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An analytic proof of the theorems of Pappus and Desargues

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Abstract. In this article we give an analytic proof of Pappus' theorem and an analytic proof of Desargues' theorem over a not necessarily commutative field.

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Introduction

In this article we give an analytic proof of Pappus' theorem and an analytic proof of Desargues' theorem over a not necessarily commutative field. Both are listed as exercises in [4, p. 76 and p. 78]. The original version of Pappus' theorem appeared in Pappus' $\Sigma\upsilon\nu\alpha\gamma\omega\gamma\acute{\eta}$ or *Collection* (see, e.g., [2, pp. 270–273]). The original version of Desargues' theorem appeared in A. Bosse's "*La Perspective de Mr. Desargues*" (Paris 1648, p. 340) as the First Geometrical Proposition (see also [1, Chapter VIII]). We refer to [3] for an excellent comprehensive historical survey of these two theorems.

1 An Analytic Proof of Pappus' Theorem

For the terminology and fundamental facts used in our proof we refer to [4].

Let $\mathbf{K}$ be a commutative field. Then the *projective plane* $\mathbb{P}^2(\mathbf{K})$ is defined as the quotient of $\mathbf{K}^3 \setminus \{0\}$ by the equivalence relation

$$\sim: \qquad X, Y \in \mathbf{K}^3 \setminus \{0\}\,,\ X \sim Y \text{ if } \exists\lambda \neq 0 \text{ in } \mathbf{K} \text{ such that } Y = \lambda X.$$

Let $\pi\colon \mathbf{K}^3 \setminus \{0\} \to \mathbb{P}^2(\mathbf{K})$ denote the canonical projection, then a *point* $P = \pi\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) \in \mathbb{P}^2(\mathbf{K})$ is represented by the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in three-space, not all

$x, y, z = 0$, and for $\lambda \neq 0$ the vector $\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda z \end{bmatrix}$ represents the same point P. By abuse of language we identify P with $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and write $P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ if there is no possible confusion. We also call x, y, z the *homogeneous coordinates* of P. Since an equation of a plane $\mathcal{P}$ in $\mathbf{K}^3$ passing through the origin is of the form

$$ax + by + cz = 0, \qquad \text{not all } a, b, c = 0,$$

we say that $\pi(\mathcal{P} \setminus \{0\})$ is a *line* in $\mathbb{P}^2(\mathbf{K})$. Thus a line l in $\mathbb{P}^2(\mathbf{K})$ is the equivalence class of the row $\begin{bmatrix} a & b & c \end{bmatrix}$ and for $\lambda \neq 0$ the row $\begin{bmatrix} \lambda a & \lambda b & \lambda c \end{bmatrix}$ represents the same line l. We again identify the line l with the row $\begin{bmatrix} a & b & c \end{bmatrix}$ and call a, b, c the *homogeneous coordinates* of l. We see that a point $P \in \mathbb{P}^2(\mathbf{K})$ lies on a line $l \in \mathbb{P}^2(\mathbf{K})$ if and only if

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = ax + by + cz = 0.$$

It can easily be shown that every line is incident with at least three points. Also since we can find two triples of numbers $[a, b, c]$, $[x, y, z]$ such that $ax + by + cz \neq 0$, we are assured that there exist a point and a line not incident. Further, using the theory of sets of linear homogeneous equations, we can prove that any two points are incident with one and only one line. Then, appealing to the duality of the definitions of point and line, it can be shown that any two distinct lines intersect in a unique point.

The following Lemma 3 is an immediate consequence of the following two theorems on pages 73 and 74 of [4].

1 Theorem. *If $P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $P \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are distinct points, then for any λ the numbers $x_i + \lambda y_i$, $i = 1, 2, 3$, are not all $= 0$, and $P \begin{bmatrix} x_1 + \lambda y_1 \\ x_2 + \lambda y_2 \\ x_3 + \lambda y_3 \end{bmatrix}$ is collinear with the first two points.*

2 Theorem. *If X, Y, Z are coordinates of three distinct collinear points, then there exist λ, μ such that $Z = \lambda X + \mu Y$.*

3 Lemma. *Three distinct points $P_1, P_2, P_3 \in \mathbb{P}^2(\mathbf{K})$ are collinear if and only if for any one of these points, say P_3, there exist nonzero scalars $\lambda, \mu \in \mathbf{K}$*

such that $P_3 = \lambda P_1 + \mu P_2$. It follows that P_1, P_2, P_3 are collinear if and only if $[P_1, P_2, P_3] = 0$, where $[P_1, P_2, P_3]$ denotes the determinant formed by their coordinates.

4 Lemma. *Let P_1, P_2, P_3 be noncollinear points of $\mathbb{P}^2(\mathbf{K})$. If $Q = a_1P_1 + a_2P_2 + a_3P_3$, $R = b_1P_1 + b_2P_2 + b_3P_3$, and $S = c_1P_1 + c_2P_2 + c_3P_3$ are points of $\mathbb{P}^2(\mathbf{K})$, then Q, R, S are collinear if and only if*

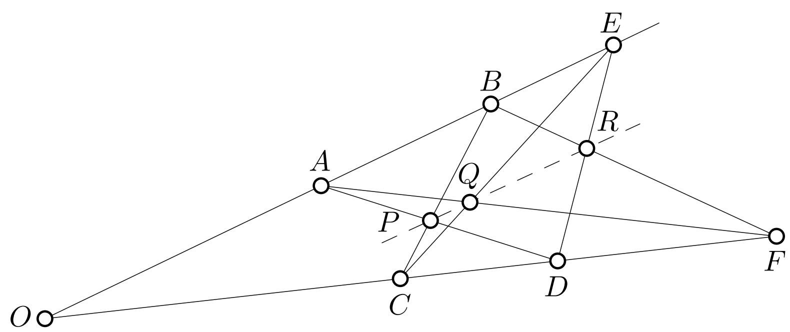
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

Proof. Let x_i, y_i, z_i denote the homogeneous coordinates of P_i, $i = 1, 2, 3$. We have

$$\begin{aligned}
[Q, R, S] &= [a_1P_1 + a_2P_2 + a_3P_3, b_1P_1 + b_2P_2 + b_3P_3, c_1P_1 + c_2P_2 + c_3P_3] \\
&= [a_1P_1, b_2P_2, c_3P_3] + [a_1P_1, b_3P_3, c_2P_2] \\
&\quad + [a_2P_2, b_1P_1, c_3P_3] + [a_2P_2, b_3P_3, c_1P_1] \\
&\quad + [a_3P_3, b_1P_1, c_2P_2] + [a_3P_3, b_2P_2, c_1P_1] \\
&= (a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1)\,[P_1, P_2, P_3] \\
&= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [P_1, P_2, P_3]\,.
\end{aligned}$$

The lemma follows from Lemma 3 and the assumption that P_1, P_2, P_3 are noncollinear. $\boxed{QED}$

5 Theorem. (Pappus) *Let l and l' be two distinct lines of $\mathbb{P}^2(\mathbf{K})$, intersecting at a point O. Let A, B, E be three distinct points on l and C, D, F three distinct points on l' such that each of these six points is different from O. Let P be the intersection of the lines AD and CB, Q the intersection of AF and CE, and R the intersection of BF and DE. Then P, Q, R are collinear.*



(Figure 1)

Proof. In drawing Pappus' configuration, we represent the points A, B, and E as vectors in three-space (using homogeneous coordinates). We use the same letters A, B, and E to represent the vectors. By Lemma 3, for A, B, and E to be collinear it is necessary and sufficient that there exist nonzero scalars s and t such that $E = sA + tB$. Since we are free to represent the point A by any nonzero scalar multiple of the vector A, we choose to represent it by sA, and similarly we represent the point B by tB. Now we rename the vector sA as A and the vector tB as B. Thus we have the vector equation $E = A + B$ in this notation. By the same reasoning there exist vectors C and D, such that C represents the point C, D represents the point D, and $C + D$ represents the point F.

Since A, B, O are distinct collinear points, and C, D, O are also distinct collinear points, there exist by Lemma 3 nonzero scalars a, b, c, d such that

$$O = aA + bB = cC + dD. \tag{1}$$

Since E is different from O we must have $a \neq b$, and since F is different from O we must have $c \neq d$. From (1) we get the relation

$$aA - dD = cC - bB. \tag{2}$$

By Lemma 3, (2) represents a point which is on both the lines AD and BC, and this intersection is named P. From (1) we also get

$$(a - b)\,A - d\,(C + D) = (c - d)\,C - b\,(A + B)\,, \tag{3}$$

which is equivalent to the equation

$$(a - b)\,A - dF = (c - d)\,C - bE. \tag*{(3)$'$}$$

Since $a - b$, $c - d$, d, and b are nonzero scalars, Lemma 3 implies that $(3)'$ represents a point which is on both the lines AF and EC. This intersection is named Q. From (1) it follows that

$$(b - a)\,B - c\,(C + D) = (d - c)\,D - a\,(A + B) \tag{4}$$

which is equivalent to the equation

$$(b - a)\,B - cF = (d - c)\,D - aE. \tag*{(4)$'$}$$

Lemma 3 implies that $(4)'$ represents a point that is on both the lines BF and DE. This intersection is named R.

By multiplying (1) by the nonzero scalar b we obtain

$$abA + b^2B - bcC - bdD = 0. \tag{5}$$

Using the right-hand side of (3), we may represent Q by $-bA - bB + (c-d)\,C$ which is equivalent to $-abA - abB + (ac-ad)\,C$ since a is different from 0. Adding the zero vector given by (5) to this representation of Q, we obtain the following representation of Q.

$$Q\colon \quad \left(b^2 - ab\right) B + (ac - ad - bc)\, C - (bd)\, D. \tag{6}$$

To prove that P, Q, and R are collinear we use (2), (6), and (4) to obtain the following representations:

$$\begin{cases} P\colon & -bB + cC, \\ Q\colon & \left(b^2 - ab\right) B + (ac - ad - bc)\, C - (bd)\, D, \\ R\colon & (b-a)\, B - cC - cD. \end{cases} \tag{7}$$

Since B, C, and D are noncollinear points of $\mathbb{P}^2(\mathbf{K})$, Lemma 4 implies that a necessary and sufficient condition for the points P, Q, and R to be collinear is that the determinant

$$\begin{vmatrix} -b & c & 0 \\ b^2 - ab & ac - ad - bc & -bd \\ b - a & -c & -c \end{vmatrix} = 0.$$

An easy calculation confirms that this determinant is zero. $\boxed{QED}$

It is well known that Pappus' theorem implies the commutativity of the multiplication in the field $\mathbf{K}$ of segment arithmetic (see the discussion in [3] and a proof of this fact in [4, pp. 76–86], for example). It is also well known (see [5]) that Pappus' theorem implies Desargues' theorem, but the converse is not true when the field $\mathbf{K}$ is not commutative. In [4, p. 75], Seidenberg gave a simple analytic proof of Desargues' theorem when $\mathbf{K}$ is commutative; in the next section we adapt his proof to the case where the field $\mathbf{K}$ is not necessarily commutative.

2 Desargues' Theorem over a Not Necessarily Commutative Field

For terminology, definitions, and basic facts, we refer to [4, Chapter III]. Let $\mathbf{K}$ be a *not necessarily commutative field*, i.e., we do not assume the rule $ab = ba$ for all $a, b \in \mathbf{K}$ (if $ab \neq ba$ for some $a, b \in \mathbf{K}$ then $\mathbf{K}$ is called a *noncommutative*, or *skew*, field). If $\mathbf{K}$ is a not necessarily commutative field then the following hold:

— There is one and only one element $e \in \mathbf{K}$ such that $ae = ea$ for all $a \in \mathbf{K}$. Such an element e is called the *multiplicative neutral element* of $\mathbf{K}$.

— For every $a \neq 0$ in $\mathbf{K}$ there exists one and only one element $a^{-1} \in \mathbf{K}$ such that $aa^{-1} = a^{-1}a = e$. This element is called the *inverse* of a.

— The equation $ax = b$, $a \neq 0$, has one and only one solution, namely $x = a^{-1}b$. Similarly, the equation $xa = b$, $a \neq 0$, has one and only one solution, namely $x = ba^{-1}$.

We now define the projective plane $\mathbb{P}^2(\mathbf{K})$ by representing a point $P \in \mathbb{P}^2(\mathbf{K})$ by a column vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, not all $x, y, z = 0$ in $\mathbf{K}$, and such that for $\lambda \neq 0$ in $\mathbf{K}$, $\begin{bmatrix} x\lambda \\ y\lambda \\ z\lambda \end{bmatrix} \equiv \begin{bmatrix} x \\ y \\ z \end{bmatrix} \lambda$ represents the same point P.

We identify P with $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Similarly we represent a line l of $\mathbb{P}^2(\mathbf{K})$ by a row vector $\begin{bmatrix} a & b & c \end{bmatrix}$, not all $a, b, c = 0$ in $\mathbf{K}$, and such that for $\mu \neq 0$ in $\mathbf{K}$, $\mu \begin{bmatrix} a & b & c \end{bmatrix} \equiv \begin{bmatrix} \mu a & \mu b & \mu c \end{bmatrix}$ represents the same line l. Again we identify the line l with the vector $\begin{bmatrix} a & b & c \end{bmatrix}$. Then we see that P lies on l if and only if for $\lambda, \mu \neq 0$

$$\mu \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \lambda = \mu ax\lambda + \mu by\lambda + \mu cz\lambda = \mu (ax + by + cz) \lambda = 0.$$

6 Lemma. *If* $P_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *and* $P_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ *are distinct points of* $\mathbb{P}^2(\mathbf{K})$ *then for* λ_1, λ_2 *not both zero scalars, the point* $P_1\lambda_1 + P_2\lambda_2$ *lies on the line through* P_2 *and* P_2.

PROOF. Obvious. $\boxed{QED}$

7 Lemma. *If* P_1, P_2, *and* P_3 *are three distinct collinear points of* $\mathbb{P}^2(\mathbf{K})$, *then for any one of these points, say* P_3, *there exist nonzero scalars* $\lambda, \mu \in \mathbf{K}$ *such that* $P_3 = P_1\lambda + P_2\mu$.

PROOF. Let x_i, y_i, z_i be homogeneous coordinates of P_i, $i = 1, 2, 3$. Since not all x_1, y_1, z_1 equal zero we may assume without loss of generality that $x_1 \neq 0$. Let $l \begin{pmatrix} a & b & c \end{pmatrix}$ be the line passing through P_1, P_2, and P_3. Then we have the system

$$\begin{cases} ax_1 + by_1 + cz_1 = 0, \\ ax_2 + by_2 + cz_2 = 0, \\ ax_3 + by_3 + cz_3 = 0. \end{cases} \tag{8}$$

The equations $x_2 = x_1\left(x_1^{-1}x_2\right)$, $y_2 = y_1\left(x_1^{-1}x_2\right)$, and $z_2 = z_1\left(x_1^{-1}x_2\right)$ cannot be all possible, for if $x_2 = 0$ then $y_2 = z_2 = 0$ which is excluded, and if $x_2 \neq 0$ then $x_1^{-1}x_2 \neq 0$ and this would imply that P_1 and P_2 coincide, contradicting our hypothesis. Since obviously $x_2 = x_1\left(x_1^{-1}x_2\right)$, we may assume without loss of generality that $y_2 \neq y_1\left(x_1^{-1}x_2\right)$. Now let us consider the system of linear equations with unknowns λ and μ

$$\begin{cases} x_1\lambda + x_2\mu = x_3, \\ y_1\lambda + y_2\mu = y_3. \end{cases} \tag{9}$$

Multiplying (to the left) the first equation of (9) by $y_1x_1^{-1}$ and subtracting the result from the second, we get $\left(y_2 - y_1x_1^{-1}x_2\right)\mu = y_3 - y_1x_1^{-1}x_3$. Since $y_2 - y_1x_1^{-1}x_2 \neq 0$ we get $\mu = \left(y_2 - y_1x_1^{-1}x_2\right)^{-1}\left(y_3 - y_1x_1^{-1}x_3\right)$. From the first equation of (9) we get $x_1\lambda = x_3 - x_2\mu$. Since $x_1 \neq 0$ we obtain $\lambda = x_1^{-1}(x_3 - x_2\mu)$. Thus the system (9) can be solved for λ and μ. Multiplying (to the right) the first equation of the system (8) by λ, the second by μ, and subtracting their sum from the last, we get

$$a\left[x_3 - (x_1\lambda + x_2\mu)\right] + b\left[y_3 - (y_1\lambda + y_2\mu)\right] + c\left[z_3 - (z_1\lambda + z_2\mu)\right] = 0.$$

Since λ and μ are solutions of the system (9), the latter is reduced to

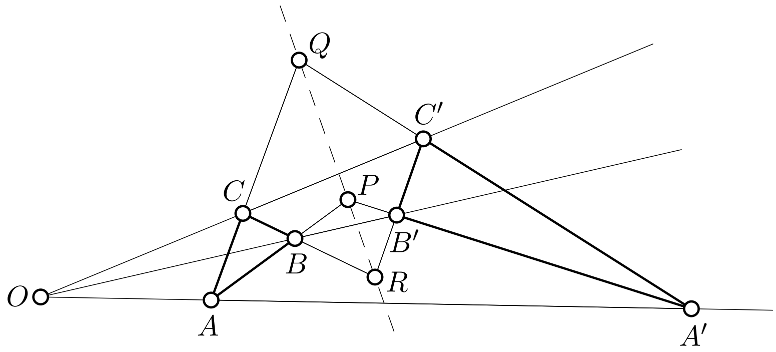
$$c\left[z_3 - (z_1\lambda + z_2\mu)\right] = 0.$$

Assuming that $c \neq 0$, we then obtain the equation $z_1\lambda + z_2\mu = z_3$, which together with the system (9) implies that $P_3 = P_1\lambda + P_2\mu$. And since by hypothesis P_3 is distinct from P_1 and P_2, λ and μ must be different from 0. Thus it remains to show that $c \neq 0$ to achieve the proof of the lemma. But if $c = 0$ then the system (8) is reduced to

$$\begin{cases} ax_1 + by_1 = 0, \\ ax_2 + by_2 = 0, \\ ax_3 + by_3 = 0. \end{cases} \tag{10}$$

Multiplying (to the right) the first equation of the system (10) by $x_1^{-1}x_2$ and subtracting the result from the second, we get $b\left(y_2 - y_1x_1^{-1}x_2\right) = 0$. Since $y_2 - y_1x_1^{-1}x_2 \neq 0$, we get $b = 0$. But then the system (10) is reduced to $ax_1 = 0$, $ax_2 = 0$, and $ax_3 = 0$. Since $x_1 \neq 0$, we get $a = 0$. Thus $a = b = c = 0$, which is not possible by the definition of the line l. $\boxed{QED}$

8 Theorem. (Desargues) *If two triangles ABC and $A'B'C'$ are in perspective from a point O distinct from the six points A, B, C, A', B', C' (i.e., the joins of A and A', B and B', C and C' intersect at O) then the intersections $P\colon AB \cap A'B'$, $Q\colon AC \cap A'C'$, and $R\colon BC \cap B'C'$ lie on a line.*



(Figure 2)

Proof. It follows from the hypothesis and Lemma 7 that

$$O = Aa + A'a' = Bb + B'b' = Cc + C'c', \tag{11}$$

for some nonzero scalars $a, a', b, b', c, c' \in \mathbf{K}$. From (11) we deduce that

$$\begin{cases} Aa - Bb = -A'a' + B'b', \\ Aa - Cc = -A'a' + C'c', \\ Bb - Cc = -B'b' + C'c'. \end{cases} \tag{12}$$

Since $-xy = x\,(-y)$ for all $x, y \in \mathbf{K}$, (12) is equivalent to

$$\begin{cases} Aa + B\,(-b) = A'\,(-a') + B'b', \\ Aa + C\,(-c) = A'\,(-a') + C'c', \\ Bb + C\,(-c) = B'\,(-b') + C'c'. \end{cases} \tag{12$'$}$$

It follows from Lemmas 6 and 7 that the first equation of $(12)'$ represents the intersection P of AB and $A'B'$, the second represents Q, and the third represents R. But then we obviously have $Q = P + R$, and by Lemma 6, P, Q, R are collinear. $\boxed{QED}$

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