Note di Matematica **24,** n. 2, 2005, 75–83.

Asymptotic stability in the large of the solutions of almost periodic impulsive differential equations

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Received: 24/4/2003; accepted: 10/1/2005.

Abstract. By means of piecewise continuous functions which are modifications of classical Lyapunov's functions, some sufficient conditions for asymptotic stability in the large of solutions of almost periodic impulsive differential equations are presented.

Keywords: Lyapunov functions, Almost periodic solutions, Impulsive differential equations

MSC 2000 classification: 34A37.

1 Introduction

Let R^n be the n-dimensional Euclidean space with norm $\|.\|$, and let

$$R =(-\infty,\infty), \tau_k \in R, \tau_k < \tau_{k+1}, k = \pm1, \pm2, \dots, \lim_{k\to\pm\infty} \tau_k = \pm\infty,$$
$$\Gamma = \{(t, x) \in R \times R^n\}, B_h = \{x \in R^n,\ ||x|| \leq h\}, h = \text{const} > 0.$$

We shall consider the following impulsive differential equations

$$\begin{cases} \dot{x}(t) = f(t, x), t \neq \tau_k, \\ \Delta x(\tau_k) = x(\tau_k + 0) - x(\tau_k - 0) = I_k(x(\tau_k - 0)),\ t = \tau_k,\ k = \pm1, \pm2, \dots, \end{cases} \tag{1}$$

where $t \in R$, $x \in R^n$, $f\colon \Gamma \to R^n$, $I_k\colon R^n \to R^n$ for $k = \pm1, \pm2, \dots,$

A solution $x(t; t_0, x_0)$ of (1) with $x(t_0 + 0; t_0, x_0) = x_0$ existing on some interval $[t_0, t_0 + \alpha)$ and undergoing impulses at the point τ_k, $t_0 < \tau_k < t_0 + \alpha$ is described as follows:

$$x(t; t_0, x_0) = \begin{cases} x(t; t_0, x_0 + 0),\ t_0 < t \leq \tau_j, \\ x_1(t; \tau_j, x_j + 0),\ \tau_j < t \leq \tau_{j+1}, \\ \dots, \\ x_k(t; \tau_k, x_k + 0),\ \tau_{j+k-1} < t \leq \tau_{j+k}, \\ \dots \end{cases}$$

The impulsive differential equations are an adequate apparatus for mathematical simulations of numerous real processes and phenomena studied in the theory of optimal control, physics, chemistry, biology, bioengineering sciences, technology, medicine, etc.

Their theory is considerably more complex than the theory of the ordinary differential equations. This is due to a number of their properties such as loss of the property of autonomy, merging of the solutions, bifurcations, etc. On the other hand, the properties listed above require the introduction of new and modification of the standard methods of investigation. This is the main reason why their theory is developing rather slowly [1, 6], [8].

In the present paper by means of piecewise continuous functions which are modifications of classical Lyapunov's functions, some sufficient conditions for asymptotically stable in the large of solutions of almost periodic impulsive differential equations are found.

2 Preliminary notes

Introduce the following conditions:

H1. The function $f(t,x)$ is continuous in Γ and has continuous partial derivatives of first order with respect to all components of x.

H2. The functions $I_k(x)$, $k = \pm1, \pm2, \ldots$, are continuously differentiable in B_h.

H3. There exists a number μ, $(0 < \mu < h)$ such that if $x \in B_\mu$, then $x + I_k(x) \in B_h$, $k = \pm1, \pm2, \ldots$.

H4. The functions $J_k(x) = x + I_k(x)$, $k = \pm1, \pm2, \ldots$, are invertible in B_h and $J_k^{-1}(x) \in B_h$ for $x \in B_h$.

H5. $f(t,0) = 0$, for $t \in R$, and $I_k(0) = 0$ for $k = \pm1, \pm2, \ldots$.

H6. There exists constant $L > 0$ such that

$$||f(t,x) - f(t,y)|| + ||I_k(x) - I_k(y)|| \leq L||x - y||,$$

for $t \in R$, $x \in R^n$, $y \in R^n$, $k = \pm1, \pm2, \ldots$.

If conditions H1-H4 are satisfied, then for each point $(t_0, x_0) \in \Gamma$ from [4] it follows that there exists a unique solution $x(t) = x(t; t_0, x_0)$, $t > t_0$, of system (1) satisfying the initial condition $x(t_0 + 0) = x_0$.

1 Definition. [7] The continuous function $f(t,x)$ is said to be *almost periodic in* t *uniformly with respect to* $x \in X$*,* $X \subseteq R^n$, if for any $\varepsilon > 0$, it is possible to find a number $l(\varepsilon)$ such that, in any interval of length $l(\varepsilon)$, there exists a number τ such that

$$\|f(t+\tau,x) - f(t,x)\| < \varepsilon,$$

for all $t \in R$, $x \in X$.

2 Definition. [7] The sequence $\{\psi_k(x)\}$, $\psi_k \colon X \to R^n$, $k = \pm 1, \pm 2, \dots$, is said to be *almost periodic in* k *uniformly with respect to* $x \in X$*,* $X \subseteq R^n$ if for any $\varepsilon > 0$ it is possible to find $l(\varepsilon)$ such that, in any interval of length $l(\varepsilon)$, there exists a number q such that

$$\|\psi_{k+q}(x) - \psi_k(x)\| < \varepsilon,$$

for all $k = \pm 1, \pm 2, \dots$, $x \in X$.

3 Definition. [1] A piecewise continuous function $\varphi \colon R \to R^n$ with discontinuities of first kind at the points τ_k is called *almost periodic*, if:

a) the set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k = \pm 1, \pm 2, \dots$, $j = \pm 1, \pm 2, \dots$ is uniformly almost periodic.

b) for any $\varepsilon > 0$ there exists a real number $\delta > 0$ such that if the points t' and t'' belong to one and the same interval of continuity of $\varphi(t)$ and satisfy the inequality $|t' - t''| < \delta$, then $|\varphi(t') - \varphi(t'')| < \varepsilon$.

c) for any $\varepsilon > 0$ there exists a relatively dense set T of ε-almost periods such that if $\tau \in T$, then $\|\varphi(t+\tau) - \varphi(t)\| < \varepsilon$ for all $t \in R$ satisfying the condition $|t - \tau_k| > \varepsilon$, $k = \pm 1, \pm 2, \dots$.

4 Definition. The number τ in Definition 1-3 is called an *ε-translation number*. Consider the sets

$$G_k = \{(t,x) \in \Gamma \colon \tau_{k-1} < t < \tau_k\}, k = \pm 1, \pm 2, \dots, G = \bigcup_{k=\pm 1, \pm 2, \dots} G_k,$$

$$S_\alpha = \{(t,x) \in \Gamma \colon x \in B_\alpha, \text{ if } (t,x) \in G \text{ } and \text{ } x + I_k(x) \in B_\alpha, \text{ if } t = \tau_k\},$$
$$0 < \alpha = \text{const}.$$

If the conditions H1-H5 hold then there exists a zero solution for system (1). [4]

5 Definition. [4] The zero solution $x(t) \equiv 0$ of the system (1) is said to be:

5.1 *stable* if

$$(\forall \varepsilon > 0)(\forall t_0 \in R)(\exists \delta > 0)(\forall x_0 \in B_\delta)(\forall t \in R, \ t > t_0) \colon ||x(t; t_0, x_0)|| < \varepsilon;$$

5.2 *uniformly stable* if

$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall t_0 \in R)(\forall x_0 \in B_\delta)(\forall t \in R,\ t > t_0)\colon ||x(t;t_0,x_0)|| < \varepsilon;$$

5.3 *asymptotically stable* if it is stable and

$$(\forall t_0 \in R)(\exists\lambda = \lambda(t_0) > 0)(\forall x_0 \in B_\delta\colon (t_0,x_0) \in S_\lambda)\colon \lim_{t\to\infty} x(t;t_0,x_0) = 0;$$

5.4 *asymptotically stable in the large*, if it is stable and every solutions of (1) tends to zero as $t \to -\infty$.

5.5 *Quasi-equi-asymptotically stable in the large*, if

$$(\forall\alpha > 0)(\forall\varepsilon > 0)(\forall t_0 \in R)(\exists T > 0)$$
$$(\forall x_0 \in B_\alpha)(\forall t \geq t_0 + T(t_0,\varepsilon,\alpha)) : ||x(t;t_0,x_0)|| < \varepsilon.$$

6 Definition. The solution $x(t;t_0,x_0)$ is called *bounded* if

$$(\exists\beta > 0), (\forall t \in R, t \geq t_0, x_0 \in B_\beta)\colon ||x(t;t_0,x_0)|| < \beta.$$

7 Definition. The solution $x(t;t_0,x_0)$ is called *equi-bounded* if

$$(\forall\alpha > 0)(\forall t_0 \in R)(\exists\beta > 0)(\forall x_0 \in B_\alpha)(\forall t \in R, t \geq t_0)\colon ||x(t;t_0,x_0)|| < \beta.$$

8 Definition. The zero solution of (1) is called *perfectly uniform-asymptotically stable in the large* if δ in Definition 5.1, T in Definition 5.5 and β in Definition 7 are independent on t_0 for all $t_0 \in R$.

Consider the following scalar impulsive differential equation

$$\begin{cases} \dot{u} = g(t,u),\ t \neq \tau_k, \\ u(\tau_k + 0) = \psi_k(u(\tau_k)),\ k = \pm1,\pm2,\ldots, \\ u(t_0 + 0) = u_0, \end{cases} \tag{2}$$

where $g : R \times R \to R$, $\psi_k\colon R \to R$, $k = \pm1,\pm2,\ldots$.

9 Lemma. *[8] Let the following conditions be fulfilled:*

(1) The function $m\colon R \to R$ is piecewise continuous with point of discontinuity of the first kind τ_k, $k = \pm1,\pm2,\ldots$, it is continuous from the left and for which the relations $\tau_k < \tau_{k+1}$, $k = \pm1,\pm2,\ldots$ and $\lim_{k\to\infty} \tau_k = \infty$.

(2) For $k = \pm 1, \pm 2, \ldots$ *inequalities*

$$D^+ m(t) \leq g(t, m(t)),\ t \neq \tau_k,$$
$$m(\tau_k + 0) \leq \psi_k(m(\tau_k)),\ k = \pm 1, \pm 2, \ldots,$$
$$m(t_0 + 0) \leq u_0$$

hold, where $g \in C[R \times R, R]$, $\psi_k \in C[R, R]$, $\psi_k(u)$ *is nondecreasing in* u *and*

$$D^+ m(t) = \lim_{h \to 0^+} \sup h^{-1}(m(t+h) - m(t)).$$

(3) The maximal solution of equation (2) *is defined in* (t_0, ∞).
Then $m(t) \leq r(t; t_0, u_0)$ *for* $t \in (t_0, \infty)$.

We introduce the classes V_0, V_1 and of piecewise continuous functions.

10 Definition. [8] We shall say that the function $V \colon \Gamma \to R^+$ belongs to the class V_0 if the following conditions hold:

(1) V is continuous in each of the sets G_k and $V(t, 0) = 0$ for $t \in R$.

(2) For each $k = \pm 1, \pm 2, \ldots$, and each point $x_0 \in B_h$ there exist and are finite the limits

$$V(\tau_k - 0, x_0) = \lim_{\substack{(t,x) \to (\tau_k, x_0) \\ (t,x) \in G_k}} V(t, x), V(\tau_k + 0, x_0) = \lim_{\substack{(t,x) \to (\tau_k, x_0) \\ (t,x) \in G_{k+1}}} V(t, x)$$

and the equality $V(\tau_k - 0, x_0) = V(\tau_k, x_0)$ holds.

(3) V is locally Lipschitz continuous in its second argument in each of the sets G_k, i. e. for each $\alpha > 0$, $x' \in B_\alpha$, $x'' \in B_\alpha$

$$|V(t, x') - V(t, x'')| \leq h(\alpha)||x' - x''||. \tag{3}$$

(4) For any $t \in R$ and any $x \in R^n$ the following inequality holds

$$V(t + 0, x + I_k(x)) \leq V(t, x),\ k = \pm 1, \pm 2, \ldots$$

11 Definition. [8] We shall say that the function $V : \Gamma \to R$ *belongs to the class* V_1 if:

(1) $V \in V_0$ and V is continuously differentiable in each of the sets G_k.

(2) For each function $V \in V_1$ we define the function

$$\dot{V}(t,x) = \frac{\partial V(t,x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial V(t,x)}{\partial x_i} f_i(t,x)$$

for $(t,x) \in G$ and

$$D^+V(t,x) = \lim_{h \to 0^+} \operatorname{suph}^{-1}(V(t+h, x(t+h;t,x)) - V(t,x))$$

is the upper right derivative of the function $V \in V_0$ with respect to system (1).

In the further considerations we shall use the following classes of functions:

K - the class of all continuous strictly increasing functions $a : R^+ \to R^+$, and $a(0) = 0$.

P - the class of all continuous strictly increasing, positive functions $b : R^+ \to R^+$.

Introduce the following conditions:

H7. The function $f(t,x)$ is almost periodic in t uniformly with respect to x, $x \in B_h$.

H8. The sequence $\{I_k(x)\}$, $k = \pm 1, \pm 2, \ldots$, is almost periodic in k uniformly with respect to x, $x \in B_h$.

H9. The set of sequences $\{\tau_k^j\}$, $\tau_k^j = \tau_{k+j} - \tau_k$, $k = \pm 1, \pm 2, \ldots$, $j = \pm 1, \pm 2, \ldots$, is uniformly almost periodic and $\inf_k \tau_k^1 = \theta > 0$.

12 Definition. If the conditions H1-H9 be fulfilled then the system (1) is called *almost periodic.*

3 Main results

13 Theorem. *Let the following conditions be fulfilled.*

(1) Conditions H1-H9 *are met.*

(2) There exist a functions $V \in V_1$ and $a, b \in P$ such that

(i) $a(||x||) \le V(t,x) \le b(||x||)$, $(t,x) \in \Gamma$;

(ii) $D^+V(t,x) \le -cV(t,x)$, *for* $(t,x) \in \Gamma$, $c = \text{const} > 0$.

Then the zero solution of (1) *is perfectly uniform-asymptotically stable in the large.*

PROOF. Let for a given $\varepsilon > 0$, $\varepsilon < 1$, choose a $\delta(\varepsilon) > 0$ such that $a(\varepsilon) > 2b(\delta)$. Let $x(t;t_0,x_0)$ be a solution of (1) so that $t_0 \in R$, $x_0 \in B_\delta$, and suppose that at some t, $||x(t;t_0,x_0)|| = \varepsilon$. Then there exists a t_1, $t_0 \leq t_1$ such that $||x(t_1;t_0,x_0)|| = \varepsilon$ and that $||x(t;t_0,x_0)|| < \varepsilon$ for $t \in [t_0,t_1]$. Clearly, there is a compact sets, for which $x(t;t_0,x_0) \in S$ for $t \in [t_0,t_1]$. Let κ denote $h(\alpha)$ in (3) for α such that the set $||x|| < \alpha$ contains S, and let τ be an $\frac{a(\delta)c}{2\kappa}$-translation number of $f(t,x)$ such that for $t \geq t_0+\tau$

$$||f(s+\tau,x) - f(t,x)|| \leq \frac{a(\delta)c}{2\kappa}, \tag{4}$$

where $s \in R$, $x \in R^n$ and there exists relatively dense set Q of integer numbers q such that

$$||I_{k+q}(x) - I_k(x)|| \leq \frac{a(\delta)c}{2\kappa}, \tag{5}$$

where $k = \pm 1, \pm 2, \dots,\ x \in R^n$.

Consider the function $V(s+\tau, x(s))$ for $s \in [t_0,t_1]$, where $x(s) = x(s;t_0,x_0)$. Denote by $x^*(t)$ the solution of (1) for which $x^*(s+\tau) = x(s)$ for any $s \neq \tau_k$, $\tau_k \in [t_0,t_1]$.

Then

$$\begin{aligned} D^+V(s+\tau,x(s)) &= \lim_{h\to 0^+} \operatorname{sup} h^{-1}(V(s+\tau+h,x(s+h)) - V(s+\tau,x(s))) \leq \\ &\leq \lim_{h\to 0^+} \operatorname{sup} h^{-1}(V(s+\tau+h,x^*(s+\tau+h)) - V(s+\tau,x(s))) + \\ &+ \lim_{h\to 0^+} \operatorname{sup} h^{-1}(V(s+\tau+h,x(s+h)) - V(s+\tau+h,x^*(s+\tau+h))). \end{aligned}$$

On the other hand from (ii) we get

$$\lim_{h\to 0^+} \operatorname{sup} h^{-1}(V(s+\tau+h,x^*(s+\tau+h)) - V(s+\tau,x(s))) \leq -cV(s+\tau,x(s)). \tag{6}$$

From (3),(4) it follows

$$\begin{aligned} &\lim_{h\to 0^+} \operatorname{sup} h^{-1}(V(s+\tau+h,x(s+h)) - V(s+\tau+h,x^*(s+\tau+h))) \leq \\ &\leq \lim_{h\to 0^+} \operatorname{sup} h^{-1}\,\kappa||x(s+h) - x^*(s+\tau+h)|| \leq \\ &\leq \kappa||f(s,x(s)) - f(s+\tau,x(s))|| \leq \frac{a(\delta)c}{2}. \end{aligned} \tag{7}$$

Then from(6),(7) we have

$$D^+V(s+\tau, x(s)) \le -cV(s+\tau, x(s)) + \frac{a(\delta)c}{2}. \tag{8}$$

From Lemma 9 it follows

$$V(t_1+\tau, x(t_1)) \le e^{-c(t-t_0)}V(t_0+\tau, x(t_0)) + \frac{a(\delta)}{2} \le b(\delta) + \frac{a(\delta)}{2} \le 2b(\delta).$$

Since $V(t_1+\tau, x(t_1)) \ge a(\varepsilon)$, there arises contradiction, because $a(\varepsilon) > 2b(\delta)$. Thus, for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that if $t_0 \in R$ and $x_0 \in B_{\delta\varepsilon}, ||x(t, t_0, x_0)|| < \varepsilon$ for all $t \ge t_0$ which that the zero solution is perfectly uniform-stable.
For a given α, by choosing $\beta(\alpha) > 0$ so that $2b(\alpha) < a(\beta)$ and applying the same argument as above, it can verified that if $t_0 \in R$ and $x_0 \in B_\alpha$, $||x(t; t_0, x_0)|| < \beta(\alpha)$ for $t \ge t_0$.
Let $x(t; t_0, x_0)$ be a solution such that $t_0 \in R$, $x_0 \in B_\alpha$.
Then there exists a $\beta(\alpha)$ for which $||x(t; t_0, x_0)|| < \beta(\alpha)$ for all $t \ge t_0$. Let S be compact set in R^n such that $x \in B_\alpha$. As was seen above, for a given $\varepsilon > 0$, there exists a $\delta(\varepsilon)$ such that $a(\varepsilon) > 2b(\delta)$ which proves uniform-stability. Let κ be the Lipshitz constant of $V(t, x)$ for $x \in S$ and let τ be $\frac{a(\delta)c}{2\kappa}$-translation number of $f(t, x)$ for $x \in S$ and there exists a relatively dense set Q of integer numbers q such that

$$||I_{k+q}(x) - I_k(x)|| \le \frac{a(\delta)c}{2\kappa},$$

for $x \in S$ and $t_0 + \tau \ge 0$. By the same argument as in the first part of this proof, we have (8). Suppose that $||x(t; t_0, x_0)|| \ge \delta(\varepsilon)$ for all $t \ge t_0$. Then, from (8), if $t > t_0 + T$, where $T = \frac{1}{c}\ln\frac{2b(\alpha)}{a(\delta)}$, $t \ne \tau_k$, it follows

$$V(t+\tau, x(t)) \le e^{-c(t-t_0)}V(t_0+\tau, x(t_0)) + \frac{a(\delta)}{2} < \frac{a(\delta)}{2b(\alpha)}b(\alpha) + \frac{a(\delta)}{2} = a(\delta),$$

which is contradicts $a(\delta) \le V(t+\tau, x(t))$. Therefore, at some t_1 such that $t_0 \le t_1 \le t_0 + T$, we have $||x(t_1; t_0, x_0)|| < \delta(\varepsilon)$, i. e., if $t \ge t_0 + T$, then $||x(t; t_0, x_0)|| < \varepsilon$. Since T depends only on α and ε the Theorem 13 is proved. $\boxed{QED}$

In the case where the system (1) and the functions $V(t, x)$ are defined on S_h, the following theorem can be proved by the same argument used in Theorem 13.

14 Theorem. *Let the following conditions be fulfilled.*

(1) Conditions H1-H9 *are met in the set* S_h.

(2) There exist a functions $V \in V_1$ *defined on the set* $G^* = \bigcup_{k=\pm 1, \pm 2, \dots} G_k^*$,

$$G_k^* = \{(t,x) \in \Gamma^* \colon \tau_{k-1} < t < \tau_k\}, \ k = \pm 1, \pm 2, \dots,$$
$$\Gamma^* = \{(t,x) \in (R, B_h)\}$$

and $a,b \in P$ *such that*

(i) $a(||x||) \leq V(t,x) \leq b(||x||)$, $(t,x) \in \Gamma^*$;

(ii) $D^+V(t,x) \leq -cV(t,x)$, *for* $(t,x) \in \Gamma^*$, $c = \text{const} > 0$.

Then the zero solution of (1) *is perfectly uniform-asymptotically stable.*

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