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# Bohr's radii and strips – a microscopic and a macroscopic view

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**Abstract.** The Bohr-Bohnenblust-Hille theorem states that the largest possible width $S$ of the strip in the complex plane on which a Dirichlet series $\sum_n a_n 1/n^s$ converges uniformly but not absolutely, equals 1/2. In fact Bohr in 1913 proved that $S \leq 1/2$, and asked for equality. The general theory of Dirichlet series during this time was one of the most fashionable topics in analysis, and Bohr's so-called *absolute convergence problem* was very much in the focus. In this context Bohr himself discovered several deep connections of Dirichlet series and power series (holomorphic functions) in infinitely many variables, and as a sort of by-product he found his famous power series theorem. Finally, Bohnenblust and Hille in 1931 in a rather ingenious fashion answered the absolute convergence problem in the positive. In recent years many authors revisited the work of Bohr, Bohnenblust and Hille – improving this work but also extending it to more general settings, for example to Dirichlet series with coefficients in Banach spaces. The aim of this article is to report on parts of this new development.

**Keywords:** Dirichlet series, Bohr's strips, Bohr radii, inf. dim. holomorphy, Bohnenblust-Hille inequalities, Banach spaces

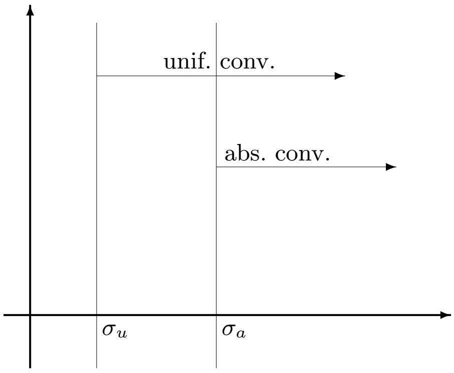
**MSC 2000 classification:** 30B50, 32A05

*Dedicated to the memory of V.B. Moscatelli*

## 1 Introduction

An ordinary Dirichlet series is a series of the form $\sum_n a_n \frac{1}{n^s}$ where $a_n$ are complex coefficients and $s$ is a complex variable. Maximal domains, where such Dirichlet series converge conditionally, uniformly or absolutely are half planes $[\mathrm{Re} > \sigma]$, where $\sigma = \sigma_c, \sigma_u$ or $\sigma_a$ are called the abscissa of conditional, uniform or absolute convergence, respectively. More precisely, $\sigma$ is the infimum of all $r \in \mathbb{R}$ such that on $[\mathrm{Re} > r]$ we have convergence of requested type.

Harald Bohr's so called *absolute convergence problem* from [6] asked for the largest possible width of the strip in $\mathbb{C}$ on which a Dirichlet series converges uniformly but not absolutely:




In other terms, Bohr asked for the precise value of the number

$$S := \sup(\sigma_a - \sigma_u),$$

the supremum taken over all possible Dirichlet series $\sum_n a_n \frac{1}{n^s}$. In [6, Satz X] he himself managed to show that

$$S \leq \frac{1}{2}.$$

Today this estimate easily follows from a Parseval type equality for Dirichlet series but here we prefer to have a look at Bohr's original method. The crucial idea was to establish the following one-to-one correspondence between Dirichlet series and (formal) power series in infinitely many variables:

$$\sum_n a_n \frac{1}{n^s} \quad \leftrightsquigarrow \quad \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha, \text{ where } a_n = c_\alpha \text{ if } n = p^\alpha \, ;$$

here $p = (p_n)$ stands for the sequence $p_1 < p_2 < \ldots$ of all prime numbers, and for each multi index $\alpha = (\alpha_1, \ldots, \alpha_k, 0, \ldots) \in \mathbb{N}_0^{(\mathbb{N})}$ we as usual write $p^\alpha = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$.

Let us describe the meaning of the absolute convergence problem in terms of power series in infinitely many variables – but instead of power series (a quite mysterious object at Bohr's time) we prefer to use the modern language of infinite dimensional holomorphy. It is well-known that every holomorphic function $f : \mathbb{D}^N \to \mathbb{C}$ in $N$ complex variables has a monomial series expansion; more precisely, for every $f \in H(\mathbb{D}^N)$ we have $f(z) = \sum_{\alpha \in \mathbb{N}_0^N} \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha$ for all $z \in \mathbb{D}^N$. From this fact we easily deduce that for every $\mathbb{C}$-valued holomorphic function $f$ defined on the open unit ball $B_{\ell_\infty}$ of $\ell_\infty$ (by definition a Fréchet differentiable function $f : B_{\ell_\infty} \to \mathbb{C}$) there is a unique family $(c_\alpha)_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}}$ of scalars such that for all finite sequences in $z \in B_{\ell_\infty}$ we have

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha \, ;$$

this power series is called the monomial series expansion of $f$, and for every $N$ and multi index $\alpha \in \mathbb{N}_0^N$ we have

$$c_\alpha = \frac{\partial^\alpha f(0)}{\alpha!} \, .$$

For which other than finite sequences in $B_{\ell_\infty}$ does this series converge (absolutely as a net) for every holomorphic $f$ on $B_{\ell_\infty}$? This independently interesting question is closely connected with Bohr's absolute convergence problem. We call

$$\operatorname{dom} H(B_{\ell_\infty}) = \bigcap_{f \in H(B_{\ell_\infty})} \{ z \in B_{\ell_\infty} \mid \sum_\alpha |c_\alpha(f) z^\alpha| < \infty \}$$

the domain of convergence of $H(B_{\ell_\infty})$, and define

$$K := \sup\{ 1 \le p \le \infty \mid \ell_p \cap B_{\ell_\infty} \subset \operatorname{dom} H(B_{\ell_\infty}) \} .$$

**Theorem 1.**

$$S = \frac{1}{K}$$

This result is due to Bohr [6, Satz IX] who formulated it in terms of power series and used the prime number theorem for its proof. Bohr was able to establish that $K \ge 2$, hence $S \le 1/2$ . His problem then was to find the exact value of $K$. He didn't even know if $K < \infty$, or in other words, he had no example of a Dirichlet series for which the abscissas $\sigma_u$ and $\sigma_a$ not coincide. In [6, p. 446] he says:

> *"Um dies Problem zu erledigen, ist ein tieferes Eindringen in die Theorie der Potenzreihen unendlich vieler Variabeln nötig, als es mir in §3 gelungen ist."*

(*"In order to solve this problem, a deeper understanding of the theory of power series in infinitely many variables is needed, more than I could do in §3."*) See [17] for an intensive study of domains of monomial convergence for holomorphic functions defined on arbitrary Banach sequence spaces, including the sequence space $\ell_\infty$. Much later, in 1931, Bohnenblust and Hille solved Bohr's absolute convergence theorem by giving an example of a Dirichlet series for which the difference $\sigma_a - \sigma_u$ equals $\frac{1}{2}$. More precisely, they created in [5, Theorem VII] for any given $0 \le \sigma \le \frac{1}{2}$ a Dirichlet series such that $\sigma_a - \sigma_u = \sigma$. This gives us what we now call the Bohr-Bohnenblust-Hille theorem.

**Theorem 2.** *The maximal width of the strip of uniform but not absolute convergence for Dirichlet series is*

$$S = \frac{1}{2}.$$

Since Bohr in 1914 didn't find any reasonable way to determine the precise value of $K$, a problem for power series (holomorphic functions) in infinitely many variables, he returned to one dimension and got as a by-product of his effort what is nowadays called Bohr's power series theorem (see [8, Section 3]):

**Theorem 3.** *For each* $f \in H(\mathbb{D})$

$$\sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right| \frac{1}{3^n} \le \|f\|_\infty ,$$

*and the value* $1/3$ *is optimal.*

In recent years quite a number of mathematicians revisited the work of Bohr, Bohnenblust and Hille – improving this work but also extending it to more general settings, for example to Dirichlet series with coefficients in Banach spaces. This article collects several interesting new results in this direction, and it in particular surveys on joint work with several of our coauthors: Frerick, García, Maestre, Ortega-Cerdà, Ounaïes, Pérez-García, Popa, Seip, Sevilla-Peris. We in a sense quantify the Bohr-Bohnenblust-Hille cycle of ideas. For Dirichlet series with complex coefficients we obtain a sort of *microscopic* view of the subject, and studying similar results for vector-valued Dirichlet series we obtain a more *macroscopic* view. The tools needed include techniques from analytic number theory, probability theory, complex analysis in one and several variables, the theory of infinite dimensional holomorphy, harmonic analysis, and local Banach space theory.

# 2 The hypercontractivity of the Bohnenblust-Hille inequality

In order to solve Bohr's absolute convergence problem, Bohnenblust and Hille in [5, Theorem I] established their famous $\frac{2m}{m+1}$-inequality – an inequality of high independent interest.

**Theorem 4** (Bohnenblust-Hille-Inequality)**.** *For every $m$-linear mapping $A : \ell_\infty^N \times \cdots \times \ell_\infty^N \longrightarrow \mathbb{C}$*

$$\left( \sum_{i_1,\ldots,i_m=1}^{N} |\, A(e_{i_1}, \ldots, e_{i_m})\,|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq m^{\frac{m+1}{2m}} \left(\sqrt{2}\right)^{m-1} \|A\|,$$

*further the exponent $\frac{2m}{m+1}$ is optimal.*

For $m = 2$ this is Littlewood's famous $\frac{4}{3}$-inequality from [27]. This inequality was long forgotten and rediscovered more than forty years later by A. Davie [11, Section 2] and S. Kaijser [25, Lemma (1.1)]. The proofs in [11] and [25] are slightly different from the original one and give, if $C_m$ stands for the best constant in the Bohnenblust-Hille inequality, the better estimate

$$C_m \leq \left(\sqrt{2}\right)^{m-1} . \tag{1}$$

Bohnenblust and Hille in fact needed a polynomial version of their inequality. By polarization they prove that for each $m$ there is a constant $D_m$ such that for every $m$-homogenous polynomial $\sum_{|\alpha|=m} c_\alpha z^\alpha$ on $\mathbb{C}^N$

$$\left( \sum_{|\alpha|=m} |c_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq D_m \sup_{z\in\mathbb{D}^N} \left| \sum_{|\alpha|=m} c_\alpha z^\alpha \right| .$$

Moreover, they show with a sophisticated argument that the exponent $\frac{2m}{m+1}$ is optimal as well.

Let us assume that $D_m$ is the optimal constant in the polynomial version of the Bohnenblust-Hille inequality. Then it is easy to deduce from (1) and an estimate of Harris [24, Theorem 1] for the polarization constant of $\ell_\infty$ that

$$D_m \leq (\sqrt{2})^{m-1} \frac{m^{\frac{m}{2}} (m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}} ;$$

see e. g. [21, Section 4]. Using Sawa's Khinchine-type inequality for Steinhaus variables, Queffélec [29, Theorem III-1] obtained the slightly better estimate

$$D_m \leq \left( \frac{2}{\sqrt{\pi}} \right)^{m-1} \frac{m^{\frac{m}{2}} (m+1)^{\frac{m+1}{2}}}{2^m (m!)^{\frac{m+1}{2m}}} .$$

A crucial point for everything following will be a substantial improvement of Defant, Frerick, Ortega-Cerdà, Ounaïes and Seip [14, Theorem 1] for the constant $D_m$. They managed to show that the polynomial Bohnenblust-Hille inequality is hypercontractive:

**Lemma 1.** *There is a constant $C \geq 1$ such that for every $m$ and every $m$-homogeneous polynomial $\sum_{|\alpha|=m} c_\alpha z^\alpha$ on $\mathbb{C}^N$ we have*

$$\left( \sum_{|\alpha|=m} |c_\alpha|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq C^m \sup_{z\in\mathbb{D}^N} \left| \sum_{|\alpha|=m} c_\alpha z^\alpha \right| .$$

In [18] Defant, Maestre, and Schwarting show a far reaching vector-valued extension of this result which we now explain. Recall first that a Banach lattice $Y$ is called $q$-concave, $1 \leq q < \infty$, if there is a constant $C$ such that for every choice of finitely many $y_1, \ldots, y_N \in Y$ we have

$$\left( \sum_{n=1}^{N} \|y_n\|^q \right)^{\frac{1}{q}} \leq C \left\| \left( \sum_{n=1}^{N} |y_n|^q \right)^{\frac{1}{q}} \right\| .$$

The concept of concavity is closely related to the notion of cotype (see section 5). A $q$-concave Banach lattice $X$ with $q \geq 2$ is of cotype $q$. Conversely, each Banach lattice of cotype 2 is 2-concave and a Banach lattice of cotype $q > 2$ is $r$-concave for all $r > q$. Moreover, a (bounded linear) operator $v : X \to Y$ between two Banach spaces is said to be $(r,1)$-summing, $1 \leq r < \infty$, if there is a constant $C$ such that for any $N$ and any choice of $N$ many vectors $x_1, \ldots, x_N$ in $X$ we have

$$\left( \sum_{n=1}^{N} \|vx_n\|_Y^r \right)^{\frac{1}{r}} \leq C \sup_{z\in\mathbb{D}^N} \left\| \sum_{n=1}^{N} x_n z_n \right\|_X ,$$

or equivalently: there is a constant $C$ such that for any $N$ and any choice of $N$ many vectors $x_\alpha \in X$, $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ with $|\alpha| = 1$

$$\left( \sum_{|\alpha|=1} \|vx_\alpha\|_Y^r \right)^{\frac{1}{r}} \leq C \sup_{z\in\mathbb{D}^N} \left\| \sum_{|\alpha|=1} x_\alpha z^\alpha \right\|_X .$$

The following lemma from [18] extends the above cited hypercontractive polynomial Bohnenblust-Hille inequality (Lemma 1), and shows under which additional assumptions on the underlying spaces such inequalities might be extended to sets of multi-indices $\alpha$ of order $m$ instead of order 1 (replacing the exponent $r$ by a larger one $\rho$).

**Lemma 2.** *Let $Y$ be a $q$-concave Banach lattice, with $2 \leq q < \infty$, and $v : X \to Y$ an $(r,1)$-summing operator with $1 \leq r \leq q$. Define for $m$ the exponent*

$$\rho := \frac{qrm}{q + (m-1)r}.$$

*Then there is a constant $C > 0$ such that for each $N$ and for any choice of $\binom{N+m-1}{m}$ many vectors $x_\alpha \in X$, $\alpha \in \mathbb{N}_0^{(\mathbb{N})}$ with $|\alpha| = m$ we have*

$$\left( \sum_{|\alpha|=m} \|vx_\alpha\|_Y^\rho \right)^{\frac{1}{\rho}} \leq C^m \sup_{z\in\mathbb{D}^N} \left\| \sum_{|\alpha|=m} x_\alpha z^\alpha \right\|_X .$$

Obviously the identity id on $\mathbb{C}$ is $(1,1)$-summing = 1-summing and $\mathbb{C}$ is 2-concave. Hence in the scalar case we have $\rho = \frac{2m}{m+1}$. This shows that Lemma 1 is a special case of Lemma 2.

# 3 Multi dimensional Bohr radii – a microscopic view

In [8, p. 2] Bohr explains the original motivation for his so called power series theorem (Theorem 3) as follows:

> "[...] *the solution of what is called the 'absolute convergence problem' for Dirichlet's series of the type $\sum a_n 1/n^s$ must be based upon a study of the relations between the absolute value of a power-series in an infinite number of variables on the one hand, and the sum of the absolute values of the individual terms on the other. It was in the course of this investigation that I was led to consider a problem concerning power-series in one variable only, which we discussed last year, and which seems to be of some interest in itself.*"

The following definition of the $N$-dimensional Bohr radius is due to Boas and Khavinson [4]. The $N$th Bohr radius $K_N$ is the supremum taken over all $0 \le r \le 1$ such that for each holomorphic function $f \in H(\mathbb{D}^N)$ we have

$$\sup_{z \in r\mathbb{D}^N} \sum_\alpha \left| \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha \right| \le \|f\|\infty .$$

Note, that with this notation Bohr's power series theorem (Theorem 3) reads

$$K_1 = \frac{1}{3} .$$

In [4, Theorem 2] Boas and Khavinson established that for $N > 1$

$$\frac{1}{3} \frac{1}{\sqrt{N}} \le K_N \le 2\sqrt{\frac{\log N}{N}}$$

(see [23, Theorem 3.2] of Dineen and Timoney for an earlier weaker version initiating the previous one), and in [3, p. 239] Boas then conjectured that *"[...] presumably this logarithmic factor, an artifact of the proof, should not really be present"*. This conjecture was disproved in [13, Theorem 1.1]:

$$\sqrt{\frac{\log N}{N \log\log N}} \prec K_N ;$$

here the notation $a_N \prec b_N$ means that there is a universal constant $C > 0$ such that $a_N \le C b_N$ for every $N$ and $a_N \asymp b_N$ means $a_N \prec b_N$ and $b_N \prec a_N$.

Finally, using the hypercontractive Bohnenblust-Hille inequality (Lemma 1), Defant, Frerick, Ortega-Cerdà, Ounaïes, and Seip in [14, Theorem 2] proved the following optimal asymptotic of multidimensional Bohr radii.

**Theorem 5.**

$$K_N \asymp \sqrt{\frac{\log N}{N}}$$

Let us sketch the proof of the lower bound: Let $K_N^m$ be the $N$th Bohr radius for $m$-homogeneous polynomials (instead of all holomorphic functions), i.e. the supremum of all $0 \le r \le 1$ such that for each $m$-homogeneous polynomial $P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha$ on $\mathbb{C}^N$ we have

$$\sup_{z \in r\mathbb{D}^N} \sum_{|\alpha|=m} |c_\alpha z^\alpha| \le |P\|_\infty ,$$

or equivalently,

$$\sum_{|\alpha|=m} |c_\alpha| \le \frac{1}{r^m} \|P\|_\infty .$$

Using Caratheodory's inequality the following estimate from [15, Corollary 2.3] can be proved:

$$\frac{1}{3}\inf_m K_N^m \le K_N\,.$$

Then by Hölder's inequality and Stirling's formula for each $m$-homogeneous polynomial $P(z) = \sum_{|\alpha|=m} c_\alpha z^\alpha$ on $\mathbb{C}^N$ we have

$$\begin{aligned}\sum_{|\alpha|=m}|c_\alpha| &\le \left(\sum_{|\alpha|=m} 1\right)^{\frac{m-1}{2m}}\left(\sum_{|\alpha|=m}|c_\alpha|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}}\\ &\le C^m\left(1+\frac{N}{m}\right)^{\frac{m-1}{2}}\left(\sum_{|\alpha|=m}|c_\alpha|^{\frac{2m}{m+1}}\right)^{\frac{m+1}{2m}},\end{aligned}$$

and hence the hypercontractivity of the Bohnenblust-Hille inequality (Lemma 1) and optimization over $m$ yield the lower bound in the above theorem.

# 4 Bohr's strips – a microscopic view

Maurizi and Queffélec [28, Theorem 2.4] observed that the maximal width

$$S = \sup_{\sum_n a_n \frac{1}{n^s}} (\sigma_a - \sigma_u),$$

of Bohr's strips equals the infimum over all $\sigma \ge 0$ for which there exists a constant $C \ge 1$ such that for each $N$ and each choice of $a_1, \dots, a_N \in \mathbb{C}$ we have

$$\sum_{n=1}^N |a_n| \le C\,N^\sigma \sup_{t\in\mathbb{R}}\left|\sum_{n=1}^N a_n n^{-it}\right|. \tag{2}$$

This motivates the following definition: Given a natural number $N$, let $Q_N$ be the best constant $D \ge 1$ such that for each choice of $a_1, \dots, a_N \in \mathbb{C}$

$$\sum_{n=1}^N |a_n| \le D \sup_{t\in\mathbb{R}}\left|\sum_{n=1}^N a_n n^{-it}\right|.$$

The following theorem gives the asymptotically optimal upper and lower estimate for $Q_N$, and it marks the endpoint of a long development started by Queffélec [29] in the mid nineties, continued by Queffélec and Konyagin [26, Theorem 4.3] in 2002 and by de la Bretèche [12, Théorème 1.1] in 2008. The final result was proved in [14, Theorem 3] by Defant, Frerick, Ortega-Cerdà, Ounaïes, and Seip; its proof again uses the hypercontractivity of the Bohnenblust-Hille inequality (Lemma 1).

**Theorem 6.**

$$Q_N = \frac{\sqrt{N}}{e^{\left(\frac{1}{\sqrt{2}}+o(1)\right)\sqrt{\log N \log\log N}}}$$

By an important result of Bohr from [7, Satz I] for each Dirichlet series $\sum_n a_n \frac{1}{n^s}$ the abscissa $\sigma_u$ of uniform convergence equals the abscissa of boundedness; the latter is the infimum of those $r$ such that the analytic function represented by the Dirichlet series is bounded on [Re $\ge r$]. When discussing the Bohr-Bohnenblust-Hille theorem (Theorem 2), it is therefore quite natural to introduce the space $\mathcal{H}^\infty$, which consists of those bounded analytic functions $f$ in [Re $> 0$] such that $f$ can be represented by an ordinary Dirichlet series $\sum_{n=1}^\infty a_n n^{-s}$ in some half-plane (and then as a consequence even on [Re $> 0$]).

**Corollary 1.** *The supremum of the set of real numbers $c$ such that for every $f = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}^\infty$ we have*

$$\sum_{n=1}^{\infty} |a_n|\, n^{-\frac{1}{2}} \exp\left\{c\sqrt{\log n \log\log n}\right\} < \infty\,,$$

*equals* $1/\sqrt{2}$.

This result is an improved version of a theorem of Balasubramanian, Calado, and Queffélec [1, Theorem 1.2] from 2006, which says that the upper inequality holds for sufficiently small $c$. The Bohr-Bohnenblust-Hille theorem (Theorem 2) shows that the (unique) Dirichlet series associated with a function $f \in \mathcal{H}^\infty$ converges absolutely on the vertical line $[\mathrm{Re} = 1/2 + \varepsilon]$, and that the number $1/2$ here is optimal. An interesting consequence of the theorem of Balasubramanian, Calado, and Queffélec (just mentioned) is that each such Dirichlet series even converges absolutely on the vertical line $[\mathrm{Re} = 1/2]$. But the preceding corollary gives a lot more; it adds a level precision that enables us to extract much more precise information about the absolute values $|a_n|$ than what is obtained from the solution of the Bohr-Bohnenblust-Hille theorem.

# 5 Bohr's radii – a macroscopic view

The study of Bohr radii for *vector-valued* holomorphic functions in several complex variables can be considered as sort of "macroscopic view" of the subject.

Let $X$ be a (nontrivial) Banach space and $\lambda \geq 1$. Then for $N \in \mathbb{N}$ the $N$th Bohr Radius, denoted by $K_N(X, \lambda)$, is the supremum taken over all $0 \leq r \leq 1$ such that for each $X$-valued holomorphic function $f = \sum_{\alpha \in \mathbb{N}_0^N} \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha$ on $\mathbb{D}^N$ we have

$$\sup_{z \in r\mathbb{D}^N} \sum_{\alpha \in \mathbb{N}_0^N} \left\| \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha \right\|_X \leq \lambda \|f\|_\infty\,. \tag{3}$$

We start with some comments on the case $N = 1$ and $X = \mathbb{C}$ (i.e. we only deal with complex valued functions in one complex variable). Note first that then the left side of the inequality in (3) can be rephrased as follows:

$$\sup_{z \in \frac{1}{3}\mathbb{D}} \sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} z^n \right| = \sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right| \frac{1}{3^n}\,,$$

and hence Bohr's power series theorem (Theorem 3) in our new notation reads

$$K_1(\mathbb{C}, 1) = 1/3\,.$$

For $1/3 < r < 1$ by the Cauchy-Schwarz inequality

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right| r^n &\leq \left( \sum_{n=0}^{\infty} r^{2n} \right)^{1/2} \left( \sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right|^2 \right)^{1/2} \\ &\leq (1 - r^2)^{-1/2} \|f\|_2 \leq (1 - r^2)^{-1/2} \|f\|_\infty\,; \end{aligned}$$

(here we assume without loss of generality that $f$ is defined on $\overline{\mathbb{D}}$, and then $\|f\|_2$ denotes the $L_2$-norm of $f$ with respect to the normalized Lebesgue measure on the Torus).

One now may ask for the best constant $C(r) \geq 1$ such that for all holomorphic functions $f$ on the open unit disk $\mathbb{D}$ we have

$$\sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right| r^n \leq C(r) \|f\|_\infty \,.$$

By a result of Bombieri in [9] the exact value of this constant in the range $1/3 \leq r \leq 1/\sqrt{2}$ is given by the formula

$$C(r) = \frac{1}{r}\left(3 - \sqrt{8(1-r^2)}\right) \,,$$

and later Bombieri and Bourgain proved in [10, Theorem 1.1, 1.2] that

$$C(r) < (1-r^2)^{-1/2} \;\text{ for } r > 1/\sqrt{2}\,,$$

and

$$C(r) \asymp (1-r^2)^{-1/2} \;\text{ as } r \to 1\,.$$

Note that the strictly increasing function $K_1(\mathbb{C}, \cdot) : [1, \infty[ \to [1/3, 1[$ has as its inverse the function $C(\cdot) : [1/3, 1[ \to [1, \infty[$. Hence Bombieri's result implies that for all $1 \leq \lambda \leq \sqrt{2}$

$$K_1(\mathbb{C}, \lambda) = \frac{1}{3\lambda - 2\sqrt{2(\lambda^2-1)}} \,,$$

and for $\lambda$ close to $\infty$

$$K_1(\mathbb{C}, \lambda) \asymp \frac{\sqrt{\lambda^2-1}}{\lambda} \,.$$

On the other hand Blasco [2, Theorem 1.2] showed that for $X = \ell_p^2$ (i.e. $\mathbb{C}^2$ with the $p$-norm) we have

$$K_1(X, 1) = 0 \;\text{ for every } 1 \leq p \leq \infty\,.$$

This explains why we implemented the constant $\lambda \geq 1$ in our definition of the vector-valued Bohr radii $K_N(X, \lambda)$; later we will see that for every nontrivial Banach space $X$ and $\lambda > 1$ the inequality $K_N(X, \lambda) \geq K_1(X, \lambda) > 0$ holds.

Again our aim is to find optimal lower and upper asymptotic bounds for $K_N(X, \lambda)$. Note first that for each Banach space $X$ and $\lambda > 1$ we have

$$\frac{1}{N} \prec K_N(X, \lambda) \prec \sqrt{\frac{\log N}{N}}$$

(see [18]), but in more concrete situations we can say much more. Recall the following well-established notion from local Banach space theory: A Banach space $X$ is said to have cotype $p$, $2 \leq p < \infty$, whenever there is some constant $C > 0$ such that for each choice of finitely many vectors $x_1, \ldots, x_N \in X$ we have

$$\left( \sum_{n=1}^{N} \|x_n\|^p \right)^{1/p} \leq C \left( \int_0^1 \left\| \sum_{n=1}^{N} r_n(t) x_n \right\|^2 dt \right)^{1/2} ;$$

here as usual $r_n$ stands for the $n$th Rademacher function on $[0, 1]$. Every Banach space $X$ by definition has cotype $\infty$, and we as usual write

$$\mathrm{Cot}(X) := \inf\{\, 2 \leq p \leq \infty \mid X \text{ has cotype } p \,\} ;$$

this infimum is sometimes called *optimal* cotype of $X$ although it in general is not attained. For all in a sense natural $X$ the value of $\mathrm{Cot}(X)$ is known – in particular, we have that

$$\mathrm{Cot}(\ell_p) = \begin{cases} 2 & p \leq 2 \\ p & p > 2 \end{cases}$$

(compare again with the notion of $p$-concavity which we recalled in section 2). The following theorem is one of the main results from [18].

**Theorem 7.** *Let $X$ be a Banach space and $\lambda > 1$. With constants depending only on $X$ and $\lambda$ we have:*

*(1) For finite dimensional $X$*

$$K_N(X, \lambda) \asymp \sqrt{\frac{\log N}{N}}\,.$$

*(2) For infinite dimensional $X$*

$$\frac{1}{N^{1-\frac{1}{\mathrm{Cot}(X)+\varepsilon}}} \prec K_N(X, \lambda) \prec \frac{1}{N^{1-\frac{1}{\mathrm{Cot}(X)}}}\;;$$

*if $\mathrm{Cot}(X)$ is attained, then the inequality even holds with left $\varepsilon = 0$.*

Note that in contrast to the finite dimensional case in the infinite dimensional case, e.g. for $\ell_p$-spaces, no logarithmic term appears.

**Corollary 2.** *With constants depending only on $p$ and $\lambda$ we have*

$$K_N(\ell_p, \lambda) \asymp \frac{1}{N^{1-\frac{1}{\max\{p,2\}}}}\,.$$

Clearly, we can also define $K_N(v, \lambda)$, where $v : X \to Y$ is some non-zero (bounded and linear) operator: The $N$th Bohr radius of $v$ and $\lambda$, denoted by $K_N(v, \lambda)$, is the best $0 \leq r \leq 1$ such that for each $X$-valued holomorphic function $f = \sum_{\alpha \in \mathbb{N}_0^N} c_\alpha z^\alpha$ on $\mathbb{D}^N$ we have

$$\sup_{z \in r\mathbb{D}^N} \sum_{\alpha \in \mathbb{N}_0^n} \|v(c_\alpha)\|_Y \leq \lambda \|f\|_\infty\,.$$

A priori we have

$$\frac{1}{N} \prec K_N(v, \lambda) \prec \sqrt{\frac{\log N}{N}}\,.$$

But in certain situations we know much more precise estimates – we mainly focus our interest on the following two scales of operators:

- $v$ any of the embeddings $\ell_p \hookrightarrow \ell_q$ with $1 \leq p \leq q < \infty$
- $v$ any operator $\ell_1 \to \ell_q$ with $1 \leq q < \infty$

Combining Lemma 2 with deep Grothedieck type inequalities from the theory of summing operators (e.g. the fact that by Littlewood's $4/3$-inequality the embedding $\ell_1 \hookrightarrow \ell_{4/3}$ is $(4/3, 1)$-summing or that by Grothendieck's theorem every operator $v : \ell_1 \to \ell_2$ is $(1, 1)$-summing together with its important improvements by Bennett, Carl and Kwapień) we know from [18] that

**Theorem 8.** *Let $1 \leq p < q < \infty$. Then with constants depending only on $p, q$ and $\lambda$:*

$$K_N(\ell_p \hookrightarrow \ell_q, \lambda) \asymp \begin{cases} \sqrt{\frac{\log N}{N}} & \text{if } p < 2, \\ \frac{1}{N^{1-\frac{1}{p}}} & \text{if } p \geq 2. \end{cases}$$

**Theorem 9.** *For every operator $v : \ell_1 \to \ell_q$, $1 \le q < \infty$*

$$\left(\frac{\log N}{N}\right)^{1-\frac{1}{\max\{2,q\}}} \prec K_N(v,\lambda) \prec \sqrt{\frac{\log N}{N}}.$$

In the second result for $q \le 2$ we have equality, and looking at the embedding $\ell_1 \hookrightarrow \ell_q$ we see that for $q \ge 2$ the right side in general can not be improved. The upper estimates in both theorems are consequences of the following more abstract theorem.

**Theorem 10.** *Let $v : X \to Y$ be a non-zero operator and $\lambda > \|v\|$. With constants depending only on $v$ and $\lambda$:*

*(1) If $Y$ is a Banach space of cotype $p$, then*

$$\frac{1}{N^{1-\frac{1}{p}}} \prec K_N(v,\lambda)\,.$$

*(2) If $Y$ is a $p$-concave Banach lattice with $p \ge 2$ and if there is $r < p$ such that $v$ is $(r,1)$-summing, then*

$$\left(\frac{\log N}{N}\right)^{1-\frac{1}{p}} \prec K_N(v,\lambda)\,.$$

# 6 Bohr's strips – a macroscopic view

Here we try a sort "macroscopic view" on the Bohr-Bohnenblust-Hille theorem (Theorem 2) – "macroscopic" since we regard Dirichlet series with coefficients in a Banach space. Clearly, for each Dirichlet series $\sum_n a_n \frac{1}{n^s}$ with coefficients $a_n$ in some fixed Banach space $X$ we again can define the abscissa of absolute and the abscissa of uniform convergence. Then $S(X)$ obviously stands for the largest possible strip in which such a $X$-valued Dirichlet series converges uniformly but not absolutely. The main result from [16, Theorem 1] is a formula on $S(X)$ in terms of the optimal cotype $\mathrm{Cot}(X)$ of the underlying space $X$.

**Theorem 11.**

$$S(X) = \begin{cases} \frac{1}{2} & \textit{if} \quad \dim X < \infty \\ 1 - \frac{1}{\mathrm{Cot}(X)} & \textit{if} \quad \dim X = \infty \end{cases}$$

From (5) we immediately deduce that

**Corollary 3.**

$$S(\ell_p) = 1 - \frac{1}{\max\{p,2\}}$$

More generally, we define for each non-zero (bounded and linear) operator $v : X \to Y$ the number

$$S(v) := \sup\left[\sigma_a\left(\sum v(a_n)\frac{1}{n^s}\right) - \sigma_u\left(\sum a_n \frac{1}{n^s}\right)\right],$$

the supremum taken over all Dirichlet series $\sum a_n \frac{1}{n^s}$ with coefficients in $X$. Obviously, we have $S(\mathrm{id}_X) = S(X)$ and in particular $S = S(\mathrm{id}_{\mathbb{C}})$. It can be seen easily that

$$\frac{1}{2} = S \le S(v) \le 1\,, \tag{4}$$

and as above we concentrate for $1 \le p \le q < \infty$ on the study of embeddings $\ell_p \hookrightarrow \ell_q$ and arbitrary non-zero operators $\ell_1 \to \ell_q$.

**Corollary 4.**

*(1)* $S(v : \ell_1 \to \ell_q) = \dfrac{1}{2}$

*(2)* $S(\ell_p \hookrightarrow \ell_q) = 1 - \dfrac{1}{\max\{p,2\}}$

The lower estimate in (1) is a consequence of (4), and the upper estimate follows easily from Corollary 3 and the fact that $S(v : \ell_1 \to \ell_q) \leq S(\mathrm{id}_{\ell_1})$. With the same arguments we get the upper estimate in (2). On the other hand one can also regard the strip $S_m(v) = \sup(\sigma_a(vA) - \sigma_u(A))$ where the supremum is taken over all $m$-homogeneous Dirichlet series $A$. It is shown in [22, Theorem 1.1] that for $p \geq 2$ we have $S_m(\ell_p \hookrightarrow \ell_q) = 1 - \frac{1}{p}$. This and the fact that $S_m(v) \leq S(v)$ give the lower estimate of (2).

Looking at the scalar case it seems natural to consider in the operator case the following "graduation" of $S(v)$. For given $N$, the number $Q_N(v)$ by definition stands for the best constant $D \geq 1$ such that for all choices of vectors $a_1, \ldots, a_N \in X$

$$\sum_{n=1}^{N} \|va_n\|_Y \leq D \sup_{t\in\mathbb{R}} \left\| \sum_{n=1}^{N} a_n n^{-it} \right\|_X .$$

Of course, we abbreviate $Q_N(\mathrm{id}_X)$ by $Q_N(X)$, and clearly we have $Q_N = Q_N(\mathrm{id}_{\mathbb{C}})$. Motivated by the history of the results in the scalar case we (can not resist to) call $Q_N(v)$ the $N$th Queffélec number of the operator $v$. Why do these numbers graduate $S(v)$? The following lemma is a vector-valued variant of the Maurizi-Queffélec result (2) mentioned above (see [20]).

**Lemma 3.** *Let $v : X \to Y$ be a non-zero operator. Then $S(v)$ equals the infimum over all $\sigma \geq 0$ for which there exists a constant $C \geq 1$ such that for each $N$ and each choice of $a_1, \ldots, a_N \in X$ we have that*

$$\sum_{n=1}^{N} \|va_n\| \leq C\, N^{\sigma} \sup_{t\in\mathbb{R}} \left\| \sum_{n=1}^{N} a_n n^{it} \right\|_X .$$

This allows us to prove (as a sort of corollary) the following formula for the the widths of Bohr's strips $S(v)$ (see [20]).

**Lemma 4.**

$$S(v) = \limsup_{N\to\infty} \frac{\log Q_N(v)}{\log N}.$$

Again we have an "a priori upper and an a priori lower estimate":

$$\frac{\sqrt{N}}{e^{(\frac{1}{\sqrt{2}}+o(1))\sqrt{\log N \log\log N}}} \prec Q_N(v) \prec N; \tag{5}$$

the lower estimate is a consequence of Theorem 6, for the upper one see [20]. But again for special operators $v$ we sometimes know much more precise inequalities. Our main interest as above lies in the asymtotics of the Queffélec numbers $Q_N(X)$, $Q_N(\ell_p \hookrightarrow \ell_q)$, and $Q_N(\ell_1 \to \ell_q)$. Analyzing the article [12] of de la Bretèche and combining it with Lemma 1 (as in [14]) the following theorem was proved in [20].

**Theorem 12.** *Let $X$ be a Banach space. Then with constants depending only on $X$ we have:*

*(1) For finite dimensional $X$*

$$Q_N(X) = \frac{\sqrt{N}}{e^{(\frac{1}{\sqrt{2}}+o(1))\sqrt{\log N \log\log N}}} .$$

*(2) For infinite dimensional $X$ and any $\varepsilon$*

$$N^{1-\frac{1}{\mathrm{Cot}(X)}} \prec Q_N(X) \prec N^{1-\frac{1}{\mathrm{Cot}(X)+\varepsilon}}\,.$$

We conjecture that in (2) it is possible to take $\varepsilon = 0$ provided the optimal cotype $\mathrm{Cot}(X)$ is attained. This in particular means that in the following corollary no $\varepsilon$ would be needed (the fact that in the case $p \geq 2$ no $\varepsilon$ is needed is proved in [20]).

**Corollary 5.** *With constants depending only on $p$ we have:*

$$N^{1-\frac{1}{\max\{p,2\}}} \prec Q_N(\ell_p) \prec \begin{cases} N^{1-\frac{1}{2+\varepsilon}} & \text{if } p < 2, \\ \\ N^{1-\frac{1}{p}} & \text{if } p \geq 2. \end{cases}$$

If we replace the identities $v = \mathrm{id}_{\ell_p}$ by the embeddings $\ell_p \hookrightarrow \ell_q$, then we obtain the following result.

**Theorem 13.** *Let $1 \leq p < q < \infty$. Then with constants depending only on $p, q$ we have:*

$$Q_N(\ell_p \hookrightarrow \ell_q) \prec \begin{cases} \dfrac{\sqrt{N}}{e^{\left(\sqrt{\frac{1}{p}-\max\{\frac{1}{2},\frac{1}{q}\}}+o(1)\right)\sqrt{\log N \log\log N}}} & \text{if } p < 2\,, \\ \\ N^{1-\frac{1}{p}} & \text{if } p \geq 2\,. \end{cases}$$

The question whether this result is optimal is open. For operators $v : \ell_1 \to \ell_q$ we have

**Theorem 14.** *Let $v : \ell_1 \to \ell_q$ be a non-zero operator an $1 \leq q \leq 2$. Then*

$$Q_N(v) \leq \frac{\sqrt{N}}{e^{\left(\sqrt{1-\frac{1}{q}}+o(1)\right)\sqrt{\log N \log\log N}}}\,.$$

Obviously, (5) shows that this estimate for $p = 1$ and $q = 2$ can not be improved – but the optimality in the general case again is unclear. As above (see Theorem 10) both of the preceding theorems can be seen as consequences of a more abstract theorem.

**Theorem 15.** *Let $Y$ be a $q$-concave Banach lattice and $v : X \to Y$ and $(r,1)$-summing operator with $1 \leq r < q$. Then*

$$Q_N(v) \leq \frac{N^{\frac{q-1}{q}}}{e^{\left(2\frac{q-1}{q}\sqrt{\frac{1}{r}-\frac{1}{q}}+o(1)\right)\sqrt{\log N \log\log N}}}\,.$$

# 7 Philosophy

Recall from Bohr's vision (which we repeated in the introduction) that in principle every definition or result from the theory of Dirichlet series can be reformulated in terms of power series in infinitely many variables (or equivalently, in terms of holomorphic functions on the open unit ball of $\ell_\infty$). In this sense for a given non-zero operator $v : X \longrightarrow Y$ the counterpart of the number $S(v)$ is the number $K(v)$:

$$K(v) := \sup p\,,$$

where the supremum is taken over all $1 \le p \le \infty$ such that for all $z \in \ell_p \cap B_{\ell_\infty}$ and all $f \in H(B_{\ell_\infty}, X)$ we have that

$$(v \circ f)(z) = \sum_\alpha v\left(\frac{f^{(\alpha)}(0)}{\alpha!}\right) z^\alpha.$$

The following theorem can be shown with Bohr's methods from [7] (needing the prime number theorem).

**Theorem 16.**

$$S(v) = \frac{1}{K(v)}.$$

All given estimates on Queffélec numbers $Q_N$, Bohr radii $K_N$ as well as $S$, $K$ in the scalar case, for Banach spaces $X$ or operators $v : X \to Y$ suggest the following *meta-theorem*.

**Meta-Theorem.** For each non-zero operator $v : X \to Y$ and each $\lambda > 1$ up to small terms in $N$ (terms like $N^\varepsilon$, $\log N$, $\log\log N$ etc....) and constants only depending on $v, \lambda$ we have

$$\begin{aligned} Q_N(v) &\asymp \frac{1}{K_N(v,\lambda)} \\ Q_N(v) &\asymp N^{S(v)} \\ K_N(v,\lambda) &\asymp \tfrac{1}{N^{1/K(v)}} \end{aligned}$$

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