) = q_6, j \in \{0, 1, 6, 7\}$.
Finally $H(q_i, 3) = q_4$, $i \in [0, 7]_{\mathbb{Z}}$.
Therefore $\pi_1^8(W_1, p_2) \cong \pi_1^8(MSC_8)$ from (Step 1) and (Step 2). QED

9 Corollary. *If there are k_0, k_1 such that $\pi_1^{k_0}(X, p)$ is not isomorphic to $\pi_1^{k_1}(Y, q)$ then X and Y are not digitally (k_0, k_1)-homeomorphic to each other.*

PROOF. A digital (k_0, k_1)-continuous map $h : (X, p) \to (Y, q)$ induces a digital fundamental group homomorphism $h_* : \pi_1^{k_0}(X, p) \to \pi_1^{k_1}(Y, q)$ defined by $h_*([f]) = [h \circ f]$. It is easy to see that h_* and h_*^{-1} are bijective homomorphisms. Thus a digital (k_0, k_1)-homeomorphism $h : (X, p) \to (Y, q)$ induces a digital fundamental group isomorphism. By the contraposition of the above statement we get the proof. QED

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