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A characterization of the affine Hall triple systems defined by groups of exponent 3

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Abstract. In this paper we show that if $(\mathbf{G}, \mathcal{L})$ is the Hall triple system associated with a group of exponent 3 $(\mathbf{G}, +)$, then $(\mathbf{G}, \mathcal{L})$ is an affine space if and only if $(\mathbf{G}, +)$ is nilpotent of class at most 2.

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1 Introduction

Let $(\mathbf{G}, \mathcal{L})$ be a pair in which $\mathbf{G}$ is a non empty set of points and $\mathcal{L}$ is a set of subsets of $\mathbf{G}$ called lines. $(\mathbf{G}, \mathcal{L})$ is a *Steiner triple system* if two different points determine a line and any line contains precisely three different points.

Any Steiner triple system determines a quasigroup (*Steiner quasigroup*) $(\mathbf{G}, \triangledown)$ if for $a \neq b$ one puts $a \triangledown b = c$, where c is the third point of the line determined by a and b; additionally one puts $a \triangledown a = a$. Hence we have:

(j) $\forall x, y \in \mathbf{G} : x \triangledown (x \triangledown y) = y$.

Thus the Steiner quasigroup $(\mathbf{G}, \triangledown)$ is an idempotent totally symmetric quasigroup (cf. [9], p. 122; [3], p. 64). Conversely, every idempotent totally symmetric quasigroup is determined by a unique Steiner triple system.

We point out that if p and $\mathbf{l}$ are respectively a point and a line of $(\mathbf{G}, \mathcal{L})$, with $p \notin \mathbf{l}$, then $\mathbf{l}$ and $p \triangledown \mathbf{l}$ are disjoint subsets of $\mathbf{G}$.

A subspace of $(\mathbf{G}, \mathcal{L})$ [a subquasigroup of $(\mathbf{G}, \triangledown)$] is a subset of $\mathbf{G}$, which is closed with respect to the join operation for the points [with respect to $\triangledown$]. In particular, the empty set, the singletons of points and the lines are subspaces.

If a_1, ..., a_n are points, then we represent by $((a_1, ..., a_n))$ the minumum subspace containing them [the *subspace generated by* $a_1, ..., a_n$]. One calls a *plane* any subspace generated by three non-collinear points.

Clearly, if $a_1 \neq a_2$, then $((a_1, a_2))$ is a line [*the line generated by* a_1 and a_2].

If $\mathbf{l}$ and $\mathbf{l}'$ are lines of $\mathbf{G}$, then $((\mathbf{l}, \mathbf{l}'))$ shall represent the subspace generated by $\mathbf{l} \cup \mathbf{l}'$; moreover, if a is a point, then $((a, \mathbf{l}))$ shall be the subspace generated by $\{a\} \cup \mathbf{l}$.

Definition 1. If $(\mathbf{G}, \mathcal{L})$ is a Steiner triple system, then one says that two lines $\mathbf{l}$ and $\mathbf{l}'$ are *parallel* [in symbols, $\mathbf{l}//\mathbf{l}'$] whenever either $\mathbf{l} = \mathbf{l}'$, or $\mathbf{l}$, $\mathbf{l}'$ are disjoint and $((\mathbf{l}, \mathbf{l}'))$ is a plane. Thus // represents a reflexive and symmetric relation.

If // is also transitive, then one says that $(\mathbf{G}, \mathcal{L})$ is *affine*.

We recall that a loop [i. e. a quasigroup with an identity 0] $\mathbf{G}$ is said a *Moufang loop* if, for all x, y, z in $\mathbf{G}$, it satisfies one of the following *Moufang identities* [which are equivalent in any loop]: $z + (x + (z + y)) = ((z + x) + z) + y$, $x + (z + (y + z)) = ((x + z) + y) + z$, $(z + x) + (y + z) = (z + (x + y)) + z$.

Remark 1. It is known that if (a, b, c) is a triple of elements in a Moufang loop such that $(a+b)+c = a+(b+c)$, then a, b, c generate an associative subloop; i.e. a group. In particular, by $(a + b) + 0 = a + b = a + (b + 0)$, the subloop generated by any two elements of a Moufang loop is a group.

If $(\mathbf{G}, \mathcal{L})$ is a Steiner triple system [$(\mathbf{G}, \triangledown)$ is a Steiner quasigroup] and 0 is a fixed element of $\mathbf{G}$, then it is natural to define another commutative binary operation $\oplus$ on $\mathbf{G}$ by putting, for any $x, y \in \mathbf{G}$:

(1) $x \oplus y := 0 \triangledown (x \triangledown y)$; then $2x := x \oplus x = 0 \triangledown x$.

For any $x, y \in \mathbf{G}$, we immediately get the following properties:

(2) $x \oplus 0 = x$, $2x \triangledown x = (0 \triangledown x) \triangledown x = 0$ and $2x \oplus x = 0 \triangledown (2x \triangledown x) = 0$.

Hence the set $\{0, x, 0 \triangledown x\}$ is an abelian group with respect to $\oplus$. One can see that $(\mathbf{G}, \oplus)$ is a loop (*the Steiner loop* of $(\mathbf{G}, \mathcal{L})$; cf. [5], p. 23).

2 Some remarks on the Hall triple systems

In this section we will deal with *autodistributive* Steiner triple systems.

One says that a Steiner triple system $(\mathbf{G}, \mathcal{L})$ [a Steiner quasigroup $(\mathbf{G}, \triangledown)$] is autodistributive whenever the operation $\triangledown$ is autodistributive. This is equivalent to say that if p and $\mathbf{l}$ are respectively a point and a line of $(\mathbf{G}, \mathcal{L})$, then $p \triangledown \mathbf{l}$ is a line too.

If $\oplus$ is the operation defined in section 1, then the following properties hold:

(3) $2x \oplus 2y$ [$= (0 \triangledown x) \oplus (0 \triangledown y)$] $= x \triangledown y$.

(4) By (1) and (3), any subset of $\mathbf{G}$ containing 0 is a subspace of $(\mathbf{G}, \mathcal{L})$ if and only if it is closed under $\oplus$.

Theorem 1. *If p is a point and $\mathbf{l}$ is a line of $(\mathbf{G}, \mathcal{L})$, then $p \triangledown \mathbf{l} \,//\, \mathbf{l}$.*

Proof. This is obvious if $p \in \mathbf{l}$. If $p \notin \mathbf{l}$, then $((p, \mathbf{l}))$ is a plane including the line $p \triangledown \mathbf{l}$, with $p \triangledown \mathbf{l}$ disjoint from $\mathbf{l}$. Hence we have the claim. QED

An example of autodistributive Steiner triple system is given by a group $(\mathbf{G}, +)$ of exponent 3, where the set of the points is $\mathbf{G}$ and the set $\mathcal{L}$ of the lines is given by the cosets of the subgroups of order 3. Clearly, it is not necessary to specify if one considers left or right cosets. Indeed, if $\mathbf{H}$ is a subgroup, any left coset $b + \mathbf{H}$ coincides with the right coset $(b + \mathbf{H} - b) + b$.

Remark 2. For any $x, y \in \mathbf{G}$, we have the following properties:

a) $x \triangledown y = y - x + y$. If $x = y$, this is trivial, by $x \triangledown x = x$. If $x \neq y$, the claim is a consequence of the fact that $\{y, x, y - x + y\}$ [$= \{0, x - y, y - x\} + y$] is the unique coset containing x and y.

b) If the element 0 fixed in Section 1 is the *zero* of $(\mathbf{G}, +)$, then we get:

(5) $x \oplus x = 0 \triangledown x = -x = x + x$;

(6) $x \oplus y = 0 \triangledown (x \triangledown y) = (0 \triangledown x) \triangledown (0 \triangledown y) = (-x) \triangledown (-y) = -x + y - x$. □

A Steiner triple system in which any 3 non-collinear points generate a plane over the Galois field $GF(3)$ is said a Hall triple system. It is clear that every Hall triple system is autodistributive. A finite Hall triple system has order 3^n, but it is not necessarily an affine space over $GF(3)$. It is known that the smallest Hall system which is not an affine space has order 81.

One can verify that if a Steiner triple system $(\mathbf{G}, \mathcal{L})$ is autodistributive, then $(\mathbf{G}, \oplus)$ is a commutative Moufang loop of exponent 3. Thus, if $a, b \in \mathbf{G}$, the Moufang subloop $\mathbf{H}$ of $(\mathbf{G}, \oplus)$ generated by a and b is an elementary abelian 3-group [see Remark 1], hence $\mathbf{H} = ((0, a, b))$ [cf. property (4) above].

Therefore, if the points a and b above are not collinear with 0, then $\mathbf{H}$ is a plane of $(\mathbf{G}, \mathcal{L})$ with nine points. Moreover, the structure of Steiner triple system $(\mathbf{H}, \mathcal{L}_{\oplus})$ associated with $(\mathbf{H}, \oplus)$, coincides with the structure of $\mathbf{H}$ as a plane of $(\mathbf{G}, \mathcal{L})$. In fact, if $x, y \in \mathbf{H}$, then we get $x \oplus < x \oplus 2y > = \{x, y, 2x \oplus 2y\} = \{x, y, x \bigtriangledown y\}$ [see property (3) above]. Whence the assertion. Thus $(\mathbf{G}, \mathcal{L})$ is a Hall triple system.

In particular, if $\oplus$ is associative, then $(\mathbf{G}, \oplus)$ is a commutative group of exponent 3. Thus $(\mathbf{G}, \mathcal{L})$ coincides with the Hall triple system $(\mathbf{G}, \mathcal{L}_{\oplus})$ associated with $(\mathbf{G}, \oplus)$ and hence $(\mathbf{G}, \mathcal{L})$ is an affine space (see Definition 1) over $GF(3)$.

On the contrary, it is known that if $(\mathbf{G}, \mathcal{L})$ is affine, then $(\mathbf{G}, \oplus)$ is a commutative group. For the benefit of the reader, by means of the following Theorem 2, we will have a direct proof of this latter property.

Meanwhile we remark that if p is a point and $\mathbf{l}$ is a line of an affine Hall triple system, then $p \oplus \mathbf{l} \,//\, \mathbf{l}$. Indeed $p \oplus \mathbf{l} = 0 \bigtriangledown (p \bigtriangledown \mathbf{l}) \,//\, p \bigtriangledown \mathbf{l} \,//\, \mathbf{l}$ (see Theorem 1). Since $//$ is transitive, we get $p \oplus \mathbf{l} \,//\, \mathbf{l}$.

Theorem 2. *If $(\mathbf{G}, \mathcal{L})$ is an affine Hall triple system, then $(\mathbf{G}, \oplus)$ is an elementary abelian 3-group.*

Proof. We have only to prove that $\oplus$ is associative. Thus consider three arbitrary points $a, b, c \in \mathbf{G}$ and prove that $[a \oplus b] \oplus c = a \oplus [b \oplus c]$.

If $0, a, b, c$ are coplanar, we already have seen that the claim holds. Hence let $0, a, b, c$ be not coplanar. Therefore $((0,\, a))$ and $((0,\, c))$ are distinct and not parallel lines. Moreover, by Theorem 1 we have:

$((b \oplus c,\; a \oplus [b \oplus c])) \,//\, ((0,\, a)) \,//\, ((b,\; a \oplus b)) \,//\, ((b \oplus c,\; [a \oplus b] \oplus c))$.

Thus $[a \oplus b] \oplus c$ and $a \oplus [b \oplus c]$ belong to the line $\mathbf{l_1}$ containing $b \oplus c$ and parallel to $((0,\, a))$. Analogously, $[a \oplus b] \oplus c$ and $a \oplus [b \oplus c]$ belong to the line $\mathbf{l_2}$ containing $a \oplus b$ and parallel to $((0,\, c))$.

Since $((0,\, a))$ and $((0,\, c))$ are not parallel, we have $\mathbf{l_1} \neq \mathbf{l_2}$. As a consequence, $[a \oplus b] \oplus c = a \oplus [b \oplus c]$. *QED*

3 Our characterization of the affine Hall triple systems associated with groups of exponent 3

In the sequel $(\mathbf{G}, +)$ shall be a group of exponent 3. Thus, for any $x, z \in \mathbf{G}$, $-x + z - x = -z + x - z$ and hence the following property holds:

(7) $\forall x, z \in \mathbf{G} : x + z - x - z = -x - z + x + z$.

Lemma 1. *The following properties are equivalent:*

(8) $\forall x, y, z \in \mathbf{G} : -x - z + y - z - x = -z - x + y - x - z;$

(8′) $\forall x, y, z \in \mathbf{G} : x + z - x - z + y = y - x - z + x + z;$

(8″) $\forall x, y, z \in \mathbf{G} : -x - z + x + z + y = y - x - z + x + z.$

Proof. It is trivial that (8) and (8′) are equivalent. On the other hand, (8′) and (8″) are equivalent by property (7). *QED*

Remark 3. We point out that (8") in Lemma 1 means that $(\mathbf{G}, +)$ is a nilpotent group of class at most 2 (cf. [10], p. 122).

We conclude with the following Theorem 3, which gives our characterization.

Theorem 3. *Let* $(\mathbf{G}, \mathcal{L}_+)$ *be the Hall triple system associated with* $(\mathbf{G}, +)$*. Then the following properties are equivalent:*

i) $\mathbf{G}$ *is also the support of an elementary abelian 3-group with the same Hall triple system as* $(\mathbf{G}, +)$*;*

ii) $(\mathbf{G}, \mathcal{L}_+)$ *is an affine space;*

iii) $(\mathbf{G}, +)$ *is a nilpotent group of class at most 2.*

Proof. *i*) and *ii*) are trivially equivalent. In order to prove that also *ii*) and *iii*) are equivalent, it is sufficient to verify that the operation $\oplus$ is associative if and only if property $(6'')$ in Lemma 1 is true [cf. Remark 3].

Being $\oplus$ commutative, $\oplus$ is associative if and only if, for any $x, y, z \in \mathbf{G}$, $x \oplus (z \oplus y) = z \oplus (x \oplus y)$. This latter property means that (8) in Lemma 1 holds. Therefore *ii*) and *iii*) are equivalent, since Lemma 1 ensures that (8) and $(8'')$ are equivalent. QED

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