Note di Matematica **24,** n. 2, 2005, 41–57.

Orbit constructions for translation planes of order 81 admitting $SL(2,5)$

Norman L. Johnson [i]
Mathematics Dept., University of Iowa,
Iowa City, Iowa 52242, USA
njohnson@math.uiowa.edu

Alan R. Prince
School of Mathematical and Computer Sciences, Heriot-Watt University,
Edinburgh, Scotland
a.r.prince@ma.hw.ac.uk

Received: 9/12/2004; accepted: 9/12/2004.

Abstract. The authors have classified all translation planes of order 81 that admit $SL(2,5)$, where the 3-elements are elations, with the use of the computer. In this article, it is shown that the spreads in $PG(3,9)$ may be obtained directly from the group $SL(2,5)$. In the process, there is a construction of a replaceable 12-nest of reguli of a Desarguesian plane.

Keywords: spread, $SL(2,5)$ elation group, Baer group

MSC 2000 classification: primary 51E23, secondary 51A40

1 Introduction

Recently, the authors [8] have determined the translation planes of order 81 admitting $SL(2,5)$, generated by elations, using the computer. In particular, there are five mutually non-isomorphic non-Desarguesian planes with spreads in $PG(3,9)$, of which only the Prohaska plane was previously known. The question is how much of the computer use is actually required for the construction of these spreads. Of particular interest is that one of the new planes may be constructed from a Desarguesian plane by replacement of a 12-nest, a set of 12 reguli that overlap such that each component lies on two reguli. In this setting, the replacement consists of 5, i. e. 'half', of the lines of each opposite regulus. This is a very rare situation. In this article, we show that all of the planes can be constructed without the use of the computer and classified as to their isomorphism type. Furthermore, with the assumption that when $SL(2,5)$ acts

[i]The ideas for this article were developed when the first author was visiting Caledonian University and Heriot-Watt University in May of 2004. The authors are grateful to the Universities and to the London Mathematical Society for support on this research and the visit to Scotland.

as above there are exactly six orbits of components of length 12, then we may determine all planes with spreads in $PG(3,9)$, admitting $SL(2,5)$, using only the group $SL(2,5)$.

2 The constructions

Let Σ denote an affine Desarguesian plane of order 81 coordinatized by K isomorphic to $GF(81)$. Let b in $F \subseteq K$, F isomorphic to $GF(9)$, such that $b^2 = -1$. Then we note the following:

1 Lemma.

(1) $\left\langle \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\rangle \simeq SL(2,5)$. *The group induces* A_5 *on the parallel classes, the central involution of* $SL(2,5)$ *is the kernel involution of* Σ.

(2) The 10 *Sylow* 3*-subgroups of* $SL(2,5)$ *are elations in* Σ *and the set of elation axes defines a regulus net* R *of* Σ*, all of whose Baer subplanes incident with the zero vector are fixed by* $SL(2,5)$.

(3) The six Sylow 5*-subgroups each fix exactly two components of* $\Sigma - R$. *Hence, there is an orbit* Γ_{12} *of length* 12 *under* $SL(2,5)$*, as the normalizer of a Sylow* 5*-subgroup* S_5 *of order* 20 *interchanges the two fixed components of* S_5.

(4) There is a component orbit Γ_{60} *of length* 60 *and* $\Sigma = R \cup \Gamma_{12} \cup \Gamma_{60}$.

(5) Γ_{12} *does not contain a regulus net.*

(6) Γ_{60} *contains a unique regulus invariant by the normalizer of a given Sylow* 5*-subgroup. There are six such reguli whose union is* Γ_{60}.

Proof. Most of this is established in Prohaska [9]. QED

2 Lemma. *Let* τ *be an element of order* 5 *in* $SL(2,5)$*, as above. There is a unique Desarguesian spread* $\Sigma^{\langle\tau\rangle}$ *consisting of* $\langle\tau\rangle$*-invariant* 2*-dimensional* F*-subspaces.*

In particular, $\Sigma^{\langle\tau\rangle}$ *contains the opposite regulus* R^{Opp} *and the two* $\langle\tau\rangle$*-invariant components of* Σ.

$\Sigma^{\langle\tau\rangle}$ *admits as a collineation group, the normalizer of* $\langle\tau\rangle$ *in* $SL(2,5) \times Z_{80}$.

Proof. Note that 5 is a 3-primitive divisor of 81 and $SL(2,5)$ fixes all Baer subplanes of R, incident with the zero vector. Then, by Johnson [7], there is a unique Desarguesian spread $\Sigma^{\langle\tau\rangle}$, of τ-invariant linesize subspaces. QED

3 Lemma. *Let $\mathcal{H}^\tau$ denote the linear set of $q-1$ mutually disjoint reguli union the two $\langle\tau\rangle$-invariant components L and M of Σ, whose union is $\Sigma^{\langle\tau\rangle}$; i. e. the carrying lines of the hyperbolic fibration are L and M. Then there are exactly two reguli of $\mathcal{H}^\tau$ that are invariant under an element of order 4 whose square is the kernel involution of Σ, that interchanges L and M.*

Proof. Choose a representation for Σ^τ such that L and M are $x=0$, $y=0$. Then we have that the reguli are the standard André reguli $A_\delta = \{y = xm\ ;\ m^{q+1} = \delta\}$, for $\delta \in F - \{0\}$. The opposite lines have the form $y = x^q n$; $n^{q+1} = \delta$. The involution interchanging $x=0$ and $y=0$ is $(x,y) \longmapsto (-y,x)$ and maps A_δ onto $A_{\delta^{-1}}$, and hence fixes exactly two; where $\delta = \pm 1$. QED

4 Lemma. *Let P^τ denote the unique regulus of Γ_{60} that is left invariant under $N_{SL(2,5)}(\langle\tau\rangle)$. Then $P^{\tau Opp}$ is a regulus of Σ^τ.*

Proof. Let π_o be any component of $P^{\tau Opp}$ and assume that π_o is not a component of Σ^τ. Then π_o is a Baer subplane and defines a regulus P_1 of Σ^τ. Let Z_{80} denote the kernel homology group of Σ that now acts on Σ^τ as a collineation group having orbits of length 10 on $\Sigma^\tau - \Sigma$. It follows easily that $P^{\tau Opp} = \pi_o Z_{80}$, so the subplanes lie across $P_1 = P^{\tau Opp}$.

Now we have that Σ^τ contains R^{Opp}, P_1, union L and M. There is a unique linear hyperbolic fibration generated by R^{Opp}, P_1 with carrying lines L and M, since L and M are inverted by the normalizer of $\langle\tau\rangle$ in $SL(2,5)$. Replace all of the $q-1$ reguli of this linear hyperbolic fibration, obtaining R, P_1^{Opp}, L and M are components. However, there is a unique Desarguesian spread containing R and L. Hence, it follows that R, L, M and $P^{\tau Opp}$ are in Σ (define components of Σ), a contradiction. Hence, $P^{\tau Opp}$ is in Σ^τ. QED

5 Lemma. *Let the $q-1=8$ reguli of the plane Σ multiply-derived from Σ^τ be denoted by R, P^τ, R_3, R_4, R_5, R_6, R_7, R_8, where we may assume that R_5, R_6, R_7, R_8 are subnets of Γ_{60}. Moreover, we may assume that R_5 and R_6, and R_7 and R_8 are interchanged by the normalizer of $\langle\tau\rangle$.*

Proof. Let N be any component other than L,M of Γ_{12}. Then there exists a unique regulus of the linear set, say R_3 that contains N. Since all of these nets are τ-invariant, this says that R_3 shares at least 5 components with Γ_{12}. However, by a lemma above, Γ_{12} does not contain a regulus. Hence, the other five components of $\Gamma_{12} - \{L, M\}$ are contained in another unique regulus from the linear set, say R_4. Note that R_3 and R_4 are then interchanged by the normalizer of $\langle\tau\rangle$, since we know that this normalizer fixes exactly two reguli of the linear set, namely R and P^τ. QED

6 Lemma. *Γ_{60} has exactly two orbits Δ_i, $i = 1,2$, of 1-dimensional F-subspaces of lengths $60 \cdot 5$ under $SL(2,5) \times Z_5$.*

Proof. The order of $SL(2,5) \times Z_5$ is $120 \cdot 5$ and the kernel involution of $SL(2,5)$ fixes every 1-dimensional F-subspace. If X_o is a 1-dimensional F-subspace in Γ_{60}, it lies on a unique component of Γ_{60}, which is in an orbit of length 60. But Z_5 is a kernel homology subgroup and cannot fix any 1-dimensional subspace, but fixes each component. Hence, the orbit lengths are as maintained. QED

7 Lemma. *Consider any of the τ-invariant reguli R_5, R_6, R_7, R_8 that lie in Γ_{60}, and let π_o be a subplane of any of these reguli and note that any subplane of R_i, $i = 5, 6, 7, 8$, is τ-invariant, since these arise from Σ^τ, where τ acts as a kernel homology group.*

Let $\pi_o = C_1 \cup C_2$, where C_1 and C_2 are orbits of 1-spaces of π_o under $\langle\tau\rangle$.

Then C_1 and C_2 are in distinct $SL(2,5) \times Z_5$ orbits.

This is also true of any τ-invariant subplane of P^τ.

Proof. This may be easily determined by use of the computer, as we have done previously. We sketch how this would be proven without the computer. Map $x = 0$, $y = 0$, $y = x$ of R in Σ to the André net $A_1 = \{y = xm \,;\, m^{q+1} = 1\}$ by mapping $y = 0$ to $y = -x$, $y = x$ to $y = xz_0^{-1}$ and $x = 0$ to $y = xz_0$ such that $z_o^{q+1} = 1$, z_o not ± 1. This may be accomplished by the collineation:

$$\begin{bmatrix} z_o - 1 & 1 - z_o \\ 1 & z_o \end{bmatrix} = j.$$

If one works out the two unique components $y = xM_i$, $i = 1, 2$, fixed by

$$\tau = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(which actually has order 10), we see that these components are uniquely determined by the following quadratic equation:

$$M_i^2 + M_i b - b = 0.$$

Now choose w so that $w^2 = b - 1$ and let $z = w + (1 - b)$.

Then it may be verified that $(w + (1 - b))^{q+1} = 1$.

Then it also may be verified that $y = xM_i$ map to $x = 0$, $y = 0$. What this means is that we may assume that there is an element j so that

$$j^{-1} \left\langle \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\rangle j \times Z_5$$

acts so that $j^{-1}\tau j$ fixes $x = 0$ and $y = 0$, R becomes A_1 and P^τ becomes A_{-1}. In this setting, we may easily calculate the new orbit $j^{-1}\Gamma_{12} j$. Note that now the

$j^{-1}\tau j$-invariant Baer subplanes of $j^{-1}R_i j$ for $i = 5, 6, 7, 8$ have the nicer form $y = x^q t$, for t in $K - \{0\}$. In this form, we see that $j^{-1}\tau j\colon (x, y) \longmapsto (xa, ya^q)$; a has order 5. Furthermore, the normalizer element of order 4 has the form: Let ω have order 4 in K, then $(x, y) \longmapsto (-y\omega^{-i}, x\omega^i)$. Note that A_1 and A_{-1} are the only André nets invariant under this element.

Choose any $y = x^q t$ in $j^{-1}R_i j$, for $j = 5, 6, 7, 8$, and letting $C_1 \cup C_2 = (y = x^q t)$, by a calculation it may be shown that C_1 and C_2 are in distinct $j^{-1}SL(2, 5)j \times Z_5$ orbits.

To see that this is also valid for the τ-invariant subplanes of the regulus P^τ, we note that if π_o is in P^τ, then the normalizer of $\langle\tau\rangle$ in $SL(2, 5)$ leaves P^τ invariant and maps π_o to another subplane of P^τ. It follows that in $SL(2, 5) \times Z_5$, there are 10 subplanes of P^τ; all of the subplanes incident with the zero vector. The assertion regarding the orbit structure of the subplanes is then clear. QED

8 Lemma. $\Delta_1 = (SL(2, 5)\times Z_5)C_1$, $\Delta_2 = (SL(2, 5)\times Z_5)C_2$. *If* $\pi_o = C_1\cup C_2$ *is a* τ*-invariant Baer subplane of* R_i*, for* $i = 5, 6, 7, 8$*, then* Γ_{60} *has a replacement of* $(SL(2, 5) \times Z_5)\pi_o$.

Proof. Suppose that $C_1 g\cap C_1$ in a 1-space X_o. Then there exists a 1-space Y_o in C_1 such that $X_o\tau^j = Y_o$. Thus, $Y_o g = X_o$, implying that $\tau^j g$ fixes Y_o. Hence, $\tau^j g$ is either trivial or the kernel involution i_2. In any case, g is in $\langle\tau, i_2\rangle$. But this group leaves C_1 invariant. Hence, $C_1 g\cap C_1$ is either C_1 or is the empty set (on 1-subspaces).

This then also means that $C_2 g = C_2$ or is disjoint from C_2. Now consider π_o and $\pi_o g = (C_1 \cup C_2)g = (C_1 g \cup C_2 g)$ and assume that $\pi_o \cap \pi_o g$ non-trivially in a 1-subspace X_o, where g is in $SL(2, 5) \times Z_5$. Since C_1 and C_2 are in distinct $SL(2, 5)\times Z_5$-orbits, it follows that $C_1\cap C_2 g$ is necessarily trivial as is $C_2\cap C_1 g$. Therefore, if there is an intersection, it can only be between C_1 and $C_1 g$ or between C_2 and $C_2 g$, which we have seen above implies that $C_1 g = C_1$, or $C_2 = C_2 g$. Assume the former. Then the regulus R_i containing π_o and the regulus $R_i g$ now share at least five components and hence are equal. But now π_o and $\pi_o g$ are in the same regulus net and share C_1 so are equal: $\pi_o = \pi_o g$. But note this also says that g is in $\langle\tau, i_2\rangle$, since the normalizer 4-element does not fix R_i. This says that there are exactly 60 disjoint images of π_o under the group $SL(2, 5) \times Z_5$, so we obtain a replacement. QED

9 Theorem. *There is a unique translation plane of order* 81 *admitting* $SL(2, 5) \times Z_5$ *that may be obtained from a Desarguesian affine plane of order* 81 *by* 12*-nest replacement.*

Proof. In the previous lemma, we have, if we choose any subplane π_o of R_i, for $i = 5, 6, 7, 8$, a replacement for Γ_{60} consisting of images of π_o under the group

$SL(2,5) \times Z_5$. Note that there are 40 possible subplanes. In each replacement net, there are five images of π_o under Z_5 that lie in the same original regulus net R_i. Since R_i and R_j, for $i \neq j$, are interchanged by the normalizer of $\langle \tau \rangle$, it follows that there are five subplanes of a second regulus R_j, any of which will produce the same replacement set.

If we take the kernel homology group Z_{80}, this sets up an isomorphism between the replacement using π_o or any subplane of $\pi_o Z_5$ with the subplanes of R_i in the second Z_5-orbit of subplanes of R_i. This means that if we take any of 20 different subplanes of R_i and R_j we obtain an isomorphic replacement set.

Now consider the original group representation and the collineation θ:

$$(x, y) \longmapsto (x^3, y^3).$$

Note that:

$$\theta^{-1} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \theta = \begin{bmatrix} 1 & b^3 \\ 0 & 1 \end{bmatrix}$$

and since $b^2 = -1$, $b^3 = -b$, so that θ normalizes $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ and similarly normalizes $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Hence, θ will normalize $SL(2,5) \times Z_5$ in the original representation. But this means that there is a subgroup of order 80 that normalizes $\langle \tau \rangle$ and normalizes $SL(2,5) \times Z_5$. Thus, this group acts on Γ_{60} and permutes five τ-invariant reguli and must fix one, namely P^τ (it is easy to verify that 1-subspaces of each Baer subplane of P^τ are in the same $SL(2,5) \times Z_5$ orbit, so this group must fix P^τ). Hence, we have a group of order 80 that permutes four reguli. We claim that this group is transitive. In order to see this, we change representations again and look at the André linear set with carrying lines $x = 0$, $y = 0$.

Consider the André linear hyperbolic fibration with carrying lines $x = 0$, $y = 0$. Here, we have the group $(x, y) \longmapsto (x^3, y^3)$ mapping A_δ onto A_{δ^3} and the mapping $(x, y) \longmapsto (-y\omega^i, x\omega^{-i})$, such that $\omega^4 = 1$ taking A_δ onto $A_{\delta^{-1}}$. Since it follows that $\delta^{-1} = \delta^3$, for δ in $GF(9)$, if and only if $\delta^4 = 1$, we see that we have two orbits of length 2 and one orbit of length 4 of the 8 reguli. Note that this group has order 2^4. Note that we can similarly show that any group of order 2^4 that has two orbits of length 2 on reguli will have an orbit of length 4. Since this general situation will be similarly represented under our representation, we see that we have that $\{R_5, R_6, R_7, R_8\}$ is an orbit under a group that normalizes $SL(2,5) \times Z_5$. Hence, it follows that any subplane of one of these regulus nets will produce an isomorphic replacement set.

It remains to show that we have a 12-nest replacement. Given a subplane π_o, say of R_5, we obtain using Z_5 that each component N of R_5 is contained in exactly two reguli. Since R_5 is inverted with R_6 using the normalizer of $\langle\tau\rangle$, it follows that we have used exactly 12 reguli to cover Γ_{60} and each component lies in exactly two reguli and we have used $(q+1)/2$, i. e. half of the lines of each opposite regulus in the replacement procedure. $\boxed{QED}$

10 Theorem. *There are exactly four translation planes of order* 81 *and kernel* $GF(9)$ *that admit* $SL(2,5)\times Z_5$*:*

(1) the Prohaska, and

(2) 12*-nest plane, where* $SL(2,5)$ *is generated by elations, and*

(3) the derived Prohaska, and

(4) the derived 12*-nest plane, where* $SL(2,5)$ *is generated by Baer* 3*-elements.*

Proof. When the kernel is $GF(9)$ and the 3-elements are Baer, Jha and Johnson [6] have shown that the Baer axes line up into a derivable net and hence a regulus net. Hence we may assume that the 3-elements are elations. Furthermore, Jha and Johnson [6] have shown that the 3-elements are always Baer or elations. And when the 3-elements are Baer, Jha and Johnson [6] have shown that the Baer subplanes pointwise fixed by the 3-elements are disjoint as subspaces.

In any case, Z_5 must fix at least 10 linesize $GF(9)$-subspaces. So, by Johnson [7], there is a unique Desarguesian plane Σ containing the Z_5-fixed subspaces and the normalizer of Z_5 acts as a collineation group of Σ. Hence, $SL(2,5)\times Z_5$ acts on Σ, a Desarguesian plane of order 81. Consider the plane π and note that any component of π that is not in Σ becomes a Baer subplane of Σ. We know that $SL(2,5)$ is generated by elations acting on Σ and there are orbits of components of lengths 10, 12, 60 in Σ. Let L be a component of π that is a Baer subplane of Σ. We know that the orbit Γ_{10} of Σ is also an orbit of π, the set of 10 elation axes. Furthermore, since $SL(2,5)\times Z_5$ acts on π, we know that Z_5 permutes the set of 72 components external to the net Γ_{10} of 10 elation axes. Thus, Z_5 fixes at least two components of $\pi-\Gamma_{10}$. Thus, π shares components with Σ in an orbit of length 12 or of length 60. Assume the latter. So, we have π a plane with spread in $PG(3,9)$ that shares 72 components with Σ, so that either $\pi=\Sigma$ or Γ_{12} is a replaceable net that does not contain a regulus. This is a contradiction to Bruen [3]. Hence, π shares Γ_{12} with Σ. If L is a component of π that is a Baer subplane of Σ, we know that the components that L lies over in Σ form a regulus R_L. Now R_L is a regulus embedded in Γ_{60} of Σ. So, the union of the $SL(2,5)$-orbits of $\pi-\Gamma_{10}\cup\Gamma_{12}$ form a replacement for the net Γ_{60}. There are exactly 10 $SL(2,5)$-orbits on 1-dimensional $GF(9)$-subspaces, each of length 60. Since we have Z_5 acting as a kernel homology group of Σ, the net R_L

has five Baer subplanes which are lines of π. We consider the orbit of R_L under $SL(2,5)$. Suppose it has length > 12. Then the 5 subplanes per image would force a larger than 60 partial spread. Hence, the orbit of R_L is of length ≤ 12. If the orbit has length 12, we have a 12-nest and if the orbit has length 6, we obtain a Prohaska spread. Since we have $SL(2,5)$ acting and note that the orbit length is divisible by 3, we may have only these two possibilities. It now remains to show that any translation plane admitting $SL(2,5) \times Z_5$ has six component orbits of length 12. Clearly, we have an orbit of length 10 and at least one of length 12. Whenever we have a component L as above, we construct a regulus R_L of Γ_{60} of Σ. In this case, since Z_5 acts as a kernel homology of Σ, it follows that we have five Baer subplanes of R_L that are components of π. Clearly, the orbit of R_L then has length 6 or 12 and we then see that Z_5 permutes one of the orbits of length 12; that is, we must have an additional orbit of length 12 and hence five more. QED

11 Corollary. *Let Σ be a Desarguesian plane of order 81 that admits $SL(2,5)$ as a collineation group. Let π_o be any subplane of a τ-invariant regulus net that sits in Γ_{60} then $SL(2,5)\pi_o$ is a partial spread of cardinality 12 that contains exactly two τ-invariant components (π_o and $\pi_o N_{SL(2,5)}(\langle\tau\rangle)$).*

Proof. Since $SL(2,5) \times Z_5\pi_o$ is a partial spread of cardinality 60, we have the proof of first part of the corollary, using the proof of the previous theorem. QED

3 The orbit constructions

By the previous section, we know that there are two orbits O_1 and O_2 in Γ_{60} of $SL(2,5) \times Z_5$. Hence, O_i has five $SL(2,5)$ orbits O_i^j, for $j = 1,2,3,4,5$ and $i = 1,2$, that are permuted cyclically by Z_5.

Moreover, for any τ-invariant subplane π_o, there are exactly two τ-invariant subplanes in $SL(2,5)\pi_o$, and each is a partial spread of cardinality 12. However, we shall be interested in the 'τ-5-orbits' or 'τ-5's', the images of 1-dimensional subspaces of Γ_{60} under $\langle\tau\rangle$.

12 Lemma. *There are exactly 12 τ-5's in each orbit O_i^j, $j = 1,2,3,4,5$, $i = 1,2$, of which there are two each in P^τ, R_5, R_6, R_7, R_8 and one each in R_3, R_4.*

Proof. Simply note that there are $60 \cdot 10/5 = 12 \cdot 10$ $\langle\tau\rangle$-orbits of length 5 and these must be partitioned equally into the 10 orbits of $SL(2,5)$. QED

13 Lemma. *For a given O_i^s, consider the 10 τ-5's that are in $\{P^\tau, R_5, R_6, R_7, R_8\}$.*

Given any τ-5 A in O_i^s, there is a second τ-5 B in O_i^s such that there are unique corresponding τ-5's C and D in certain O_j^k's such that $A \cup C$ and $B \cup D$ are τ-invariant subplanes such that $SL(2,5)(A \cup C) = SL(2,5)(B \cup D)$.

PROOF. We know that the τ-invariant subplanes π_o split into two τ-5's in different $SL(2,5) \times Z_5$-orbits. And, we know that $SL(2,5)\pi_o$ contains exactly two τ-invariant subplanes π_o and π_o' such that $SL(2,5)\pi_o = SL(2,5)\pi_o'$. QED

14 Notation. In O_1^j, label the τ-5's in pairs $\{C_{1,k}^j, \widehat{C_{1,k}^j}\ ; k = 1,2,3,4,5\}$ that have corresponding τ-5's in various O_2^ws such that these pairs of pairs generate the same $SL(2,5)\pi_{k,j}$. Also note that $SL(2,5)\pi_{k,j}$ is a union of two $SL(2,5)$ orbits, one is O_1^j and one is O_2^w, for some w.

15 Lemma. *Choose two τ-5's in O_1^j, $C_{1,k}^j$ and $C_{1,r}^j$. Then the uniquely defined τ-5's in O_2, say $B_{2,j}^k$ and $B_{2,j}^r$, cannot lie in the same O_2^w. This implies that the 5 τ-5's in O_1^j, $C_{1,k}^j$, for $k = 1,2,3,4,5$, have corresponding τ-5's, one each in the orbits O_2^w, for $w = 1,2,3,4,5$.*

Hence, we choose the notation so that the corresponding τ-5 of $C_{1,k}^j$ is denoted by $B_{2,j}^k$ in O_2^k, for $k = 1,2,3,4,5$.

PROOF. Suppose so; then $SL(2,5)\left(C_{1,k}^j \cup B_{2,j}^k\right) = SL(2,5)\left(C_{1,r}^j \cup B_{2,j}^r\right)$. However, this says that there is a partial spread of cardinality 12 that has a proper replacement. By Bruen [2], this says that there must be a derivable net within $SL(2,5)$, a contradiction, or the smallest replaceable net has cardinality $2(q-1) = 2(8) = 16$, also a contradiction. QED

16 Notation. We emphasize the following notation: For each $j = 1, 2, 3, 4, 5$, in O_1^j there are five τ-5's $C_{1,k}^j$, for $k = 1, 2,3, 4, 5$. The corresponding τ-5, $B_{2,j}^k$, is in O_2^k. Hence O_2^k contains $B_{2,j}^k$ such that $j = 1,2,3,4,5$ is the corresponding τ-5 of $C_{1,k}^j$ in O_1^j.

3.1 The orbit replacement theorem

17 Theorem.

(1) *For O_1^1, choose any of the five C_{1,k_1}^1 and locate the corresponding $B_{2,1}^{k_1}$ so that*

$$SL(2,5)\left(C_{1,k_1}^1 \cup B_{2,1}^{k_1}\right)$$

is a partial spread of degree 12, which is the union of two $SL(2,5)$ orbits O_1^1 and $O_2^{k_1}$.

(2) Then for O_1^2, choose any of the five C_{1,k_2}^2 whose corresponding $B_{2,2}^{k_2}$ is not in $O_2^{k_1}$; there are four possible choices. Then

$$SL(2,5)\left(C_{1,k_2}^2 \cup B_{2,2}^{k_2}\right)$$

is a partial spread of degree 12 and

$$SL(2,5)\left(C_{1,k_1}^1 \cup B_{2,1}^{k_1}\right) \cup SL(2,5)\left(C_{1,k_2}^2 \cup B_{2,2}^{k_2}\right)$$

is a partial spread of degree 24.

(3) For O_1^3, choose any of the five C_{1,k_3}^3 whose corresponding $B_{2,3}^{k_3}$ are not in $O_2^{k_1}$ or $O_2^{k_2}$; there are three possible choices. Then:

$$SL(2,5)\left(C_{1,k_1}^1 \cup B_{2,1}^{k_1}\right) \cup SL(2,5)\left(C_{1,k_2}^2 \cup B_{2,2}^{k_2}\right) \cup SL(2,5)\left(C_{1,k_3}^3 \cup B_{2,3}^{k_3}\right)$$

is a partial spread of degree 36.

(4) Similarly for O_1^4, choose any of the two C_{1,k_4}^4 whose corresponding $B_{2,4}^{k_4}$ are not in $O_2^{k_s}$, for $s = 1,2,3$, and finally for O_1^5, choose the remaining C_{1,k_5}^5, whose corresponding $B_{2,5}^{k_5}$ is not in $O_2^{k_s}$, for $s = 1,2,3,4$. Then

$$\bigcup_{j=1}^{5}\bigcup_{s=1}^{5} SL(2,5)\left(C_{1,k_s}^j \cup B_{2,j}^{k_s}\right)$$

is a partial spread of degree 60 that replaces Γ_{60}.

(5) For any permutation σ of $\{1,2,3,4,5\}$, there is a translation plane π_σ of order 81 with spread in $PG(3,9)$ that admits $SL(2,5)$, where the 3-elements are elations. π_σ has spread:

$$\Gamma_{10} \cup \Gamma_{12} \cup \bigcup_{j=1}^{5}\bigcup_{s=1}^{5} SL(2,5)\left(C_{1,k_s}^j \cup B_{2,j}^{k_s}\right)$$

where $\sigma(s) = k_s$.

Hence, there are $5!$ possible spreads.

PROOF. We note that in each $SL(2,5)\left(C_{1,k_s}^j \cup B_{2,j}^{k_s}\right)$, we have a union of two distinct $SL(2,5)$ 1-space orbits. Since when we vary across O_1^j, we choose the C_{1,k_s}^j so that the corresponding $B_{2,j}^{k_s}$ lies in an orbit $O_2^{k_s}$, not previously selected, we are simply taking the union of the $SL(2,5)$ orbits, pairs at a time. Since these are disjoint, any union of these forms a partial spread of degree 12, 24, 36, 48, or 60. QED

18 Corollary. *Let π_o be any τ-invariant 2-dimensional $GF(9)$-subspace. Then $SL(2,5)\pi_o$ is a partial spread of degree 12.*

PROOF. Since this is true for the τ-invariant subplanes lying in Γ_{60}, we may consider the Desarguesian spreads containing Γ_{10} and realize that this result is more generally true in the vector space. $\boxed{QED}$

19 Theorem. *There are exactly $6!$ non-Desarguesian spreads in $PG(3,9)$ admitting $SL(2,5)$, where the 3-elements are elations, constructed by the replacement of six $SL(2,5)\pi_i$, $i = 1,2,3,4,5,6$, where π_i is a τ-invariant 2-dimensional $GF(9)$-subspace.*

PROOF. If we consider the group $SL(2,5)$ acting on the 4-dimensional vector space V_4, where the $SL(2,5)$ acts so that the 3-elements are elations, then there are 12 $SL(2,5)$-orbits on $V_4 - \Gamma_{10}$, $O_1^1, \ldots, O_1^6$ and $O_2^1, \ldots, O_2^6$, where each τ-invariant 2-dimensional $GF(9)$-subspace is a union of two τ-5's in different $SL(2,5) \times Z_5$ orbits. What this means is that each $SL(2,5)$-orbit contains 12 τ-5's and each τ-5 in O_1^i corresponds to a τ-5 in one of the O_2^j's. However, no two of the 12 τ-5's in O_1^i have a corresponding τ-5 in the same O_2^j. Hence, we may repeat the proof of the Orbit Replacement Theorem and construct a set of six $SL(2,5)\pi_i$'s. Note that there are two τ-invariant subspaces in each $SL(2,5)\pi_i$, which means that the τ-5's are paired just as before. Hence, for each $j = 1,2,3,4,5,6$, we take six $C_{1,k}^j$ τ-5's in O_1^j with corresponding τ-5's $B_{2,j}^k$ in O_2^k, for $k = 1,2,3,4,5,6$; then, using the avoidance principle established in the previous theorem, we have six possible choices for C_{1,k_1}^1, then five choices for C_{1,k_2}^2, etc., producing a set of exactly $6!$ partial spreads of degree 72 which when unioned with Γ_{10} are spreads admitting $SL(2,5)$. This completes the proof. $\boxed{QED}$

Note we now know how to choose the Prohaska plane. In particular, we need to choose the unique τ-5 in P^τ, for each O_1^j. Furthermore, there are four distinct ways to choose a 12-nest spread, all of which are isomorphic. What this means is that if $C_{1,1}^j$, for $j = 1,2,3,4,5$, denotes the choice for the Prohaska spread P^τ (using the same notation for two different sets), and $C_{1,k}^j$, for $j = 1,2,3,4,5$, denotes the four 12-nest spreads N_k for $k = 2,3,4,5$, we have an indexing forming the following 5×5 matrix:

$$\begin{bmatrix} C_{1,1}^1 & C_{1,2}^1 & C_{1,3}^1 & C_{1,4}^1 & C_{1,5}^1 \\ C_{1,5}^2 & C_{1,1}^2 & C_{1,2}^2 & C_{1,3}^2 & C_{1,4}^2 \\ C_{1,4}^3 & C_{1,5}^3 & C_{1,1}^3 & C_{1,2}^3 & C_{1,3}^3 \\ C_{1,3}^4 & C_{1,4}^4 & C_{1,5}^4 & C_{1,1}^4 & C_{1,2}^4 \\ C_{1,2}^5 & C_{1,3}^5 & C_{1,4}^5 & C_{1,5}^5 & C_{1,1}^5 \end{bmatrix}$$

In this matrix, the choice of any set consisting of five elements, one element from each row and column, produces the $5!$ spreads. The notation is possible due to the selection of corresponding τ-5's. The selection from row 1 uses O_1^1 and a corresponding $O_2^{k_1}$, the selection from row 2 cannot use this particular $O_2^{k_1}$ so one of the choices of τ-5's of O_1^2 is restricted. Thus, the choice that must be avoided is the one that we place directly below our earlier choice. In this matrix, there is a unique way to obtain the Prohaska P^τ and four ways to obtain a 12-nest; a unique way to obtain N_k, for $k = 2, 3, 4, 5$.

20 Corollary. *There are exactly* 36 *Desarguesian spreads:*

$$\Sigma_i,\ i = 1, 2, \ldots, 36,$$

containing Γ_{10} *and admitting* $SL(2,5)$ *as a collineation group, each of which produces* $5!$ *non-Desarguesian spreads in* $PG(3,9)$*.*

(1) These 36 *Desarguesian spreads correspond to taking any of the* 36 $SL(2,5)\pi_i$ *partial spreads and finding the unique Desarguesian spread containing* $SL(2,5)\pi_i$ *and* Γ_{10}*. Hence, with multiplicity* 6*, there are* $36\ (5!)$ *spreads.*

(2) There are exactly 36 *Prohaska spreads, a unique Prohaska spread defined by each Desarguesian spread* Σ_i*.*

(3) There are exactly 144 12*-nest spreads,* 4 *defined by each Desarguesian spread* Σ_i*.*

If a translation plane of order 81 *with spread in* $PG(3,9)$ *has six orbits of length* 12*, the plane must be constructed as above.*

Proof. In any translation plane of order 81 admitting $SL(2,5)$, where the 3-elements are elations, where the spread is in $PG(3,9)$, it is possible to show that the component orbits have lengths 12, 30 or 60. Furthermore, there must be a 12-orbit, Γ_{12}, and the orbit Γ_{10} of elation axes into a unique Desarguesian spread Σ. Recall that there is an orbit Γ_{60} of length 60 under $SL(2,5)$ in Σ. The remaining part of the translation plane must lie over Γ_{60} so this partial spread is a replacement net for Γ_{60}. There is a unique τ-invariant Desarguesian plane Σ^τ containing the τ-invariant 2-dimensional $GF(9)$-subspaces. Since some of the latter must lie in Γ_{60} as Baer subplanes, it follows that τ must fix a set of five reguli that lie in Γ_{60}.

However, we simply avoid this situation by assumption.

So, any plane has $SL(2,5)$ as a normal subgroup (unless $SL(2,9)$ is generated, implying that the plane is Desarguesian) and there are 6 $SL(2,5)\pi_i$'s that are permuted by the full collineation group as these are component orbits of $SL(2,5)$. Suppose that g is a collineation of π that fixes each $SL(2,5)\pi_i$. Then clearly g is a collineation of Σ and since we have g normalizing $SL(2,5)$, g acts on 10, 12, 60 components of Σ. Furthermore, since Γ_{60} is an orbit, we may assume that g fixes a component of Γ_{60}. If the order of g is 3, we clearly have a contradiction, as then g would be an elation of Σ. Hence, g fixes two components of Γ_{60}. Assume that g is in $GL(2,81)$ acting on Σ. Then either g is a kernel homology group of Σ or g fixes exactly two components of Σ and then the order of g divides $(80,58,10,12) = 2$. So, we may assume that we have an affine homology with axis and coaxis in Γ_{60}. However, the normalizer of $SL(2,5)$, modulo $SL(2,5)$, in $GL(4,q)$, centralizes $SL(2,5)$, so that this cannot occur. Hence, g is a kernel homology of Σ that leaves each $SL(2,5)\pi_i$ invariant. However, there are exactly two τ-invariant subplanes of Σ in each $SL(2,5)\pi_i$, so that g has order dividing 16. Furthermore, unless the plane is Prohaska, the two subplanes in $SL(2,5)\pi_i$, for some i, are in different τ-invariant reguli sitting in Σ. This means that g is in the $GF(9)$-kernel homology group, when the spread is not Prohaska. So, if the spread is not Prohaska, the kernel of the action in $GL(4,9)$ is the $GF(9)$-kernel homology group.

The normalizer of $SL(2,5)$, modulo $SL(2,5)$, is $\langle GL(2,9),\alpha\rangle$, where α is the collineation arising from the Frobenius automorphism of $GF(9)$, of order 2 acting in $\Gamma L(4,9)$.

We have seen that there is a unique way to choose a Prohaska spread from each Desarguesian spread containing Γ_{10} (where the $SL(2,5)\pi_o$ that is in the Desarguesian spread is left invariant under Z_5). Hence, there are exactly 36 Prohaska spreads, all isomorphic since the 36 Desarguesian spreads form an orbit under the normalizer of $SL(2,5)$. Similarly, there are exactly four ways to choose a 12-nest spread in a Desarguesian plane, so that are exactly $36\cdot 4 = 144$ 12-nest spreads, all isomorphic. $\boxed{QED}$

We now consider the $SL(2,5)$-spreads that are not Prohaska or 12-nest spreads. Since there are $6!$ possible spreads, this leaves $720-36-144 = 540$ spreads to consider. We know that a collineation group of any associated translation plane π normalizes $SL(2,5)$ and permutes a set of six $SL(2,5)\pi_i$'s. Suppose that there is a collineation of order 5. Then this collineation centralizes $SL(2,5)$ and we obtain a collineation group isomorphic to $SL(2,5)\times Z_5$. However, this means that the spread is Prohaska or a 12-nest spread. Therefore, the orbit of isomorphic spreads must be divisible by 5 and, of course, by 9. We consider the action of the collineation group on the six $SL(2,5)\pi_i$'s, as a subgroup of S_6 that contains no elements of odd order. Hence, the group induced on the six $SL(2,5)\pi$'s is an even-order subgroup. Suppose that the order is at least 8. Then there is a subgroup of order 4 that fixes two $SL(2,5)\pi_i$'s and hence may be considered a collineation g of some Desarguesian plane (actually two Desarguesian planes) containing Γ_{10}. In this case, we have seen above that $GL(2,81)$ contains only the $GF(9)$-kernel homology group. But then we can only have that g is α of order 2 or in the $GF(9)$-kernel homology group. In the latter case, then g is trivial acting on the six $SL(2,5)\pi_i$'s. Hence, the collineation group of π that fixes some $SL(2,5)\pi_1$ has order at most 2 modulo the $GF(9)$-kernel homology group. So, the collineation group of π has order at most 8, module $SL(2,5)\times Z_8$. Since the normalizer of $SL(2,5)$ modulo $SL(2,5)$ has order $9\cdot 80\cdot 8\cdot 2$, it follows that there are at least $9\cdot 80\cdot 8\cdot 2/(8\cdot 8) = 180$ planes isomorphic to π. Since there are 540 spreads remaining, we have at most 3 mutually non-isomorphic planes. Furthermore, there are either 3 planes or 2 planes since the Sylow 3-subgroup of $GL(2,9)$ has order 9. However, we may choose the set of six $SL(2,5)\pi_i$'s relative to some Desarguesian plane in $120-5$ ways to get one of these possibly three spreads. We note that there is always an orbit of length less than or equal 2 of $SL(2,5)\pi_i$'s. We then may choose the $SL(2,5)\pi_i$'s to have a group of order 8, modulo the $GF(9)$-kernel homology group.

Hence, there are exactly three mutually non-isomorphic planes.

21 Corollary. *There are exactly six mutually non-isomorphic spreads admitting $SL(2,5)$, where the 3-elements are elations, provided in the non-Desarguesian case that there are six component orbits of length 12:*

(1) Desarguesian,

(2) Prohaska,

(3) the 12-nest spread, and

(4) three spreads arising from 24 reguli in a Desarguesian affine plane.

There are exactly six mutually non-isomorphic spreads π in $PG(3,9)$ that admit $SL(2,5)$, generated by elations.

Proof. Let Γ_{10} denote the net defined by the ten axes of elations in $SL(2,5)$. Then since $SL(2,5)$ is generated by central collineations, it follows that $SL(2,5)$ leaves invariant each of the 10 Baer subplanes incident with the zero vector. Hence, by Johnson [7], there is a unique Desarguesian affine plane Σ^{τ} consisting of τ-invariant linesized subspaces, where τ has order 10. The normalizer of $\langle\tau\rangle$ in $SL(2,5)$ has order 20, and we may consider τ to be a kernel homology group of Σ^{τ}. It follows that the involution of $SL(2,5)$ is the kernel involution of π. We note that τ^2 must fix at least two components L and M of $\pi-\Gamma_{10}$. Hence, L and M are also components of Σ^{τ} and the normalizer of order 20 acts on the Desarguesian plane and is dihedral on the line at infinity. Hence, L and M are inverted by a collineation in the normalizer of $\langle\tau\rangle$. Let $\mathcal{H}$ denote the linear set of

$q-1$ reguli of Σ^τ with carrying lines L and M. Hence, Γ_{10}^* (derived) is in $\mathcal{H}$. If we multiply derive $\mathcal{H}$, we obtain a unique Desarguesian plane Σ containing Γ_{10}, L and M and admitting $SL(2,5)$. It follows that the L and M are in the unique orbit Γ_{12} of length 12 of $SL(2,5)$ acting on Σ. Hence, it follows that Γ_{12} is also a subnet of π. Hence, π and Σ share Γ_{10}, Γ_{12} and $SL(2,5)$ has an orbit Γ_{60} of length 60 on Σ and has five more orbits of length 12 on π. Hence, five orbits of length 12 of π lie across Γ_{60}. Each orbit of length 12 in π can only contain and must contain exactly two τ-invariant components. These are τ-invariant subplanes of Γ_{60}. However, we cannot be certain that we have constructed the plane using Γ_{60}. That is, there are exactly six orbits of length 12 for each plane. If we choose any of these orbits of length 12, we may embed Γ_{10} and this orbit in a unique Desarguesian spread containing Γ_{10}. The normalizer of $SL(2,5)$, modulo $SL(2,5)$, contains the group that centralizes $SL(2,5)$ and acts as an elation group on the Γ_{10}^{Opp}. That is, there is a subgroup isomorphic to $SL(2,9)$ that does this. Furthermore, the group $\alpha\colon (x,y) \longmapsto \left(x^3,y^3\right)$ of Σ normalizes $SL(2,5)$. In any case, since we are stabilizing a regulus, the group is a central product of $\Gamma L(2,9)GL(2,9)$ by a group of order 8. The normalizer of $SL(2,5)$ then is generated by $GL(2,9)$ and the Frobenius automorphism. We note that we are interested in the normalizer of $\langle\tau\rangle$, which has index 6 in $\langle\alpha, GL(2,9)\rangle$.

This shows that any isomorphism g between π_σ and π_ρ may be considered a Desarguesian collineation that fixes Γ_{10} and Γ_{12} and normalizes $SL(2,5)$. Since we may assume that the $SL(2,5)\pi_{\sigma,i}$ pieces are mapped by g onto the corresponding $SL(2,5)\pi_{\rho,k}$ pieces, and these are orbits themselves, we may assume that g maps some τ-invariant subplane of Σ in Γ_{60} to another τ-invariant subplane. Consider $\Sigma^\tau g$. Since g maps Γ_{10}^{Opp} back into itself and maps one τ-invariant subspace to another τ-invariant subspace, it follows that $\Sigma^\tau g = \Sigma^\tau$. Hence, g is a collineation of the two Desarguesian subspaces Σ and Σ^τ and normalizes $SL(2,5)$.

Since Z_5 is also normalized by g (automatically), and modulo $SL(2,5)$, the order of the group that leaves O_1 and O_2 invariant is exactly 20, it follows that we have exactly five mutually non-isomorphic planes, all constructed from a Desarguesian plane Σ by the method stated in the Orbit Replacement Theorem. Hence, in total, counting the Desarguesian spread, there are exactly six spreads admitting $SL(2,5)$, where the 3-elements are elations.

So, in general it would remain to show that any non-Desarguesian translation plane admitting $SL(2,5)$ generated by elations has exactly six component orbits of length 12. The above argument shows that there must be an orbit Γ_{10} of length 10, the elation axes, and at least one orbit Γ_{12} of length 12. Furthermore, it follows from arguments in Jha and Johnson [6] that there cannot be an orbit of length 6 or 15. Hence, all orbits are of length 12, 30 or 60. Furthermore, there is a unique Desarguesian spread Σ containing Γ_{10} and Γ_{12}, and, assuming that π is not Σ, all other orbits of components of π are orbits of Baer subplanes of Γ_{60} in the Desarguesian plane Σ. Consider the orbit of R_L, the regulus of Γ_{60} containing L. If R_L is not a τ-invariant regulus, for some element τ of order 5, then the orbit length is divisible by 3 and 5 and hence is either 15, 30, or 60. If the orbit length of R_L is 30, then the orbit length of L is either 30 or 60.

Again, we may take this as a hypothesis to simplify the argument. QED

4 The Baer case

Now assume that we have $SL(2,5)$ acting on a translation plane π of order 81. By Jha and Johnson [6], we note that the 3-elements are elations or Baer. Furthermore, in the Baer case, we note the following:

22 Remark. If whenever there is an orbit of length 12, all orbits of components have lengths 1, 12 or 60, then the Baer axes line up into a derivable net.

It is known by computer that this is exactly the situation when there is an orbit of length 12, however, we do not have a proof of this without the use of the computer.

If we make this assumption in the dimension 2 case, we have a computer-free construction of all translation planes of order 81 with spread in $PG(3,9)$ that admit $SL(2,5)$ as a collineation group.

23 Theorem. *Let π be a translation plane of order 81 with spread in $PG(3,9)$ that admits $SL(2,5)$ as a collineation group. Acting on the vector space V_4, assume that when there is a partial spread orbit of length 12, Γ_{12}, all partial spread orbits disjoint from Γ_{12} have length 1, 12 or 60, and there is only an orbit of length 60 in the Desarguesian case.*

Then π is one of the following twelve planes:

I. *The 3-elements are elations and π is one of the following six planes:*

   (1) *Desarguesian,*
   (2) *Prohaska*
   (3) *a 12-nest plane,*
   (4) *one of three planes obtained from a Desarguesian plane using 24 reguli;*

II. *The 3-elements are Baer and π is one of the following six planes:*

   (1) *Hall,*

(2) *derived Prohaska,*

(3) *derived* 12*-nest plane,*

(4) *the derived planes of the three planes of* (I4) *above.*

24 Remark. The computer will tell us that, in fact, any non-Desarguesian translation plane of order 81 and spread in $PG(3,9)$ admitting $SL(2,5)$ does have the property that there are six orbits of components of length 12. It may be possible to prove this fact without the use of the computer, thereby completely determining the translation planes with spreads in $PG(3,9)$ in a completely analytical manner.

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