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(LB)-spaces and quasi-reflexivity

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Abstract. Let (X_n) be a sequence of infinite-dimensional Banach spaces. For E being the space $\bigoplus_{n=1}^{\infty} X_n$, the following equivalences are shown: 1. Every closed subspace Y of E, with the Mackey topology $\mu(Y, Y')$, is an (LB)-space. 2. Every separated quotient of E' $[\mu(E', E)]$ is locally complete. 3. X_n is quasi-reflexive, $n \in \mathbb{N}$. Besides this, the following two properties are seen to be equivalent: 1. E' $[\mu(E', E)]$ has the Krein-Šmulian property. 2. X_n is reflexive, $n \in \mathbb{N}$.

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Dedicated to the memory of V.B. Moscatelli

1 Introduction and notation

The linear spaces that we shall use here are assumed to be defined over the field $\mathbb{K}$ of real or complex numbers, and the topologies on them will all be Hausdorff. As usual, $\mathbb{N}$ represents the set of positive integers. If $\langle E, F\rangle$ is a dual pair, then $\sigma(E, F)$, $\mu(E, F)$ and $\beta(E, F)$ denote the weak, Mackey and strong topologies on E, respectively; we shall write $\langle\cdot,\cdot\rangle$ for the bilinear functional associated to $\langle E, F\rangle$. If E is a locally convex space, E' is its topological dual and $\langle E, E'\rangle$, and also $\langle E', E\rangle$, denote the standard duality. E'' stands for the dual of $E'[\beta(E', E)]$. We identify, in the usual manner, E with a linear subspace of E''. If B is a subset of E, then $\tilde{B}$ is the closure of B in $E''[\sigma(E'', E')]$. If A is an arbitrary subset of E, by A° we denote the subset of E' given by the polar of A.

If X is a Banach space, then $B(X)$ will denote its closed unit ball, X^* is the Banach space conjugate of X, and X^{**} is its second conjugate, that is, the conjugate of X^*. In the usual manner, we suppose that X is a subspace of X^{**}. We say that X is quasi-reflexive when it has finite codimension in X^{**}. In [2], R. C. James gives an example of a quasi-reflexive Banach space which is not reflexive.

If A is a bounded absolutely convex subset of a locally convex space E, by E_A we mean the linear span of A with the norm provided by the gauge of A; the space E is said to be locally complete whenever E_A is complete for each closed bounded absolutely convex subset A of E. If E is sequentially complete, in particular if E is complete, then E is locally complete. We write ω to denote the space $\mathbb{K}^{\mathbb{N}}$ with the product topology.

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Following [6] (see also [1, p. 299]), we say that a locally convex space E is B-complete if each subspace F of $E'[\sigma(E',E)]$ is closed provided it intersects every closed absolutely convex subset which is equicontinuous in a closed set.

A locally convex space E is said to be an (LB)-space if it is the inductive limit of a sequence of Banach spaces, or, equivalently, if E is the separated quotient of the topological direct sum of a sequence of Banach spaces.

In [7], we obtain the following result: a) *Let* (X_n) *be a sequence of infinite-dimensional Banach spaces. If* $E := \bigoplus_{n=1}^{\infty} X_n$, *then the following are equivalent: 1.* $E'[\mu(E',E)]$ *is B-complete. 2. Every separated quotient of* $E'[\mu(E',E)]$ *is complete. 3.* X_n *is quasi-reflexive,* $n \in \mathbb{N}$.

In Section 2 of this paper, we obtain a theorem which adds new equivalences to the three before stated.

A locally convex space E is said to have the Krein-Šmulian Property whenever a convex subset A of E' is $\sigma(E',E)$-closed provided that, for each absolutely convex $\sigma(E',E)$-closed and equicontinuous subset M of E', the set $A \cap M$ is $\sigma(E',E)$-closed. Krein-Šmulian's theorem asserts that every Fréchet space has the Krein-Šmulian Property [1, p. 246].

In this paper, we characterize when E' $[\mu(E',E)]$ has the Krein-Šmulian Property, when $E := \bigoplus_{n=1}^{\infty} X_n$, with X_n, $n \in \mathbb{N}$, being a Banach space of infinite dimension.

2 (LB)-spaces and quasi-reflexivity

Theorem 1. *Let* (X_n) *be a sequence of infinite-dimensional Banach spaces. For* E *being* $\bigoplus_{n=1}^{\infty} X_n$, *the following are equivalent:*

(1) Every separated quotient of E' $[\mu(E',E)]$ *is locally complete.*

(2) Every closed subspace Y *of* E, *with the Mackey topology* $\mu(Y',Y)$, *is an (LB)-space.*

(3) X_n *is quasi-reflexive,* $n \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$, we write $E_n := \bigoplus_{j=1}^{n} X_j$ and we consider, in the usual way, that E_n is a subspace of E. We set

$$B_n := \ n \bigoplus_{j=1}^{n} B(X_j), \quad n \in \mathbb{N}.$$

$1 \Rightarrow 2$. Let us assume that 2. is not satisfied for a certain closed subspace Y of E. We put $Y_n := E_n \cap Y$ and $A_n := B_n \cap Y$, $n \in \mathbb{N}$. We find a linear functional v on Y such that it is not continuous although its restriction to each subspace Y_n is continuous. After Hahn-Banach's extension theorem we obtain, for each $n \in \mathbb{N}$, an element u_n of Y' such that

$$u_{n|Y_n} \ = \ v_{|Y_n}.$$

For an arbitrary x of Y, we find $n_0 \in \mathbb{N}$ such that $x \in Y_{n_0}$. Then

$$\langle x, u_n \rangle \ = \ \langle x, v \rangle, \quad n \geq n_0,$$

and thus $\{u_n : \ n \in \mathbb{N}\}$ is a bounded subset of Y' $[\sigma(Y',Y)]$. If T denotes the polar subset in Y of $\{u_n : n \in \mathbb{N}\}$, we have that T is a barrel in Y that absorbs each of the sets A_n, $n \in \mathbb{N}$. We now find a sequence of positive integers (j_n) such that

$$\frac{1}{j_n} A_n \ \subset \ T, \quad n \in \mathbb{N}.$$

Let A be the convex hull of

$$\cup\{\frac{1}{j_n 2^n} A_n : \ n \in \mathbb{N}\}.$$

Since A is absorbing in Y, we have that A° is a closed bounded absolutely convex subset of $Y'[\sigma(Y', Y)]$.We prove next that Y'_{A° is not a Banach space. Let $\| \cdot \|$ denote the norm in Y'_{A°. Given $\varepsilon > 0$, we find $n_0 \in \mathbb{N}$ such that $\frac{1}{2^{n_0}} < \frac{\varepsilon}{8}$. We take two integers p, q such that $p > q > n_0$. We can find an element z of A for which

$$\| u_p - u_q \| \leq \ 4 \ \mid \langle z, u_p - u_q \rangle \mid .$$

z may be written in the form

$$\sum_{n=1}^{\infty} \alpha_n z_n, \quad \alpha_n \geq 0, \quad z_n \in \frac{1}{j_n 2^n}{}_n, \quad n \in \mathbb{N}, \quad \sum_{n=1}^{\infty} \alpha_n = 1,$$

where the terms of the sequence (α_n) are all zero from a certain subindex on. Then

$$\begin{aligned} \| u_p - u_q \| \ &\leq \ 4 \mid \langle z, u_p - u_q \rangle \mid \leq 4 \sum_{n=1}^{\infty} \mid \langle \alpha_n z_n, u_p - u_q \rangle \mid \\ &= \ 4 \sum_{n=n_0+1}^{\infty} \alpha_n \mid \langle z_n, u_p - u_q \rangle \mid \leq \ 4 \sum_{n=n_0+1}^{\infty} \alpha_n (\mid \langle z_n, u_p \rangle \mid \ + \ \mid \langle z_n, u_q \rangle \mid) \\ &\leq \ 8 \sum_{n=n_0+1}^{\infty} \frac{1}{2^n} \ = \ \frac{8}{2^{n_0}} \ < \ \varepsilon. \end{aligned}$$

Consequently, (u_n) is a Cauchy sequence in Y'_{A°. If this were a Banach space, this sequence would converge to a certain element u of Y'_{A°. Clearly, u should coincide with v, which is a contradiction. $2 \Rightarrow 3$. Assuming that 3 does not hold, after result *a*), there is a closed subspace Z of $E'[\mu(E', E)]$ such that $E'[\mu(E', E)]/Z$ is not complete. Let Y represent the subspace of E orthogonal to Z. Let w be a linear functional on Y which belongs to the completion of $E'[\mu(E', E)]/Z$ but does not belong to $E'[\mu(E', E)]/Z$. From the theorem of Ptàk-Collins, [4, p. 271], $w^{-1}(0)$ intersects every weakly compact absolutely convex subset of Y in a closed subset, hence w is bounded in every bounded subset of Y. Since w is not continuous in Y, we deduce from above that $Y[\mu(Y, Y')]$ is not an (LB)-space. $3 \Rightarrow 1$. After result *a*), every separated quotient of $E'[\mu(E', E)]$ is complete and thus it is locally complete. QED

In the previous theorem, we have considered closed subspaces Y of $E = \bigoplus_{n=1}^{\infty} X_n$ endowed with the Mackey topology $\mu(Y, Y')$. It may happen that for some closed subspace Y of E, Y is not an (LB)-space and nevertheless $Y[\mu(Y, Y')]$ is indeed an (LB)-space. In Theorem 2, this property is considered when X_n is a reflexive Banach space, $n \in \mathbb{N}$.

We shall then use the following result that we obtained in [8]: *b) Let F be a Fréchet space such that for each closed subspace G of F and each bounded subset B of F/G there is a bounded subset A of F for which $\varphi(A) = B$, where φ is the canonical projection from F onto F/G, then one of the following assertions holds: 1. F is a Banach space. 2. F is a Schwartz space. 3. F is the product of a Banach space by ω.*

Theorem 2. *Let (X_n) be a sequence of reflexive Banach spaces of infinite dimension. Then, there is a closed subspace Y of $E := \bigoplus_{n=1}^{\infty} X_n$ whose topology is not that of Mackey's $\mu(Y, Y')$.*

PROOF. By applying result *b*) we obtain a closed subspace G of $F := \prod_{n=1}^{\infty} X_n^*$ and a closed bounded absolutely convex subset B of F/G such that there is no bounded subset A of F with $\varphi(A) = B$, where φ is the canonical projection from F onto F/G. Clearly, F/G is reflexive and so B is weakly compact. We have that $F'[\mu(F',F)] = E$. We identify, in the usual manner, $(F/G)'$ with the subspace Y of E orthogonal to G. If B° is the polar set of B in Y, then B° is a zero-neighborhood in $Y[\mu(Y,Y')]$. Now, given that B is not the image by φ of any bounded subset of F, there is no zero-neighborhood U in E for which $U \cap Y \subset B^\circ$. Therefore, the subspace Y of E does not have the Mackey topology. $Y[\mu(Y,Y')]$ is an (LB)-space in light of our former theorem. QED

3 The Krein-$\breve{S}$mulian Property

Theorem 3. *Let (X_n) be a sequence of Banach spaces of infinite dimension. If E is $\bigoplus_{n=1}^{\infty} X_n$, then $E'[\mu(E',E)]$ has the Krein-Šmulian property if and only if X_n is reflexive, $n \in \mathbb{N}$.*

Before giving the proof of this theorem, we shall obtain some previous results. For the next four propositions, we shall consider the sequence (Z_n) of infinite-dimensional separable Banach spaces such that Z_1 is quasi-reflexive non-reflexive and Z_n is reflexive, for $n = 2,3,\ldots$ we put $F := \bigoplus_{n=1}^{\infty} Z_n$ and $F_n := \bigoplus_{j=1}^{n} Z_j$, and identify, in the usual fashion, F_n with a subspace of F and $\tilde{F}_n$ with F_n'', $n \in \mathbb{N}$. We take a vector y in $Z_1^{**} \setminus Z_1$. We fix now j in $\mathbb{N}$. In $\tilde{F}_{j+1}\ [\sigma(\tilde{F}_{j+1}, F'_{j+1})]$, $\tilde{F}_j + B(Z_{j+1})$ is a closed subset whose intersection with Z_{j+1} coincides with $B(Z_{j+1})$ and, since $B(Z_{j+1})$ is not a weak neighborhood of zero in Z_{j+1}, we have that $\tilde{F}_j + B(Z_{j+1})$ has no interior points. On the other hand,

$$\frac{1}{j}y \ \in \ \tilde{F}_1 \ \subset \ \tilde{F}_j \ + \ B(Z_{j+1})$$

and $\tilde{F}_{j+1}\ [\beta(\tilde{F}_{j+1}, F'_{j+1})]$ is separable, so there is a sequence (z_n) in

$$F_{j+1} \setminus (\tilde{F}_j + \ B(Z_{j+1}))$$

which converges to $\frac{1}{j}y$ in $\tilde{F}_{j+1}[\sigma(\tilde{F}_{j+1}, F'_{j+1})]$. We may now find a subsequence (z_{jn}) of (z_n) which is basic in F_{j+1}, [5]. Let T_{j+1} be the projection from F_{j+1} onto Z_{j+1} along F_j. Then, $T_{j+1}z_{jn} \notin B(Z_{j+1})$, $n \in \mathbb{N}$, and the sequence $(T_{j+1}z_{jn})$ converges weakly to the origin in Z_{n+1}. Consequently, we may find in (z_{jn}) a subsequence (y_{jn}) such that $(T_{j+1}y_{jn})$ is basic in Z_{j+1}, [3, p. 334]. In $\tilde{F}_{j+1}[\sigma(\tilde{F}_{j+1}, F'_{j+1})]$, we put A_j for the closed convex hull of $\{y_{jn} : n \in \mathbb{N}\}$. We have that $\{y_{jn} : n \in \mathbb{N}\} \cup \{\frac{1}{j}y\}$ is compact and hence A_j is also compact. We choose in F'_{j+1} a sequence (u_{jn}) such that

$$\langle y_{jn}, u_{jn}\rangle \ = \ 1, \quad \langle y_{jm}, u_{jn}\rangle = 0, \quad m \neq n, \ m,n \in \mathbb{N}.$$

Proposition 1. *An element z of $\tilde{F}_{j+1}$ is in A_j if and only if it can be represented as*

$$z \ = \ \sum_{n=1}^{\infty} a_n y_{jn} \ + \ \frac{1}{j}ay, \quad a \geq 0, a_n \geq 0, n \in \mathbb{N}, \sum_{n=1}^{\infty} a_n \ + a = 1,$$

where the coefficients a and a_n, $n \in \mathbb{N}$, are univocally determined by z.

Proof. Clearly, if an element z of $\tilde{F}_{j+1}$ has the representation above given, then it belongs to A_j.

An arbitrary element of the convex hull M_j of $\{y_{jn} : n \in \mathbb{N}\} \cup \{\frac{1}{j}y\}$ has the form

$$\sum_{n=1}^{\infty} a_n y_{jn} \ + \ \frac{1}{j}\ ay, \quad a \geq 0, a_n \geq 0, n \in \mathbb{N}, \ \sum_{n=1}^{\infty} a_n + a \ = 1,$$

where the terms of the sequence (a_n) are all zero except for a finite number of them. Given z in A_j, we find a net

$$\{\sum_{n=1}^{\infty} a_n^{(i)} y_{jn} + \frac{1}{j} a^{(i)} y \ : \ i \in I, \ \succeq\}$$

in M_j such that it $\sigma(\tilde{F}_{j+1}, F'_{j+1})$-converges to z. Given r in $\mathbb{N}$, we have that

$$\langle \sum_{n=1}^{\infty} a_n^{(i)} y_{jn} + \frac{1}{j} a^{(i)} y, u_r \rangle \ = \ a_r^{(i)} + \frac{1}{j} a^{(i)} \langle y, u_r \rangle$$

$$= \ a_r^{(i)} + a^{(i)} \lim_n \langle y_{jn}, u_r \rangle = \ a_r^{(i)},$$

thus, in $\mathbb{R}$,

$$\lim_i a_r^{(i)} \ = \ \langle z, u_r \rangle \ =: \ a_r.$$

Clearly, $\sum_{r=1}^{\infty} a_r \leq 1$. Let $a := 1 - \sum_{r=1}^{\infty} a_r$. We consider the vector

$$\sum_{n=1}^{\infty} a_n y_{jn} + \frac{1}{j} a y$$

of $\tilde{F}_{j+1}[\sigma(\tilde{F}_{j+1}, F'_{j+1})]$ and we proceed to show that it coincides with z. Given u in F'_{j+1}, having in mind that $\{y_{jn} : n \in \mathbb{N}\}$ is bounded in F_{j+1}, we find $\lambda_j > 0$ such that

$$|\ \langle y_{jn}, u \rangle\ | < \ \lambda_j, \quad n \in \mathbb{N}, \quad |\ \langle y, u \rangle\ | < \ \lambda_j.$$

Given $\varepsilon > 0$, we find $s \in \mathbb{N}$ such that

$$|\ \langle y_{jn} - \frac{1}{j} y, u \rangle\ | < \frac{\varepsilon}{6}, \quad n \geq s.$$

We now determine i_0 in I such that, for $i \succeq i_0$,

$$|\ a_n - a_n^{(i)}\ | < \ \frac{\varepsilon}{6\lambda_j s}, \quad n = 1, 2, \ldots, s,$$

$$|\ \langle z - (\sum_{n=1}^{\infty} a_n^{(i)} y_{jn} + \frac{1}{j} a^{(i)} y), u \rangle\ | < \ \frac{\varepsilon}{3}.$$

Then, for such values of i,

$$\begin{aligned}
&|\langle z - (\sum_{n=1}^{\infty} a_n y_{jn} + \frac{1}{j} a y), u \rangle\ | \leq |\ \langle z - (\sum_{n=1}^{\infty} a_n^{(i)} y_{jn} + \frac{1}{j} a^{(i)} y), u \rangle\ | \\
&+ \ |\ \langle \sum_{n=1}^{\infty} a_n^{(i)} y_{jn} + \frac{1}{j} a^{(i)} y - (\sum_{n=1}^{\infty} a_n y_{jn} + \frac{1}{j} a y), u \rangle\ | \ < \ \frac{\varepsilon}{3} \\
&+ \ |\ \langle \sum_{n=1}^{\infty} (a_n^{(i)} - a_n) y_{jn} + \frac{1}{j} (a^{(i)} - a) y, u \rangle\ | = \ \frac{\varepsilon}{3} \\
&+ \ |\ \langle \sum_{n=1}^{\infty} (a_n^{(i)} - a_n) y_{jn} + \frac{1}{j} (1 - \sum_{n=1}^{\infty} a_n^{(i)} - (1 - \sum_{n=1}^{\infty} a_n)) y, u \rangle\ | \\
&= \ \frac{\varepsilon}{3} + \ |\ \langle \sum_{n=1}^{\infty} (a_n^{(i)} - a_n) y_{jn} + \frac{1}{j} \sum_{n=1}^{\infty} (a_n - a_n^{(i)}) y, u \rangle\ |
\end{aligned}$$

$$\leq \frac{\varepsilon}{3} + |\langle\sum_{n=1}^{s}(a_n^{(i)} - a_n)y_{jn} + \frac{1}{j}\sum_{n=1}^{s}(a_n - a_n^{(i)})y, u\rangle|$$
$$+ |\langle \sum_{n=s+1}^{\infty}(a_n^{(i)} - a_n)y_{jn} + \frac{1}{j}\sum_{n=s+1}^{\infty}(a_n - a_n^{(i)})y, u\rangle|$$
$$\leq \frac{\varepsilon}{3} + 2\lambda_j\sum_{n=1}^{s}|a_n - a_n^{(i)}| + \sum_{n=s+1}^{\infty}|a_n - a_n^{(i)}|\cdot|\langle y_{jn} - \frac{1}{j}y, u\rangle|$$
$$\leq \frac{\varepsilon}{3} + 2\lambda_j s\frac{\varepsilon}{6\lambda_j s} + 2\frac{\varepsilon}{6} = \varepsilon,$$

from where we deduce that, in $\tilde{F}_{j+1}[\sigma(\tilde{F}_{j+1}, F'_{j+1})]$,

$$z = \sum_{n=1}^{\infty} a_n y_{jn} + \frac{1}{j}ay.$$

Besides, it is plain that

$$a_n = \langle z, u_n\rangle, \quad n\in\mathbb{N}, \quad a = 1 - \sum_{n=1}^{\infty}\langle z, u_n\rangle.$$

QED

Corollary 1. *We have that*

$$A_j \cap F_{j+1} = \{\sum_{n=1}^{\infty} a_n y_{jn} : \ a_n \geq 0, n\in\mathbb{N}, \sum_{n=1}^{\infty} a_n = 1\}.$$

Corollary 2. *If $z \in A_j$, then z may be univocally expressed as*

$$z = bz_1 + \frac{1}{j}cy, \quad z_1 \in A_j \cap F_{j+1}, \quad b \geq 0, \quad c \geq 0, \quad b + c = 1.$$

In the sequel, we put D for the convex hull of

$$\cup\{A_j \cap F_{j+1} : \ j \in \mathbb{N}\}$$

and D_r for the convex hull of

$$\cup\{A_j \cap F_{j+1} : \ j = 1, 2, \ldots, r\}, \quad r \in \mathbb{N}.$$

Proposition 2. *For each $r \in \mathbb{N}$, we have that*

$$D_r = D \cap F_{r+1}.$$

Proof. Given a positive integer s, we take an element z of D_{s+1}. Then, z may be written in the form

$$z = \sum_{j=1}^{s+1}\alpha_j z_j, \quad z_j \in A_j \cap F_{j+1}, \quad \alpha_j \geq 0, \ j = 1, 2, \ldots, s+1, \sum_{j=1}^{s+1}\alpha_j = 1.$$

Let us first assume that $\alpha_{s+1} \neq 0$. After Corollary 1, z_{j+1} can be written as

$$\sum_{n=1}^{\infty} a_n y_{(s+1)n}, a_n \geq 0, n\in\mathbb{N}, \sum_{n=1}^{\infty} a_n = 1.$$

We have that $(T_{s+2}(y_{(s+1)n}))$ is a basic sequence in Z_{s+2} and thus the vector of Z_{s+2}

$$T_{s+2}(z_{s+1}) \;=\; T_{s+2}(\sum_{n=1}^{\infty} a_n y_{(s+1)n}) \;=\; \sum_{n=1}^{\infty} a_n T_{s+2}(y_{(s+1)n})$$

is non-zero. Then

$$\alpha_{s+1} z_{s+1} \;\notin F_{s+1}$$

and since

$$\sum_{j=1}^{s} \alpha_j z_j \;\in\; F_{s+1},$$

it follows that

$$z \;=\; \sum_{j=1}^{s+1} \alpha_j z_j \;\notin\; F_{s+1}.$$

On the other hand, if $\alpha_{s+1} = 0$, we have that z belongs to D_s.

We deduce from above that

$$D_{s+1} \cap F_{s+1} \;\subset\; D_s$$

and, since D_s is clearly contained in $D_{s+1} \cap F_{s+1}$, it follows that

$$D_s \;=\; D_{s+1} \cap F_{s+1}.$$

Finally, given $r \in \mathbb{N}$, we have that

$$D_r \;=\; D_{r+1} \cap F_{r+1} \;=\; D_{r+2} \cap F_{r+2} \cap F_{r+1} \;=\; D_{r+2} \cap F_{r+1}$$

and, proceeding recurrently, we have that, for each $m \in \mathbb{N}$,

$$D_r \;=\; D_{r+m} \cap F_{r+1},$$

from where we conclude that

$$D_r \;=\; (\cup_{m=1}^{\infty} D_{r+m}) \cap F_{r+1} \;=\; D \cap F_{r+1}$$

QED

Proposition 3. *For each $r \in \mathbb{N}$, D_r is closed in F_{r+1}.*

PROOF. We write C_r for the convex hull of $\cup\{A_j \;:\; j = 1, 2, \ldots, r\}$. Clearly, C_r is $\sigma(\tilde{F}_{r+1}, F'_{r+1})$-compact and so it suffices to show that D_r coincides with $C_r \cap F_{r+1}$. We take z in C_r. After Corollary 2, z may be written in the form

$$\sum_{j=1}^{r} \alpha_j (a_j z_j + \frac{1}{j} b_j y_j), \quad a_j \geq 0,\; b_j \geq 0,\; \alpha_j \geq 0,$$

$$a_j + b_j = 1, \quad z_j \in A_j \cap F_{j+1}, \quad j = 1, 2, \ldots, r, \; \sum_{j=1}^{r} \alpha_j = 1.$$

If z belongs to F_{r+1}, then $\sum_{j=1}^{r} \frac{1}{j} \alpha_j b_j = 0$ and thus $\alpha_j b_j = 0$, $j = 1, 2, \ldots, r$. Then

$$z \;=\; \sum_{j=1}^{r} \alpha_j a_j z_j \;=\; \sum_{j=1}^{r} \alpha_j (1 - b_j) z_j \;=\; \sum_{j=1}^{r} \alpha_j z_j,$$

from where we deduce that z is in D_r. Therefore

$$C_r \;\cap\; F_{r+1} \;\subset\; D_r.$$

On the other hand, it is immediate that D_r is contained in $C_r \cap F_{r+1}$ and the result follows.

QED

Proposition 4. *In F, each weakly compact absolutely convex subset intersects D in a closed set. Besides, D is not closed in F.*

PROOF. Let M be a weakly compact absolutely convex subset of F. Then there is $r \in \mathbb{N}$ such that M is contained in F_{r+1}. Then

$$M \cap D = M \cap F_{r+1} \cap D = M \cap D_r$$

and, after the previous proposition, we have that $M \cap D_r$ is closed in F_{r+1}, from where we get that $M \cap D$ is closed in F. On the other hand, the origin of F is not in D. We consider a weak neighborhood U of the origin in F. We find an open neighborhood V of the origin in F'' $[\sigma(F'', F')]$ such that $V \cap F \subset U$. We find $s \in \mathbb{N}$ so that $\frac{1}{s}y \in V$. Now, since V is a neighborhood of $\frac{1}{s}y$ in F'' $[\sigma(F'', F')]$ and (y_{sn}) converges in this space to $\frac{1}{s}y$, there is $m \in \mathbb{N}$ for which $y_{sm} \in V$. Consequently, $U \cap D \neq \emptyset$, thus the weak closure of D in E contains the origin and hence D is not closed in F. QED

Finally, we give the proof of Theorem 3, but for that we shall need the following result to be found in [9]: *c) Let X be an infinite-dimensional Banach space such that X^{**} is separable. Let T be a closed subspace of X^{**} containing X. Then there is an infinite-dimensional closed subspace Y of X such that $X + \tilde{Y} = T$.*

PROOF. If X_n is reflexive, $n \in \mathbb{N}$, then E is the Mackey dual of the space E' $[\mu(E', E)]$ and so this space has the Krein-$\check{S}$mulian Property. If some of the spaces X_n, $n \in \mathbb{N}$, is not quasi-reflexive, then we apply result a) to obtain that E' $[\mu(E', E)]$ is not B-complete and so it does not have the Krein-$\check{S}$mulian Property. It remains to consider the case in which all the spaces X_n, $n \in \mathbb{N}$, are quasi-reflexive and there is at least one of them which is not reflexive. More precisely, let us assume that X_1 is not reflexive. From Eberlein's theorem, $B(X_1)$ is not weakly countably compact and so there is a sequence (x_n) in $B(X_1)$ with no weak cluster points in X_1. Let Z_1 be the closed linear span in X_1 of $\{x_n : n \in \mathbb{N}\}$. Then, Z_1 is a separable Banach space which is quasi-reflexive but not reflexive. For each $n \in \mathbb{N}$, $n > 1$, we find in X_n a separable closed subspace Y_n of infinite dimension. Since Y_n is quasi-reflexive, it follows that Y_n^{**} is separable, from where, applying result c) for the case $T = X = Y_n$, we have that there is a separable closed subspace Z_n of Y_n, with infinite dimension, such that $Y_n + \tilde{Z}_n = Y_n$, that is, $\tilde{Z}_n \subset Y_n$ and so Z_n is reflexive. We have that $F := \bigoplus_{n=1}^{\infty} Y_n$ is a closed subspace of $E = \bigoplus_{n=1}^{\infty} X_n$. On the other hand, after Proposition 4, there is a convex subset D of F, not closed, which meets each weakly compact absolutely convex subset of F in a closed set. Then D is a convex non-closed subset of E that meets each weakly compact absolutely convex subset of E in a closed subset of E. Consequently, E' $[\mu(E', E)]$ does not have the Krein-$\check{S}$mulian Property. QED

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