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P-adic Measures and P-adic Spaces of Continuous Functions

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Abstract. For X a Hausdorff zero-dimensional topological space and E a Hausdorff non-Archimedean locally convex space, let $C(X, E)$ (resp. $C_b(X, E)$) be the space of all continuous (resp. bounded continuous) E-valued functions on X. Some of the properties of the spaces $C(X, E)$, $C_b(X, E)$, equipped with certain locally convex topologies, are studied. Also, some complete spaces of measures, on the algebra of all clopen subsets of X, are investigated.

Keywords: Non-Archimedean fields, zero-dimensional spaces, Banaschewski compactification, locally convex spaces

MSC 2000 classification: 46S10, 46G10

Introduction

Let $\mathbb{K}$ be a complete non-Archimedean valued field and let $C(X, E)$ be the space of all continuous functions from a zero-dimensional Hausdorff topological space X to a non-Archimedean Hausdorff locally convex space E. We will denote by $C_b(X, E)$ (resp. by $C_{rc}(X, E)$) the space of all $f \in C(X, E)$ for which $f(X)$ is a bounded (resp. relatively compact) subset of E. The dual space of $C_{rc}(X, E)$, under the topology t_u of uniform convergence, is a space $M(X, E')$ of finitely-additive E'-valued measures on the algebra $K(X)$ of all clopen , i.e. both closed and open, subsets of X. Some subspaces of $M(X, E')$ turn out to be the duals of $C(X, E)$ or of $C_b(X, E)$ under certain locally convex topologies.

In section 2 of this paper, we study some of the properties of the so called Q-integrals, a concept given by the author in [14]. In section 3, we identify the dual of $C_b(X, E)$ under the strict topology β_1. In section 4, we prove that the dual space of $C(X, E)$, under the topology of uniform convergence on the bounding subsets of X, is the space of all $m \in M(X, E')$ which have a bounding support. In section 5 it is shown that the space $M_s(X)$ of all separable members of $M(X)$, under the topology of uniform convergence on the uniformly bounded equicontinuous subsets of $C_b(X)$, is complete. The same is proved in section 6 for the space $M_{sv_o}(X)$ of those separable m for which the support of the extension m^{β_o}, to all of the Banaschewski compactification $\beta_o X$ of X, is contained in the $\mathbf{N}$-repletion $v_o X$ of X, if we equip $M_{sv_o}(X)$ with the topology of uniform convergence on the pointwise bounded equicontinuous subsets of $C(X)$.

1 Preliminaries

Throughout this paper, $\mathbb{K}$ will be a complete non-Archimedean valued field, whose valuation is non-trivial. By a seminorm, on a vector space over $\mathbb{K}$, we will mean a non-Archimedean seminorm. Similarly, by a locally convex space we will mean a non-Archimedean locally convex space over $\mathbb{K}$ (see [25]). Unless it is stated explicitly otherwise, X will be a Hausdorff zero-dimensional topological space , E a Hasusdorff locally convex space and $cs(E)$ the set of all continuous seminorms on E. The space of all $\mathbb{K}$-valued linear maps on E is denoted by $E^\star$, while E' denotes the topological dual of E. A seminorm p, on a vector space G over $\mathbb{K}$, is called polar if $p = \sup\{|f| : f \in G^\star, |f| \leq p\}$. A locally convex space G is called polar if its topology is generated by a family of polar seminorms. A subset A of G is called absolutely convex if $\lambda x + \mu y \in A$ whenever $x, y \in A$ and $\lambda, \mu \in \mathbb{K}$, with $|\lambda|, |\mu| \leq 1$. We will denote by $\beta_o X$ the Banaschewski compactification of X (see [5]) and by $\upsilon_o X$ the $\mathbf{N}$-repletion of X, where $\mathbf{N}$ is the set of natural numbers. We will let $C(X, E)$ denote the space of all continuous E-valued functions on X and $C_b(X, E)$ (resp. $C_{rc}(X, E)$) the space of all $f \in C(X, E)$ for which $f(X)$ is a bounded (resp. relatively compact) subset of E. In case $E = \mathbb{K}$, we will simply write $C(X), C_b(X)$ and $C_{rc}(X)$ respectively. For $A \subset X$, we denote by χ_A the $\mathbb{K}$-valued characteristic function of A. Also, for $X \subset Y \subset \beta_o X$, we denote by $\bar{B}^Y$ the closure of B in Y. If $f \in E^X, p$ a seminorm on E and $A \subset X$, we define

$$\|f\|_p = \sup_{x \in X} p(f(x)), \quad \|f\|_{A,p} = \sup_{x \in A} p(f(x)).$$

The strict topology β_o on $C_b(X, E)$ (see [9]) is the locally convex topology generated by the seminorms $f \mapsto \|hf\|_p$, where $p \in cs(E)$ and h is in the space $B_o(X)$ of all bounded $\mathbb{K}$-valued functions on X which vanish at infinity, i.e. for every $\epsilon > 0$ there exists a compact subset Y of X such that $|h(x)| < \epsilon$ if $x \notin Y$.

Let $\Omega = \Omega(X)$ be the family of all compact subsets of $\beta_o X \setminus X$. For $H \in \Omega$, let C_H be the space of all $h \in C_{rc}(X)$ for which the continuous extension h^{β_o} to all of $\beta_o X$ vanishes on H. For $p \in cs(E)$, let $\beta_{H,p}$ be the locally convex topology on $C_b(X, E)$ generated by the seminorms $f \mapsto \|hf\|_p, \quad h \in C_H$. For $H \in \Omega, \beta_H$ is the locally convex topology on $C_b(X, E)$ generated by the seminorms $f \mapsto \|hf\|_p, \quad h \in C_H, p \in cs(E)$. The inductive limit of the topologies $\beta_H, H \in \Omega$, is the topology β. Replacing Ω by the family Ω_1 of all $\mathbb{K}$-zero subsets of $\beta_o X$, which are disjoint from X, we get the topology β_1. Recall that a $\mathbb{K}$-zero subset of $\beta_o X$ is a set of the form $\{x \in \beta_o X : g(x) = 0\}$, for some $g \in C(\beta_o X)$. We get the topologies β_u and β'_u replacing Ω by the family Ω_u of all $Q \in \Omega$ with the following property: There exists a clopen partition $(A_i)_{i \in I}$ of X such that Q is disjoint from each $\overline{A_i}^{\beta_o X}$. Now β_u is the inductive limit of the topologies $\beta_Q, \quad Q \in \Omega_u$. The inductive limit of the topologies $\beta_{H,p}$, as H ranges over Ω_u, is denoted by $\beta_{u,p}$, while β'_u is the projective limit of the topologies $\beta_{u,p}, \quad p \in cs(E)$. For the definition of the topology β_e on $C_b(X)$ we refer to [12].

Let now $K(X)$ be the algebra of all clopen subsets of X. We denote by $M(X, E')$ the space of all finitely-additive E'-additive measures m on $K(X)$ for which the set $m(K(X))$ is an equicontinuous subset of E'. For each such m, there exists a $p \in cs(E)$ such that $\|m\|_p = m_p(X) < \infty$, where, for $A \in K(X)$,

$$m_p(A) = \sup\{|m(B)s|/p(s) : p(s) \neq 0, \quad A \supset B \in K(X)\}.$$

The space of all $m \in M(X, E')$ for which $m_p(X) < \infty$ is denoted by $M_p(X, E')$. For $m \in M_p(X, E')$ we define $N_{m,p}$ on X by

$$N_{m,p}(x) = \inf\{m_p(V) : x \in V \in K(X)\}.$$

In case $E = \mathbb{K}$, we denote by $M(X)$ the space of all finitely-additive bounded $\mathbb{K}$-valued measures on $K(X)$. An element m of $M(X)$ is called τ-additive if $m(V_\delta) \to 0$ for each decreasing net (V_δ) of clopen subsets of X with $\bigcap V_\delta = \emptyset$. In this case we write $V_\delta \downarrow \emptyset$. We denote by $M_\tau(X)$ the space of all τ-additive members of $M(X)$. Analogously, we denote by $M_\sigma(X)$ the space of all σ-additive m, i.e. those m with $m(V_n) \to 0$ when $V_n \downarrow \emptyset$. For an $m \in M(X, E')$ and $s \in E$, we denote by ms the element of $M(X)$ defined by $(ms)(V) = m(V)s$. A subset G of X is called a support set of an $m \in M(X, E')$ if $m(V) = 0$ for each $V \in K(X)$ disjoint from G.

Theorem 1 ([17]. *, Theorem 2.1] Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$, for all $s \in E$, and let $p \in cs(E)$ with $\|m\|_p < \infty$. Then :*

(1) $m_p(V) = \sup_{x \in V} N_{m,p}(x)$ for every $V \in K(X)$.

(2) The set

$$supp(m) = \bigcap \{V \in K(X) : m_p(V^c) = 0\}$$

is the smallest of all closed support sets for m.

(3) $supp(m) = \overline{\{x : N_{m,p}(x) \neq 0\}}$.

(4) If V is a clopen set contained in the union of a family $(V_i)_{i \in I}$ of clopen sets, then

$$m_p(V) \leq \sup\{m_p(V_i) : i \in I\}.$$

Next we recall the definition of the integral of an $f \in E^X$ with respect to an $m \in M(X, E')$. For a non-empty clopen subset A of X, let $\mathcal{D}_\mathcal{A}$ be the family of all $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\}$, where $\{A_1, \ldots, A_n\}$ is a clopen partition of A and $x_k \in A_k$. We make $\mathcal{D}_\mathcal{A}$ into a directed set by defining $\alpha_1 \geq \alpha_2$ iff the partition of A in α_1 is a refinement of the one in α_2. For an $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\} \in \mathcal{D}_\mathcal{A}$ and $m \in M(X, E')$, we define

$$\omega_\alpha(f, m) = \sum_{k=1}^{n} m(A_k) f(x_k).$$

If the limit $\lim \omega_\alpha(f, m)$ exists in $\mathbb{K}$, we will say that f is m-integrable over A and denote this limit by $\int_A f\, dm$. We define the integral over the empty set to be 0. For $A = X$, we write simply $\int f\, dm$. It is easy to see that if f is m-integrable over X, then it is m-integrable over every clopen subset A of X and $\int_A f\, dm = \int \chi_A f\, dm$. If τ_u is the topology of uniform convergence, then every $m \in M(X, E')$ defines a τ_u-continuous linear functional ϕ_m on $C_{rc}(X, E)$, $\phi_m(f) = \int f\, dm$. Also every $\phi \in (C_{rc}(X, E), \tau_u)'$ is given in this way by some $m \in M(X, E')$.

For $p \in cs(E)$, we denote by $M_{t,p}(X, E')$ the space of all $m \in M_p(X, E')$ for which m_p is tight, i.e. for each $\epsilon > 0$, there exists a compact subset Y of X such that $m_p(A) < \epsilon$ if the clopen set A is disjoint from Y. Let

$$M_t(X, E') = \bigcup_{p \in cs(E} M_{t,p}(X, E').$$

Every $m \in M_{t,p}(X, E')$ defines a β_0-continuous linear functional u_m on $C_b(X, E)$, $u_m(f) = \int f\, dm$. The map $m \mapsto u_m$, from $M_t(X, E')$ to $(C_b(X, E), \beta_o)'$, is an algebraic isomorphism. For $m \in M_\tau(X)$ and $f \in \mathbb{K}^X$, we will denote by $(VR) \int f\, dm$ the integral of f , with respect to m, as it is defined in [25]. We will call $(VR) \int f\, dm$ the (VR)-integral of f.

For all unexplained terms on locally convex spaces, we refer to [23] and [25].

2 Q-Integrals

We will recall next the definition of the Q-integral which was given in [14]. Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for all $s \in E$. This in particular happens if $m \in M_\tau(X, E')$. For $f \in E^X$ and $x \in X$, we define

$$Q_{m,f}(x) = \inf_{x \in V \in K(X)} \sup\{|m(B)f(x)| : V \supset B \in K(X)\}, \quad \|f\|_{Q_m} = \sup_{x \in X} Q_{m,f}(x).$$

Let $S(X, E)$ be the linear subspace of E^X spanned by the functions $\chi_A s, \ s \in E, \ A \in K(X)$, where χ_A is the $\mathbb{K}$-characteristic function of A. We will write simply $S(X)$ if $E = \mathbb{K}$.

Lemma 1. *If* $g \in S(X, E)$, *then*

$$\|g\|_{Q_m} = \sup_{x \in X} Q_{m,g}(x) < \infty.$$

Proof: The proof was given in [14], Lemma 7.2. Note that, if $\|m\|_p < \infty$ and $d \geq \|g\|_p$, then $Q_{m,g}(x) \leq d \cdot m_p(X)$.

Lemma 2. *For* $g \in S(X, E)$, *we have*

$$\left| \int g \, dm \right| \leq \|g\|_{Q_m}.$$

Proof: Assume first that $g = \chi_A s, \ A \in K(X)$. Then

$$|m(A)s| \leq |ms|(A) = \sup_{y \in A} N_{ms}(y).$$

But, for $y \in A$, we have

$$N_{ms}(y) = \inf_{y \in V \in K(X)} \sup_{V \supset B \in K(X)} |m(B)s| = \inf_{y \in V \in K(X)} \sup_{V \supset B \in K(X)} |m(B)g(y)| = Q_{m,g}(y).$$

Thus $|m(A)s| \leq \sup_{y \in A} Q_{m,g}(y)$. In the general case, there are pairwise disjoint clopen sets $A_1, \ldots, A_n$ covering X and $s_k \in E$ with $g = \sum_{k=1}^n \chi_{A_k} s_k$. Thus,

$$\left| \int g \, dm \right| = \left| \sum_{k=1}^n m(A_k) s_k \right| \leq \max_{1 \leq k \leq n} |m(A_k) s_k| \leq \sup_{x \in X} Q_{m,g}(x) = \|g\|_{Q_m}.$$

Definition 1. Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for all $s \in E$. A function $f \in E^X$ is said to be Q-integrable with respect to m if there exists a sequence (g_n) in $S(X, E)$ such that $\|f - g_n\|_{Q_m} \to 0$. In this case, the Q-integral of f is defined by

$$(Q) \int f \, dm = \lim_{n \to \infty} \int g_n \, dm.$$

If f is Q-integrable with respect to m, then for $A \in K(X)$ the function $\chi_A f$ is also Q-integrable. We define

$$(Q) \int_A f \, dm = (Q) \int \chi_A f \, dm.$$

As it is proved in [14], the Q-integral is well defined. If $\mu \in M_\tau(X)$ and $g \in \mathbb{K}^X$, then $Q_{\mu,g}(x) = |g(x)| N_\mu(x)$. Thus the Q-integral with respect to μ coincides with the integral as it is defined in [23], which we will call (VR)-integral. Hence

$$(VR) \int g \, d\mu = (Q) \int g \, d\mu.$$

Lemma 3. *If $f \in E^X$ is Q-integrable with respect to an $m \in M(X, E')$ and if (g_n) is a sequence in $S(X, E)$, with $\|f - g_n\|_{Q_m} \to 0$, then*

$$\|f\|_{Q_m} = \lim_{n\to\infty} \|g_n\|_{Q_m} < \infty, \quad \textit{and} \quad \left|(Q)\int f\, dm\right| \le \|f\|_{Q_m}.$$

Proof: Since

$$Q_{m,h+g}(x) \le \max\{Q_{m,g}(x), Q_{m,h}(x)\},$$

it follows that

$$\|h + g\|_{Q_m} \le \max\{\|h\|_{Q_m}, \|g\|_{Q_m}\}.$$

Thus

$$\|f\|_{Q_m} \le \max\{\|f - g_n\|_{Q_m}, \|g_n\|_{Q_m}\} \le \|f - g_n\|_{Q_m} + \|g_n\|_{Q_m} < \infty.$$

It follows that

$$|\|f\|_{Q_m} - \|g_n\|_{Q_m}| \le \|f - g_n\|_{Q_m} \to 0.$$

Moreover,

$$\left|(Q)\int f\, dm\right| = \lim_{n\to\infty} \left|\int g_n\, dm\right| \le \lim_{n\to\infty} \|g_n\|_{Q_m} = \|f\|_{Q_m}.$$

Hence the result follows.

Theorem 2. *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for all $s \in E$, and let $f \in E^X$ be Q-integrable. Define*

$$m_f : K(X) \to \mathbb{K}, \quad m_f(A) = (Q)\int_A f\, dm.$$

Then $m_f \in M_\tau(X)$.

Proof: Since $|m_f(A)| \le \|f\|_{Q_m}$, it is easy to see that $m_f \in M(X)$. Let now $V_\delta \downarrow \emptyset$ and $\epsilon > 0$. Choose a $g = \sum_{k=1}^n \chi_{A_k} s_k \in S(X, E)$ such that $\|f - g\|_{Q_m} < \epsilon$. Then

$$\int_{V_\delta} g\, dm = \sum_{k=1}^n (ms_k)(V_\delta \cap A_k) \to 0.$$

Let δ_o be such that $\left|\int_{V_\delta} g\, dm\right| < \epsilon$ if $\delta \ge \delta_o$. Now, for $\delta \ge \delta_o$, we have

$$\begin{aligned} \left|(Q)\int_{V_\delta} f\, dm\right| &\le \max\{\left|(Q)\int_{V_\delta} (f - g)\, dm\right|, \ \left|\int_{V_\delta} g\, dm\right|\} \\ &\le \max\{\|f - g\|_{Q_m}, \ \left|\int_{V_\delta} g\, dm\right|\} < \epsilon. \end{aligned}$$

Thus $m_f(V_\delta) \to 0$.

Lemma 4. *If $f \in E^X$ is Q-integrable with respect to an $m \in M(X, E')$, then the map $x \to Q_{m,f}(x)$ is upper semicontinuous.*

Proof: We need to show that, for each $\alpha > 0$, the set

$$V = \{x : Q_{m,f}(x) < \alpha\}$$

is open. So let $x \in V$ and choose $\epsilon > 0$ such that $Q_{m,f}(x) < \alpha - 2\epsilon$. Let $g \in S(X, E)$ be such that $\|f - g\|_{Q_m} < \epsilon$. Let $A_1, \ldots, A_n$ be a clopen partition of X and $s_k \in E$ such that $g = \sum_{k=1}^n \chi_{A_k} s_k$. Let k be such that $x \in A_k$. There exists a clopen set B, containing x and contained in A_k, such that $|m(D)g(x)| < Q_{m,g}(x) + \epsilon$ for every clopen set D contained in B. If $y \in B$, then for $B \supset D \in K(X)$ we have

$$\begin{aligned}|m(D)g(y)| &= |m(D)g(x)| < Q_{m,g}(x) + \epsilon \\ &\leq \max\{Q_{m,g-f}(x), \quad Q_{m,f}(x)\} + \epsilon \\ &\leq Q_{m,f}(x) + 2\epsilon.\end{aligned}$$

Thus $Q_{m,g}(y) \leq Q_{m,f}(x) + 2\epsilon < \alpha$. Hence $x \in B \subset V$ and the result follows.

Lemma 5. *If $f \in E^X$ is Q-integrable with respect to an $m \in M(X, E')$, then $N_{m_f} \leq Q_{m,f}$.*

Proof: Let $x \in X$ and $\epsilon > 0$. In view of the preceding Lemma, there exists a clopen neighborhood V of X such that $Q_{m,f}(y) \leq Q_{m,f}(x) + \epsilon$ for all $y \in V$. If $V \supset B \in K(X)$, then

$$|m_f(B)| \leq \sup_{y \in B} Q_{m,f}(y) \leq Q_{m,f}(x) + \epsilon$$

and so

$$N_{m_f}(x) \leq |m_f|(V) \leq Q_{m,f}(x) + \epsilon.$$

Hence the result follows.

Lemma 6. *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for all $s \in E$. If $g \in S(X, E)$, then $Q_{m,g} = N_{m_g}$.*

Proof: Let $\{A_1, \ldots, A_n\}$ be a clopen partition of X and $s_k \in E$ such that $g = \sum_{k=1}^n \chi_{A_k} s_k$. Suppose that $N_{m_g}(x) < \alpha$. Then, there exists a clopen neighborhood V of x such that $|m_g|(V) < \alpha$. Let $x \in A_k$. If B is a clopen set contained in $A_k \cap V$, then

$$m_g(B) = (Q) \int_B g \, dm = \int_B g \, dm = m(B)g(x)$$

since $g = g(x)$ on B. Thus

$$Q_{m,g}(x) \leq \sup_{B \subset A_k \cap V} |m(B)g(x)| \leq |m_g|(V) < \alpha.$$

This proves that $Q_{m,g} \leq N_{m_g}$ and the result follows.

Theorem 3. *If $f \in E^X$ is Q-integrable with respect to an $m \in M(X, E')$, then $Q_{m,f} = N_{m_f}$.*

Proof: Assume that $N_{m_f}(x) < \alpha$ and let $0 < \epsilon < \alpha$. There exists a clopen neighborhood V of x such that $|m_f|(V) < \alpha$. Let $g \in S(X, E)$ be such that $\|f - g\|_{Q_m} < \epsilon$. For A clopen contained in V, we have

$$|m_f(A) - m_g(A)| = \left|(Q) \int (f - g) \, dm\right| \leq \|f - g\|_{Q_m} < \epsilon$$

and so

$$|m_g(A)| \leq \max\{\epsilon, |m_f(A)|\} < \alpha.$$

Thus

$$Q_{m,g}(x) = N_{m_g}(x) \leq |m_g|(V) \leq \alpha.$$

Now

$$Q_{m,f}(x) \leq \max\{Q_{m,f-g}(x), \quad Q_{m,g}(x)\} \leq \alpha,$$

which proves that $Q_{m,f} \leq N_{m_f}$ and the result follows by Lemma 5.

Theorem 4. *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$, for all $s \in E$, and let $f \in E^X$ be Q-integrable with respect to m. If $g \in \mathbb{K}^X$ is Q-integrable with respect to m_f, then gf is Q-integrable with respect to m and*

$$(Q)\int gf\,dm = (Q)\int g\,dm_f.$$

Proof: If $h \in \mathbb{K}^X$, then

$$Q_{m,hf}(x) = |h(x)|Q_{m,f}(x) = |h(x)|N_{m_f}(x) = Q_{m_f,h}(x).$$

Let (g_n) be a sequence in $S(X)$ such that $\|g - g_n\|_{Q_{m_f}} \to 0$. We have

$$\begin{aligned}\|g - g_n\|_{Q_{m_f}} &= \sup_{x\in X} |g(x) - g_n(x)| \cdot N_{m_f}(x) \\ &= \sup_{x\in X} Q_{m,(g-g_n)f}(x) = \|gf - g_n f\|_{Q_m}.\end{aligned}$$

If $A \in K(X)$, then $\chi_A f$ is Q-integrable with respect to m and

$$(Q)\int \chi_A f\,dm = (Q)\int_A f\,dm = m_f(A) = \int \chi_A\,dm_f.$$

It follows that, for all n, $g_n f$ is Q-integrable with respect to m and

$$(Q)\int g_n f\,dm = \int g_n\,dm_f \to (Q)\int g\,dm_f.$$

Since $g_n f$ is Q-integrable with respect to m and $\|gf - g_n f\|_{Q_m} \to 0$, it follows that gf is Q-integrable and

$$(Q)\int gf\,dm = \lim_{n\to\infty}(Q)\int g_n f\,dm = \lim_{n\to\infty}\int g_n\,dm_f = (Q)\int g\,dm_f,$$

which completes the proof.

Theorem 5. *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$, for all $s \in E$, and let $p \in cs(E)$ with $\|m\|_p < \infty$. If $f \in E^X$ is Q-integrable with respect to m, then, given $\epsilon > 0$, there exists $\alpha > 0$ such that $\left|(Q)\int_A f\,dm\right| < \epsilon$ if $m_p(A) < \alpha$.*

Proof: Let $g \in S(X, E)$ with $\|f - g\|_{Q_m} < \epsilon$. For a clopen set A, we have $\left|\int_A g\,dm\right| \le \|g\|_p \cdot m_p(A)$. Let $\alpha > 0$ be such that $\alpha \cdot \|g\|_p < \epsilon$. If $m_p(A) < \alpha$, then

$$\begin{aligned}\left|(Q)\int_A f\,dm\right| &\le \max\{\left|(Q)\int_A (f-g)\,dm\right|, \ \left|\int_A g\,dm\right|\} \\ &\le \max\{\|f-g\|_{Q_m}, \ \|g\|_p \cdot m_p(A)\} < \epsilon.\end{aligned}$$

Lemma 7. *Let $m \in M_\tau(X)$ and let $g \in \mathbb{K}^X$ be (VR)-integrable. Then, given $\epsilon > 0$, there exists $\delta > 0$ such that $\|g\|_{A,N_m} \le \epsilon$ if $|m|(A) < \delta$.*

Proof: There exists $h \in S(X)$ such that $\|g - h\|_{N_m} \le \epsilon$. It suffices to choose $\delta > 0$ such that $\delta \cdot \|h\| < \epsilon$.

Let $m \in M(X)$. For $A \subset X$, we define

$$|m|^\wedge(A) = \inf\{|m|(V) : V \in K(X), \quad A \subset V\}.$$

Recall that a sequence (g_n) in $\mathbb{K}^X$ converges in measure to an $f \in \mathbb{K}^X$, with respect to m (see [14], Definition 2.12) if, for each $\alpha > 0$, we have

$$\lim_{n\to\infty} |m|^\wedge\{x : |g_n(x) - g(x)| \ge \alpha\} = 0.$$

Theorem 6 (Dominated Convergence Theorem). *Let $m \in M_\tau(X)$ and let (f_n) be a sequence of (VR)-integrable, with respect to m, functions, which converges in measure to some $f \in \mathbb{K}^X$. If there exists a (VR)-integrable function $g \in \mathbb{K}^X$ such that $|f_n| \le |g|$ for all n, then f is (VR)-integrable and*

$$(VR)\int f\,dm = \lim_{n\to\infty}(VR)\int f_n\,dm.$$

Proof: Let $\epsilon > 0$ and choose inductively $n_1 < n_2 < \ldots$ such that $|m|^\wedge(V_k) < 1/k$, where

$$V_k = \{x : |f_{n_k}(x) - f(x) \ge 1/k\}.$$

Let $V = \bigcap_{N=1}^\infty \bigcup_{k\ge N} V_k$. If $x \in V$, then $N_m(x) = 0$. Indeed, for every N, there exists $k \ge N$ with $x \in V_k$ and so $N_m(x) \le |m|(V_k) < 1/k \le 1/N$, which proves that $N_m(x) = 0$. Also, for $x \in X \setminus V$, we have $f(x) = \lim_{k\to\infty} f_{n_k}(x)$. In fact, there exists N such that $x \notin V_k$ for $k \ge N$ and so $|f_{n_k}(x) - f(x)| < 1/k \to 0$. It follows that $|f(x)| \le |g(x)|$ when $x \notin V$. Since g is (VR)-integrable, there exists (by the preceding Lemma) $\delta > 0$ such that $\|g\|_{A,N_m} < \epsilon$ if $|m|(A) < \delta$. Let now $\alpha > 0$ be such that $\alpha \cdot \|m\| < \epsilon$. For each n, let

$$G_n = \{x : |f_n(x) - f(x)| \ge \alpha\}$$

and choose a clopen set W_n containing G_n with $|m|(W_n) < 1/n + |m|^\wedge(G_n)$. Since $|m|^\wedge(G_n) \to 0$, there exists n_o such that $|m|(W_n) < \delta$ if $n \ge n_o$. Let now $n \ge n_o$ and $x \in X$. If $x \in V$, then $N_m(x) = 0$. Suppose that $x \notin V$. Then $|f(x)| \le |g(x)|$ and so

$$|f(x) - f_n(x)|N_m(x) \le |g(x)|N_m(x).$$

If $x \in W_n$, then $|g(x)|N_m(x) \le \epsilon$, since $|m|(W_n) < \delta$, while for $x \notin W_n$ we have

$$|f(x) - f_n(x)|N_m(x) \le \alpha \cdot \|m\| < \epsilon.$$

Thus, for $n \ge n_o$, we have $\|f - f_n\|_{N_m} \le \epsilon$. Since f_n is (VR)-integrable, it follows that f is (VR)-integrable and

$$(VR)\int f\,dm = \lim_{n\to\infty}(VR)\int f_n\,dm$$

since

$$\left|(VR)\int (f - f_n)\,dm\right| \le \|f - f_n\|_{N_m} \to 0.$$

This completes the proof.

Let now τ be the topology of X and let $K_c(X)$ be the collection of all subsets A of X such that $A \cap Y$ is clopen in Y for each compact subset Y of X. It is easy to see that if A, A_1, A_2 are in $K_c(X)$, then each of the sets A^c, $A_1 \cap A_2$ and $A_1 \cup A_2$ is also in $K_c(X)$. Now $K_c(X)$ is a base for a zero-dimensional topology τ^k on X finer than τ. We will denote by $X^{(k)}$ the set X equipped with the topology τ^k. We have the following easily established

Theorem 7. *(1) τ and τ^k have the same compact sets.*

(2) τ and τ^k induce the same topology on each τ-compact subset of X.

(3) A subset B of X is τ^k-clopen iff $B \in K_c(X)$.

(4) If Y is a zero-dimensional topological space and $f : X \to Y$, then f is τ^k-continuous iff the restriction of f to every compact subset of X is τ-continuous.

Let now $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for each $s \in E$.

Lemma 8. *If $B \in K_c(X)$, $s \in E$ and $h = \chi_B s$, then h is Q-integrable with respect to m.*

Proof: Let $\epsilon > 0$. Since $ms \in M_\tau(X)$, there exists a compact subset Y of X such that $|ms|(V) < \epsilon$ for each clopen subset V of X disjoint from Y. Since $B \cap Y$ is clopen in Y and Y is compact, there exists $A \in K(X)$ with $B \cap Y = A \cap Y$ (see [25], p. 188). Let $g = \chi_A s$, $f = h - g$. If $x \in A \Delta B$, then x is not in Y and so there exists $V \in K(X)$ such that $x \in V \subset Y^c$. If $W \in K(X)$ is contained in V, then $|m(W)f(x)| = |m(W)s| \leq |ms|(V) < \epsilon$ and so $Q_{m,f}(x) \leq \epsilon$. Thus $\|h - g\|_{Q_m} \leq \epsilon$. Hence the Lemma follows.

Now for $B \in K_c(X)$, we define

$$m^{(k)}(B) : E \to \mathbb{K}, \quad m^{(k)}(B)s = (Q)\int \chi_B s \, dm.$$

Clearly $m^{(k)}$ is linear. Let $p \in cs(E)$ be such that $m_p(X) < \infty$.

Theorem 8. *Let $A \in K_c(X)$, and let $V \in K(X)$ with $A \subset V$. Then :*

(1) $|m^{(k)}(A)s| \leq |ms|(V) \leq m_p(V) \cdot p(s)$ *for all* $s \in E$.

(2) $m^{(k)} \in M_p(X^{(k)}, E')$.

(3) $m^{(k)}s \in M_\tau(X^{(k)})$ *for all* $s \in E$.

(4) If $m \in M_{t,p}(X, E')$, *then* $m^{(k)} \in M_{t,p}(X^{(k)}, E')$.

Proof: Let $s \in E$, $h = \chi_A s$ and $x \in A \subset V$. If W is a clopen subset of X contained in V, then $|m(W)h(x)| \leq |ms|(V)$ and so $Q_{m,h}(x) \leq |ms|(V)$, which implies that

$$|m^{(k)}(A)s| \leq \sup_{x \in A} Q_{m,h}(x) \leq |ms|(V) \leq m_p(V) \cdot p(s).$$

This proves that $m^{(k)}(A) \in E'$ and $\|m^{(k)}(A)\|_p \leq m_p(V)$. Clearly $m^{(k)} \in M_p(X^{(k)}, E')$ and $\|m^{(k)}\|_p \leq \|m\|_p$.

Let now $s \in E$ and $\epsilon > 0$. There exists a compact subset Y of X such that $|ms|(Z) < \epsilon$ for each $Z \in K(X)$ disjoint from Y. Let $B \in K_c(X)$ be disjoint from Y and let $x \in B$. Then $x \notin Y$ and so there exists a $D \in K(X)$ containing x ad contained in Y^c. For $h = \chi_B s$, we have $Q_{m,h}(x) \leq |ms|(D) < \epsilon$. Thus $|m^{(k)}(A)s| \leq \epsilon$. It follows that $|m^{(k)}s|(B) \leq \epsilon$ for each $B \in K_c(X)$ disjoint from Y and so $m^{(k)}s \in M_\tau(X^{(k)})$. Finally, assume that $m \in M_{t,p}(X.E)$. Given $\epsilon > 0$, there exists a compact subset Y of X such that $m_p(V) < \epsilon$ for each $V \in K(X)$ disjoint from Y. If $s \in E$, with $p(s) > 0$, then for $V \in K(X)$ disjoint from Y we have $|ms|(V) \leq m_p(V) \cdot p(s) < \epsilon \cdot p(s)$. Thus, for $B \in K_c(X)$ disjoint from Y we have $|m^{(k)}s|(B) \leq \epsilon \cdot p(s)$ and so $m_p^{(k)}(B) \leq \epsilon$. This clearly completes the proof.

Theorem 9. *Let $m \in M(X, E')$ be such that $ms \in M_\tau(X)$ for each $s \in E$. Then:*

(1) If $A \in K(X)$, *then* $|ms|(A) = |m^{(k)}s|(A)$ *for all* $s \in E$.

(2) If $m \in M_p(X, E')$, *then* $m_p(A) = m_p^{(k)}(A)$ *for each* $A \in K(X)$.

(3) If $f \in E^X$ *is Q-integrable with respect to m, then f is Q-integrable with respect to* $m^{(k)}$ *and* $Q_{m,f} \leq Q_{m^{(k)},f}$. *Moreover*

$$(Q)\int f \, dm = (Q)\int f \, dm^{(k)}.$$

Proof: Let $A \in K(X)$. Clearly $|ms|(A) \leq |m^{(k)}s|(A)$. On the other hand, let $|m^{(k)}s|(A) > \theta > 0$. There exists $D \in K_c(X)$, $D \subset A$, such that $|m^{(k)}(D)s| > \theta$. Let $h = \chi_D s$. Since $|m^{(k)}(D)s| \leq \sup_{x \in D} Q_{m,h}(x)$, there exists $x \in D$ such that $Q_{m,h}(x) > \theta$. The set $Y = \{z \in X : Q_{m,h}(z) \geq \theta\}$ is compact. Hence there exists $Z \in K(X)$ with $Z \cap Y = D \cap Y$. Since $x \in Z \cap A$ and $Q_{m,h}(X) > \theta$, there exists $W \in K(X)$ contained in $Z \cap A$ and such that $|m(W)h(x)| > \theta$. Then $h(x) = s$ and so $|m(W)s| > \theta$, which proves that $|ms|(A) > \theta$. Thus, $|ms|(A)| \geq |m^{(k)}s|(A)$. Assume next that $m_p^{(k)}(A) > \alpha > 0$. There exists $B \in K_c(X)$ contained in A and $s \in E$ with $|m^{(k)}(B)s|/p(s) > \alpha$. Now $|ms|(A) = |m^{(k)}s|(A) > \alpha \cdot p(s)$. Thus $m_p(A) \geq |ms|(A)/p(s) > \alpha$, which shows that $m_p(A) = m_p^{(k)}(A)$. Thus (1) and (2) hold.

(3). Assume that $f \in E^X$ is Q-integrable with respect to m.

Claim : If $x \in D \in K(X)$, then

$$\sup_{Z \in K_c(X), Z \subset D} |m^{(k)}(Z)f(x)| = \sup_{Z \in K(X), Z \subset D} |m(Z)f(x)|.$$

Indeed, suppose that there exists a $Z \in K_c(X)$ contained in D such that $|m^{(k)}(Z)f(x)| > \theta > 0$. For $h = \chi_Z f(x)$, we have

$$\theta < |m^{(k)}(Z)f(x)| \leq \sup_{z \in Z} Q_{m,h}(z).$$

Thus, there exists $z \in Z$ with $Q_{m,h}(z) > \theta$. Since $z \in Z \subset D$, there exists $W \in K(X)$ contained in D such that $|m(W)h(z)| = |m(W)f(x)| > \theta$. This clearly proves the claim. Now

$$\begin{aligned} Q_{m,f}(x) &= \inf_{x \in D \in K(X)} \sup_{D \supset Z \in K(X)} |m(Z)f(x)| \\ &= \inf_{x \in D \in K(X)} \sup_{D \supset Z \in K_c(X)} |m^{(k)}(Z)f(x)| \geq Q_{m^{(k)},f}(x). \end{aligned}$$

Since f is Q-integrable with respect to m, there exists a sequence $(g_n) \subset S(X,E) \subset S(X^{(k)}, E)$ such that $\|f - g_n\|_{Q_m} \to 0$. But then $\|f - g_n\|_{Q_{m^{(k)}}} \leq \|f - g_n\|_{Q_m} \to 0$. Hence f is Q-integrable with respect to $m^{(k)}$ and

$$(Q) \int f \, dm^{(k)} = \lim_{n \to \infty} \int g_n \, dm^{(k)} = \lim_{n \to \infty} \int g_n \, dm = (Q) \int f \, dm.$$

This completes the proof of the Theorem.

Next we recall the definition of the topology $\bar{\beta}_o$ which was given in [14]. Let $C_{b,k}(X,E)$ be the space of all bounded E-valued functions on X whose restriction to every compact subset of X is continuous. By Theorem 7 we have that $C_{b,k}(X,E) = C_b(X^{(k)}, E)$. For $p \in cs(E)$, we denote by $\bar{\beta}_{o,p}$ the locally convex topology on $C_{b,k}(X,E)$ generated by the seminorms $f \mapsto \|hf\|_p$, $h \in B_o(X)$. Since X and $X^{(k)}$ have the same compact sets, we have that $B_o(X) = B_o(X^{(k)})$ and so $\bar{\beta}_{o,p}$ coincides with the topology $\beta_{o,p}$ on $C_b(X^{(k)}, E)$. The topology $\bar{\beta}_o$ is defined to be the locally convex projective limit of the topologies $\bar{\beta}_{o,p}$, $p \in cs(E)$. Thus $\bar{\beta}_o$ coincides with topology β_o on $C_b(X^{(k)}, E)$.

Theorem 10. *(1) If $m \in M_t(X, E')$, then every $f \in C_{b,k}(X,E)$ is Q-integrable with respect to m and*

$$(Q) \int f \, dm = \int f \, dm^{(k)}.$$

Thus the map

$$\phi_m : C_{b,k}(X,E) \to \mathbb{K}, \quad \phi_m(f) = (Q) \int f \, dm$$

is $\bar{\beta}_o$-continuous.

(2) If E is polar, then every $\bar{\beta}_o$-continuous linear functional ϕ on $C_{b,k}(X,E)$ is of the form ϕ_m for some $m \in M_t(X, E')$.

Proof: 1. Let $p \in cs(E)$ be such that $m \in M_{t,p}(X, E')$ and $\|m\|_p < 1$. Let $d > \|f\|_p$ and $\epsilon > 0$. There exists a compact subset Y of X such that $m_p(V) < \epsilon/d$ for every $V \in K(X)$ disjoint from Y. For each $x \in Y$, the set

$$D_x = \{y \in Y : p(f(y) - f(x)) < \epsilon\}$$

is clopen in Y and $D_x = D_y$ if $D_x \cap D_y \neq \emptyset$. In view of the compactness of Y, there are $x_1, \ldots, x_n$ in Y such that the sets $D_{x_1}, \ldots, D_{x_n}$ form a partition of Y. For each k, there exists a clopen subset V_k of X such that $V_k \cap Y = D_{x_k}$. If $W_k = V_k \setminus \bigcup_{i \neq k} V_i$, then $W_k \cap Y = D_{x_k}$. Let $g = \sum_{k=1}^n \chi_{W_k} f(x_k)$. Then $\|f - g\|_{Q_m} \leq \epsilon$. Indeed, let $x \in X$.

Case I: $x \notin Y$. There is a clopen neighborhood V of x disjoint from Y. If $B \in K(X)$ is contained in V, then

$$|m(B)[f(x) - g(x)]| \leq p(f(x) - g(x)) \cdot m_p(V) \leq \epsilon$$

and so $Q_{m,f-g}(x) \leq \epsilon$.

Case II : $x \in Y$. There exists a k such that $x \in W_k$ and so $g(x) = f(x_k)$. If a clopen set B is contained in W_k, then

$$|m(B)[f(x) - g(x)]| = |m(B)[f(x) - f(x_k]| \leq m_p(V_k) \cdot p(f(x) - f(x_k)) \leq \epsilon,$$

and so again $Q_{m,f-g}(x) \leq \epsilon$. This proves that $\|f - g\|_{Q_m} \leq \epsilon$ and so f is Q-integrable. Now

$$\phi_m(f) = (Q) \int f\, dm = (Q) \int f\, dm^{(k)} = \int f\, dm^{(k)}.$$

Thus ϕ_m is $\bar{\beta}_o$-continuous on $C_{b,k}(X, E)$.

Finally assume that E is polar and let ϕ be a $\bar{\beta}_o$-continuous linear functional on $C_{b,k}(X, E)$. Since $\bar{\beta}_o$ induces the topology β_o on $C_b(X, E)$, there exists an $m \in M_t(X, E')$ such that

$$\phi(f) = \int f\, dm = (Q) \int f\, dm$$

for each $f \in C_b(X, E)$. Now ϕ and ϕ_m are both $\bar{\beta}_o$-continuous on $C_{b,k}(X, E)$ and they coincide on the $\bar{\beta}_o$-dense subspace $C_b(X, E)$ of $C_{b,k}(X, E)$. Thus $\phi = \phi_m$ and the proof is complete.

3 The Dual Space of $(C_b(X, E), \beta_1)$

For u a linear functional on $C_b(X, E)$, $p \in cs(E)$ and $h \in \mathbb{K}^X$, we define

$$|u|_p(h) = \sup\{|u(g)| : g \in C_b(X, E), \quad p \circ g \leq |h|\}.$$

Theorem 11. *For a linear functional u on $C_b(X, E)$, the following are equivalent :*

(1) u is β_1-continuous.

(2) For each sequence (V_n) of clopen sets, with $V_n \downarrow \emptyset$, there exists $p \in cs(E)$ such that $\|u\|_p < \infty$ and $\lim_{n\to\infty} |u|_p(\chi_{V_n}) = 0$.

(3) For each sequence (h_n) in $C_b(X)$, with $h_n \downarrow 0$, there exists $p \in cs(E)$ such that $\|u\|_p < \infty$ and $\lim_{n\to\infty} |u|_p(h_n) \to 0$.

Proof: (1) $\Rightarrow$ (2). Let $V_n \downarrow \emptyset$ and $H = \bigcap \overline{V_n}^{\beta_o X}$. Then $H \in \Omega_1$ and so u is $\beta_{H,p}$-continuous for some $p \in cs(E)$. Let $\epsilon > 0$ and $h \in C_H$ be such that

$$W_1 = \{f \in C_b(X, E) : \|hf\|_p \leq 1\} \subset W = \{f : |u(f)| \leq \epsilon\}.$$

It is easy to see that $\|u\|_p < \infty$. Let $M = \{x \in X : |h(x)| \geq 1\}$. There exists n_o such that $M \subset V_{n_o}^c$. Let now $n \geq n_o$ and $f \in C_b(X, E)$ with $p \circ f \leq |\chi_{V_n}|$. Let $f_1 = \chi_M f, \quad f_2 = f - f_1$. If $x \in M$, then $x \in V_n^c$ and so $p(f(x)) = 0$. This implies that $f_1 \in W_1 \subset W$. Also, if $x \notin M$, then $|h(x)| \leq 1$ and so $|h(x)|p(f(x)) \leq 1$, which proves that $f_2 \in W_1$. Thus $f = f_1 + f_2 \in W$, which shows that $|u|_p(\chi_{V_n}) \leq \epsilon$.

(2) $\Rightarrow$ (3). Let $h_n \downarrow 0$. Without loss of generality, we may assume that $\|h_1\| \leq 1$. Let $\lambda \in \mathbb{K}, \quad 0 < |\lambda| < 1$ and set

$$V_n = \{x : |h_n(x) \geq |\lambda|\}.$$

Then $V_n \downarrow \emptyset$. By (2), there exists $p \in cs(E)$ with $\|u\|_p < \infty$ and $|u|_p(\chi_{V_n}) \to 0$. We may choose p so that $\|u\|_p \leq 1$. Choose n_o such that $|u|_p(\chi_{V_n}) < |\lambda|$ if $n \geq n_o$. Let now $n \geq n_o$. We will show that $|u|_p(h_n) \leq |\lambda|$. In fact, let $f \in C_b(X, E)$ with $p \circ f \leq |h_n|$, $g_1 = \chi_{V_n} f$, $g_2 = f - g_1$. If $x \in V_n$, then $p(g_1(x)) \leq |h_n(x)|$ and so $p \circ g_1 \leq |\chi_{V_n}|$, which implies that $|u(g_1)| \leq |\lambda|$. If $x \notin V_n$, then $p(g_2(x)) = p(f(x)) \leq |h_n(x)| < |\lambda|$. Hence $|u(g_2)| \leq \|u\|_p \cdot \|g_2\|_p \leq |\lambda|$, and therefore $|u(f)| \leq |\lambda|$. This proves that $|u|_p(h_n) \leq |\lambda|$.

(3) $\Rightarrow$ (2). It is trivial.

(2) $\Rightarrow$ (1). Let

$$W = \{f \in C_b(X, E) : |u(f)| \leq 1\}$$

and let $H \in \Omega_1$. There exists a decreasing sequence (V_n) of clopen subsets of X with $\bigcap \overline{V_n}^{\beta_o X} = H$. Let $p \in cs(E)$ be such that $\|u\|_p \leq 1$ and $|u|_p(\chi_{V_n}) \to 0$. Let λ be a nonzero element of $\mathbb{K}$ and choose n so that $|u|(\chi_{V_n}) < |\lambda|^{-1}$. Now

$$W_1 = \{f \in C_b(X, E) : \|f\|_p \leq |\lambda|, \quad \|f\|_{V_n^c, p} \leq 1\} \subset W.$$

Indeed, let $f \in W_1$ and set $f_1 = \chi_{V_n} f, f_2 = f - f_1$. Since $|\lambda^{-1} f_1| \leq |\chi_{V_n}|$, we have that $|u(f_1)| \leq 1$. Also $|u(f_2)| \leq \|f_2\|_p \leq 1$, and so $|u(f)| \leq 1$, which proves that $W_1 \subset W$. By [13], Theorem 2.2, it follows that W is a $\beta_{H,p}$-neighborhood of zero. This, being true for all $H \in \Omega_1$, implies that W is a β_1-neighborhood of zero, i.e. u is β_1-continuous, which completes the proof.

Theorem 12. *For a set H of linear functionals on $C_b(X, E)$, the following are equivalent :*

(1) H is β_1-equicontinuous.

(2) If (V_n) is a sequence of clopen subsets of X which decreases to the empty set, then there exists $p \in cs(E)$ such that $\sup_{u \in H} \|u\|_p < \infty$ and $|u|_p(\chi_{V_n}) \to 0$ uniformly for $u \in H$.

(3) If (h_n) is a sequence in $C_b(X)$ with $h_n \downarrow 0$, then there exists $p \in cs(E)$ such that $\sup_{u \in H} \|u\|_p < \infty$ and $|u|_p(h_n) \to 0$ uniformly for $u \in H$.

Proof: (1) $\Rightarrow$ (2). Let $V_n \downarrow \emptyset$. Then $Z = \bigcap \overline{V_n}^{\beta_o X} \in \Omega_1$. Let $\lambda \in \mathbb{K}, \lambda \neq 0$. Since H is β_1-equicontinuous, the set λH^o is a β_1-neighborhood of zero. Thus, there exists $p \in cs(E)$ such that λH^o is a $\beta_{Z,p}$-neighborhood of zero. Let $h \in C_Z$ be such that

$$W_1 = \{f : \|hf\|_p \leq 1\} \subset \lambda H^o.$$

It follows now easily that $\sup_{u \in H} \|u\|_p < \infty$. Also, as in the proof of the implication (1) $\Rightarrow$ (2) in the preceding Theorem, we prove that $|u|_p(\chi_{V_n}) \to 0$ uniformly for $u \in H$. For the proofs of the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) we use an argument analogous to the one used in the proof of the preceding Theorem.

Theorem 13. *In the space $C_b(X)$, β_1 is the finest of all locally solid topologies γ with the following property: If $(f_n) \subset C_b(X)$ with $f_n \downarrow 0$, then $f_n \xrightarrow{\gamma} 0$.*

Proof: By [12], Theorems 3.7 and 3.8, β_1 is locally solid and $f_n \stackrel{\beta 1}{\to} 0$ when $f_n \downarrow 0$. Consider now the family $\mathcal{U}$ of all solid absolutely convex subsets W of $C_b(X)$ such that $f_n \in W$ eventually when $f_n \downarrow 0$. Clearly $\mathcal{U}$ is a base at zero for the finest locally solid topology γ_o on $C_b(X)$ having the property mentioned in the Theorem.

Claim I : γ_o is coarser than τ_u. Indeed, let $W \in \mathcal{U}$ and let $\lambda \in \mathbb{K}$, $0 < |\lambda| < 1$. For each n, let g_n be the constant function λ^n. Since $g_n \downarrow 0$, there exists an n with $g_n \in W$. If now $f \in C_b(X)$ with $\|f\| \le |\lambda|^n$, then $f \in W$, which implies that W is a τ_u-neighborhood of zero.

Claim II : β_1 is finer than γ_o and hence $\beta_1 = \gamma_o$. Indeed, let $W \in \mathcal{U}$, $Z \in \Omega_1$ and $r > 0$. There exists $\epsilon > 0$ such that

$$W_1 = \{f \in C_b(X) : \|g\| \le \epsilon\} \subset W.$$

Choose $\mu \in \mathbb{K}$ with $|\mu| \ge r$. There exists a decreasing sequence (V_n) of clopen subsets of X with $Z = \bigcap \overline{V_n}^{\beta_o X}$. Since $\mu\chi_{V_n} \downarrow 0$, there exists n such that $\mu\chi_{V_n} \in W$. Let now $f \in C_b(X)$ with $\|f\| \le r$, $\|f\|_{V_n^c} \le \epsilon$, and let $g = f \cdot \chi_{V_n}$, $h = f - g$. Then $|g| \le |\mu\chi_{V_n}|$and so $g \in W$ since W is solid. Also, $\|h\| \le \epsilon$ and so $h \in W$, which implies that $f \in W$. This proves that W is a β_Z-neighborhood of zero for all $Z \in \Omega_1$ and hence W is a β_1-neighborhood of zero. This clearly completes the proof.

The proofs of the following two Theorems are analogous to the ones of Theorems 12 and 13.

Theorem 14. *For a subset H of linear functionals on $C_b(X, E)$, the following are equivalent :*

(1) H is β-equicontinuous.

(2) For each net (V_δ), of clopen subsets of X with $V_\delta \downarrow 0$, there exists $p \in cs(E)$ such that $\sup_{u \in H} \|u\|)p < \infty$ and $|u|_p(\chi_{V_\delta}) \to 0$ uniformly for $u \in H$.

(3) For each net (h_δ) in $C_b(X)$ with $h_\delta \downarrow 0$, there exists $p \in cs(E)$ such that $\sup_{u \in H} \|u\|_p < \infty$ and $|u|_p(h_\delta) \to 0$ uniformly for $u \in H$.

Theorem 15. *In the space $C_b(X)$, β is the finest of all locally solid topologies γ with the following property: If $(f_\delta) \subset C_b(X)$ with $f_\delta \downarrow 0$, then $f_\delta \stackrel{\gamma}{\to} 0$.*

4 The Space $M_b(X, E')$

A subset A of X is called bounding if every $f \in C(X)$ is bounded on A. Note that several authors use the term bounded set instead of bounding. But in this paper we will use the term bounding to distinguish from the notion of a bounded set in a topological vector space. A set $A \subset X$ is bounding iff $\overline{A}^{\upsilon_o X}$ is compact. In this case (as it is shown in [1], Theorem 4.6) we have that $\overline{A}^{\upsilon_o X} = \overline{A}^{\beta_o X}$. Clearly a continuous image of a bounding set is bounding.

Theorem 16 ([17]**.** *, Theorem 3.4] If G is a locally convex space (not necessarily Hausdorff), then every bounding subset A of G is totally bounded.*

We denote by $M_b(X, E')$ the space of all $m \in M(X, E')$ which have a bounding support, i.e. there exists a bounding subset B of X such that $m(V) = 0$ for all clopen V disjoint from B. In case $E = \mathbb{K}$, we write simply $M_b(X)$.

Theorem 17. *If $m \in M_b(X, E')$, then every $f \in C(X, E)$ is m-integrable. Moreover, if B is a bounding support of m and $p \in cs(E)$ with $m_p(X) < \infty$, then*

$$\left| \int f \, dm \right| \le \|f\|_{B,p} \cdot \|m\|_p.$$

Proof: Let $f \in C_b(X,E)$ and let B be a bounding subset of X which is a support set for m. Since the closure of a bounding set is bounding, we may assume that B is closed. Let $p \in cs(E)$ with $m_p(X) < \infty$. The set $f(B)$ is bounding in E and hence totaly bounded by Theorem 4.1. Thus, given $\epsilon > 0$, there are $x_1, \dots, x_n$ in B such that the sets

$$V_k = \{x : p(f(x) - f(x_k)) \le \epsilon/\|m\|_p\}, \quad k = 1, \dots, n,$$

are pairwise disjoint and cover B. Let $V_{n+1} = X \setminus \bigcup_{k=1}^n V_k$ and choose $x_{n+1} \in V_{n+1}$ if $V_{n+1} \ne \emptyset$. Let $\{W_1, \dots, W_N\}$ be a clopen partition of X which is a refinement of $\{V_1, \dots, V_{n+1}\}$ and $y_j \in W_j$. We may assume that $\bigcup_{i=1}^n V_i = \bigcup_{j=1}^k W_j$. If $W_j \subset V_i$ for some $i \le n$, then

$$|m(W_j)[f(y_j) - f(x_i)]| \le \|m\|_p \cdot p(f(y_j) - f(x_i)) \le \epsilon,$$

while, for $W_j \subset V_{n+1}$, we have $m(W_j) = 0$. Thus

$$\left| \sum_{j=1}^N m(W_j) f(y_j) - \sum_{i=1}^n m(V_i) f(x_i) \right| \le \epsilon.$$

This proves that f is m-integrable and

$$\left| \int f\, dm - \sum_{i=1}^n m(V_i) f(x_i) \right| \le \epsilon.$$

Since $|m(V_i) f(x_i)| \le \|f\|_{B,p} \cdot \|m\|_p$, it follows that

$$\left| \int f\, dm \right| \le \max\{\|f\|_{B,p} \cdot \|m\|_p, \epsilon\},$$

for each $\epsilon > 0$, and the proof is complete.

We denote by τ_b the topology on $C(X,E)$ of uniform convergence on the bounding subsets of X.

Lemma 9. *The space $S(X,E)$ is τ_b-dense in $C(X,E)$.*

Proof: Let $f \in C(X,E)$, $p \in cs(E)$, $\epsilon > 0$ and B a bounding subset of X. There are $x_1, \dots, x_n$ in B such that the sets

$$V_k = \{x : p(f(x) - f(x_k)) \le \epsilon\}, \quad k = 1, \dots, n,$$

are pairwise disjoint and cover B. If $g = \sum_{k=1}^n \chi_{V_k} f(x_k)$, then $\|f - g\|_{B,p} \le \epsilon$ and the Lemma follows.

Theorem 18. *For $m \in M_b(X,E')$, let*

$$\psi_m : C(X,E) \to \mathbb{K}, \quad \psi_m(f) = \int f\, dm.$$

Then ψ_m is τ_b-continuous and $M_b(X,E')$ is algebraically isomorphic to the dual space of $(C(X,E), \tau_b)$ via the isomorphism $m \mapsto \psi_m$.

Proof: In view of Theorem 4.2, ψ_m is an element of $G = (C(X,E), \tau_b)'$. On the other hand, let $\psi \in G$. Since $\tau_b|_{C_{rc}(X,E)}$ is coarser than the topology τ_u of uniform convergence, there exists $m \in M(X,E')$ such that $\psi(f) = \int f\, dm$ for all $f \in C_{rc}(X,E)$. Let B a bounding subset of X and $p \in cs(E)$ be such that

$$\{f \in C(X,E) : \|f\|_{B,p} \le 1\} \subset \{f : |\psi(f)| \le 1\}.$$

It follows that B is a support set for m and so $m \in M_b(X, E')$. Now ψ and ψ_m are both τ_b-continuous and they coincide on the τ_b-dense subspace $S(X, E)$ of $C(X, E)$. Thus $\psi = \psi_m$ and the result follows.

Recall that, for $p \in cs(E)$, $\mathcal{M}_{u,p}(X, E')$ denotes the space of all $m \in M_p(X, E')$ such that $m_p(A_\delta) \to 0$ for each decreasing net (A_δ) of clopen subsets of X for which $\bigcap \overline{A_\delta}^{\beta_o X} \in \Omega_u$ (see [13], p. 123).

Theorem 19. *Let $m \in M_b(X, E')$. If $p \in cs(E)$ is such that $\|m\|_p < \infty$, then $m \in \mathcal{M}_{u,p}(X, E')$.*

Proof: Let B be a bounding support for m and let $(V_i)_{i \in I}$ be a clopen partition of X. The set $\overline{B}^{\theta_o X}$ is compact and

$$\overline{B}^{\theta_o X} \subset \theta_o X \subset \bigcup_i \overline{V_i}^{\beta_o X}.$$

Hence, there exists a finite subset J of I such that

$$\overline{B}^{\theta_o X} \subset \bigcup_{i \in J} \overline{V_i}^{\beta_o X}$$

and so $B \subset \bigcup_{i \in J} V_i$, which implies that $m_p(\bigcup_{i \notin J} V_i) = 0$. Thus $m \in \mathcal{M}_{u,p}(X, E')$ by [13], Theorem 5.7.

Theorem 20. *The topology induced by τ_b on $C_b(X, E)$ is coarser than β'_u.*

Proof: Let B be a bounding subset of X, $p \in cs(E)$ and $H \in \Omega_u$. There exists a clopen partition $(V_i)_{i \in I})$ of X such that

$$H \subset \beta_o X \setminus \bigcup_{i \in I} \overline{V_i}^{\beta_o X}.$$

As in the proof of the preceding Theorem, there exists a finite subset J of I such that $B \subset \bigcup_{i \in J} V_i = V$. If $h = \chi_V$, then $h^{\beta_o} = \chi_{\overline{V}^{\beta_o X}}$ vanishes on H and

$$\{f \in C_b(X, E) : \|hf\|_p \le \epsilon\} \subset \{f : \|f\|_{B,p} \le \epsilon\}$$

which clearly completes the proof.

5 $M_s(X)$ as a Completion

The space $M_s(X)$ was introduced in [12]. It is the space of the so called separable members of $M_\sigma(X)$. For $m \in M(X)$, d a continuous ultrapseudometric on X and A a d-clopen subset of X, we define

$$|m|_d(A) = \sup\{|m(B)| : B \subset A, B \quad \mathrm{d}-clopen\}.$$

For $F \subset X$, we define

$$|m|_d^\star(F) = \inf \sup_n |m|_d(A_n),$$

where the infimum is taken over the family of all sequences (A_n) of d-clopen sets which cover F. An element m of $M_\sigma(X)$ is said to be separable if, for each continuous ultrapseudometric d on X, there exists a d-closed, d-separable subset G of X such that $|m|_d^\star(X \setminus G) = 0$. As it is shown in [12], if $m \in M_s(X)$, then every $f \in C_b(X)$ is m-integrable. Let now $G = (C_b(X), \tau_u)'$, where τ_u is the topology of uniform convergence. For each $x \in X$, let δ_x be the corresponding Dirac measure. Thus $\delta_x \in G$, $\delta_x(f) = f(x)$. Let $L(X)$ be the subspace of G spanned by the set $\{\delta_x : x \in X\}$. Let $\mathcal{E}_u$ be the collection of all equicontinuous τ_u-bounded subsets of $C_b(X)$. Consider the dual pair $< C_b(X), L(X) >$.

For d a bounded continuous ultrapseudometric on X, let

$$\pi_d : X \to X_d, \quad x \mapsto \tilde{x}_d,$$

be the quotient map and let

$$T_d : (C_b(X_d), \beta) \to (C_b(X), \beta_e)$$

be the induced linear map. The dual of the space $(C_b(X), \beta_e)$ is the space $M_s(X)$ (see [12], Theorem 6.4) and

$$T_d^\star(M_s(X)) \subset M_\tau(X_d) = M_s(X_d).$$

Theorem 21. *For an $m \in M_\sigma(X)$, the following are equivalent :*

(1) $m \in M_s(X)$.

(2) For each continuous ultrapseudometric d on X, there exists a d-closed, d-separable subset G of X such that $m(V) = 0$ for each d-clopen set V disjoint from G.

Proof: (1) $\Rightarrow$ (2). Let d be a continuous ultrapseudometric on X and let $\mu = T_d^\star m \in M_\tau(X_d)$. By [12], Theorem 6.2, there exists a closed separable subset Z of X_d such that $|\mu|^\star(X_d \setminus Z) = 0$. If $z \in X_d \setminus Z$, then $N_\mu(z) = 0$. In fact, given $\epsilon > 0$, there is a sequence (A_n) of clopen subsets of X_d covering $X_d \setminus Z$ and $\sup_n |\mu|(A_n) < \epsilon$ and so $N_\mu(z) < \epsilon$. If now B is a clopen subset of X_d disjoint from Z, then $|\mu|(B) = \sup_{z \in B} N_\mu(z) = 0$. If $G = \pi_d^{-1}(Z)$, then G is d-closed, d-separable and $m(V) = 0$ for each d-clopen set V disjoint from G.

(2) $\Rightarrow$ (1). Let $(V_i)_{i \in I}$ be a clopen partition of X and let $f_i = \chi_{V_i}$. Define

$$d(x, y) = \sup_i |f_i(x) - f_i(y)|.$$

Then, d is a continuous ultrapseudometric on X. Each V_i is d-clopen and hence $\bigcup_{i \in J} V_i$ is d-clopen for each subset J of I. Since G is d-separable (and hence d-Lindelöf), there exists a countable subset $J = \{i_1, i_2, \ldots\}$ such that $G \subset \bigcup_k V_{i_k}$. Let $J_1 = I \setminus J$. The set $V = \bigcup_{i \in J_1} V_i$ is d-clopen and $m(V) = 0$. Also, $m(V_i) = 0$ for $i \in J_1$. Since m is σ-additive, we have that

$$m(X) = m(V) + \sum_{k=1}^{\infty} m(V_{i_k}) = \sum_{k=1}^{\infty} m(V_{i_k}) = \sum_{i \in I} m(V_i).$$

This (In view of [12], Theorem 6.9) proves that $m \in M_s(X)$ and the result follows.

Lemma 10. *If $B \in \mathcal{E}_u$, then the bipolar B^{oo} of B, with respect to $< C_b(X), L(X) >$, is also in $\mathcal{E}_u$.*

Proof: Let $\sigma = \sigma(C_b(X), L(X))$. By [21], Proposition 4.10, we have that $B^{oo} = \left(\overline{co(B)}^\sigma\right)^e$, where $co(B)$ is the absolutely convex hull of B , $\overline{co(B)}^\sigma$ the σ-closure of $co(B)$ and, for A an absolutely convex subset of a vector space E over $\mathbb{K}$, A^e is the edged hull of A (see [25]). Thus, if $|\lambda| > 1$, we have

$$B^{oo} \subset \lambda \overline{co(B)}^\sigma.$$

So it suffices to show that the set $B_1 = \overline{co(B)}^\sigma$ is in $\mathcal{E}_u$. But

$$\sup_{f \in B_1} \|f\| = \sup_{f \in B} \|f\| < \infty.$$

Given $x \in X$, and $\epsilon > 0$, there exists a neighborhood V of x such that $|f(x) - f(y)| \leq \epsilon$ for every $f \in B$ and every $y \in V$. It is easy to see , for $f \in B_1$ and $y \in V$, we have $|f(x)-f(y)| \leq \epsilon$. This proves that $B^{oo} \in \mathcal{E}_u$ and the result follows.

Consider now on $L(X)$ the topology e_u of uniform convergence on the members of $\mathcal{E}_u$. Thus e_u is generated by the family of seminorms p_B, $B \in \mathcal{E}_u$, where $p_B(u) = \sup_{f\in B} |u(f)|$. Let

$$\Delta : X \to L(X), \quad x \mapsto \delta_x.$$

Clearly Δ is one-to-one.

Theorem 22. *The map*

$$\Delta : X \to (\Delta(X), e_u|_{\Delta(X)})$$

is a homeomorphism.

Proof: Let (x_γ) be a net in X converging to some $x \in X$ and let $B \in \mathcal{E}_u$ and $\epsilon > 0$. There exists a neighborhood V of x such that

$$p_B(\delta_x - \delta_y) = \sup_{f\in B} |f(x) - f(y)| < \epsilon$$

if $y \in V$. Let γ_o be such that $x_\gamma \in V$ if $\gamma \geq \gamma_o$. Now, for $\gamma \geq \gamma_o$, we have that $p_B(\delta_x - \delta_{x_\gamma}) < \epsilon$, which proves that Δ is continuous. Conversely, suppose that for a net (x_γ) in X, we have that $\delta_{x_\gamma} \xrightarrow{e_u} \delta_x$ and let V be a clopen neighborhood of x. Let $f = \chi_V$, $B = \{f\} \in \mathcal{E}_u$. There exists a γ_o such that $p_B(x - x_\gamma) = |f(x) - f(y)| < 1$ when $\gamma \geq \gamma_o$. But then $x_\gamma \in V$ when $\gamma \geq \gamma_o$, which proves that $x_\gamma \to x$, and the result follows.

In view of the preceding Theorem, we may consider X as a topological subspace of $(L(X), e_u)$.

Theorem 23. *e_u is the finest of all polar locally convex topologies γ on $L(X)$ which induce on X its topology and for which X is a bounded subset of $(L(X), \gamma)$.*

Proof: The topology e_u is clearly polar. We show first that X is e_u-bounded. Indeed, let $B \in \mathcal{E}_u$ and choose $\lambda \in \mathbb{K}$ with $|\lambda| > \sup_{f\in B} \|f\|$. Since $|\delta_x(f)| \leq |\lambda|$, for all $f \in B$, we have that $X \subset \lambda B^o$, and so X is e_u-bounded. Suppose now that γ is a polar topology on $L(X)$ which induces on X its topology and for which X is γ-bounded. Let W be a polar γ-neighborhood of zero in $L(X)$ and take $B = \{\phi|_X : \phi \in W^o\}$, where W^o is the polar of W in the dual space of $(L(X), \gamma)$. Every $f \in B$ is continuous on X. Since X is γ-bounded, there exists $\lambda \in \mathbb{K}$, such that $X \subset \lambda W$ and so $\sup_{f\in B} \|f\| \leq |\lambda|$. Also, B is an equicontinuous set. In fact, let $x \in X \subset \lambda W$. Let α be a non-zero element of $\mathbb{K}$ and take $V = (x + \alpha W) \cap X$. Then V is a neighborhood of x in X. If $y \in V$, then for $\phi \in W^o$ and $f = \phi|_X$, we have $|fy) - f(x)| \leq |\alpha|$. This proves that $B \in \mathcal{E}_u$. Moreover $B^o \subset W^{oo} = W$, which proves that W is a neighborhood of zero in $L(X)$ for the topology e_u. This completes the proof.

Theorem 24. *The dual space of $F = (L(X), e_u)$ coincides with $C_b(X)$.*

Proof: Since e_u is finer than the weak topology $\sigma(L(X), C_b(X))$, it follows that $C_b(X)$ is contained in F' (considering every element of $C_b(X)$ as a linear functional on $L(X)$). On the other hand, let $\phi \in F'$ and define $f : X \to \mathbb{K}, \quad f(x) = \phi(\delta_x)$. Then f is continuous. Since X is e_u-bounded, there exists $\lambda \in \mathbb{K}$ such that $X \subset \lambda D$, where $D = \{u \in L(X) : |\phi(u)| \leq 1\}$. It follows that $\|f\| \leq |\lambda|$ and so $f \in C_b(X)$. It is now clear that $\phi(u) =< f, u >$, for all $u \in L(X)$, and the result follows.

Next we will look at the completion $\hat{F}$ of the space $F = (L(X), e_u)$. Since F is a Hausdorff polar space, $\hat{F}$ is the space of all linear functionals on $F' = C_b(X)$ which are $\sigma(C_b(X), L(X))$-continuous on each e_u-equicontinuous subset of $C_b(X)$ (by [16]). We will prove that $\hat{F}$ coincides with the space $M_s(X)$ equipped with the topology of uniform convergence on the members of $\mathcal{E}_u$.

Lemma 11. *A subset B of $C_b(X)$ is e_u-equicontinuous iff $B \in \mathcal{E}_u$.*

Proof: If $B \in \mathcal{E}_u$, then B^o is an e_u-neighborhood of zero and so B^{oo} (and hence also its subset B) is e_u-equicontinuous . Conversely, let B be an e_u-equicontinuous subset of $C_b(X)$. There exists $B_1 \in \mathcal{E}_u$ such that $B \subset B_1^{oo}$. Since $B_1^{oo} \in \mathcal{E}_u$, the same holds for B and the Lemma follows.

Theorem 25. *The completion of the space $F = (L(X), e_u)$ is the space $M_s(X)$ equipped with the topology of uniform convergence on the members of $\mathcal{E}_u$.*

Proof: Let $u \in \hat{F}$. Then u is a linear functional on $F' = C_b(X)$.

Claim I. u is τ_u-continuous. In fact, Let (f_n) be a sequence in $C_b(X)$ with $f_n \xrightarrow{\tau_u} 0$. The set $B = \{f_n : n \in \mathbf{N}\}$ belongs to $\mathcal{E}_u$ and $f_n \to 0$ in the weak topology $\sigma(C_b(X), L(X))$. Since $u \in \hat{F}$, we have that $u(f_n) \to 0$, which proves that u is τ_u-continuous.

Claim II. u is β_u-continuous. To prove this, it suffices to show that, on every member of $\mathcal{E}_u$, u is continuous with respect to the topology of simple convergence (by [12], Theorem 6.4). But the last topology coincides with $\sigma(C_b(X), L(X))$. Hence the claim follows.

By [12], Theorem 6.4, there exists an $m \in M_s(X)$ such that $u(f) = \int f\,dm$, for all $f \in C_b(X)$. Conversely, if $m \in M_s(X)$, then the linear functional u_m on $C_b(X)$, $u_m(f) = \int f\,dm$, is in $\hat{F}$ by Lemma 11 and by [12], Theorem 6.4. This clearly completes the proof.

Theorem 26. *Let E be a Hausdorff polar locally convex space and let $f : X \to E$ be continuous such that $f(X)$ is bounded. Then there exists a unique continuous linear map $T : (L(X), e_u) \to E$ such that $T = f$ on X. If E is in addition complete, then there exists a continuous linear map $T : (M_s(X), e_u) \to E$ such that $T = f$ on X.*

Proof: Let $T : (L(X), e_u) \to E$ be the unique continuous linear extension of f. We need to show that T is e_u-continuous. Let τ_o be the polar topology of E. Then $\tau_1 = T^{-1}(\tau_o)$ is polar and so the supremum $\tau_2 = e_u \vee \tau_1$ is polar. It is easy to see that X is τ_2-bounded. Also $\tau_2|_X$ coincides with the topology of X. In view of Theorem 23, τ_2 coincide with e_u which clearly implies that T is e_u-continuous. In case E is complete, T has a continuous linear extension $\hat{T} : (M_s(X), e_u) \to E$ since $(L(X), e_u)$ is a dense topological subspace of $(M_s(X), e_u)$. Hence the result follows.

A linear functional ϕ on $C_b(X)$ is said to be bounded if it is τ_u-continuous. Equivalently, ϕ is bounded if

$$\|\phi\| = \sup\{|\phi(f)|/\|f\| : f \in C_b(X), f \neq 0\} < \infty.$$

Theorem 27. *For a linear functional ϕ on $C_b(X)$ the following are equivalent :*

(1) There exists $m \in M_s(X)$ such that $\phi(f) = \int f\,dm$ for all $f \in C_b(X)$.

(2) ϕ is bounded and, for each equicontinuous net (f_δ) in $C_b(X)$, with $f_\delta \downarrow 0$, we have that $\phi(f_\delta) \to 0$.

Proof: (1) $\Rightarrow$ (2) . Let $m \in M_s(X)$ be such that $\phi = u_m$, $u_m(f) = \int f\,dm$. By Theorem 25, ϕ belongs to the completion of $F = (L(X), e_u)$. Then ϕ is bounded. Let $(f_\delta)_{\delta\in\Delta}$ be an equicontinuous net with $f_\delta \downarrow 0$. If $\delta_o \in \Delta$, then taking the subnet $(f_\delta)_{\delta\geq\delta_o}$ we see that $\{f_\delta : \delta \geq \delta_o\} \in \mathcal{E}_u$. Since $f_\delta(x) \to 0$ for all x, we have that $\phi(f_\delta) \to 0$.

(2) $\Rightarrow$ (1). Since ϕ is bounded, there exists an $m \in M(X)$ such that $\phi(f) = \int f\,dm$ for all $f \in C_{rc}(X)$.

Claim I. $m \in M_s(X)$. Indeed, let $(V_i)_{i\in I}$ be a clopen partition of X. For each finite subset J of I, let $A_J = \bigcup_{i\in J} V_i$, $B_J = A_J^c$. If $f_J = \chi_{B_J}$, then $f_J \downarrow 0$. Also (f_J) is equicontinuous and $f_J \to 0$ pointwise. By our hypothesis, $m(B_J) = \phi(f_J) \to 0$. Thus $m(X) - \sum_{i\in J} m(V_i) = m(B_J) \to 0$, and so $m \in M_s(X)$ by [12], Theorem 6.9.

Claim II. $\phi = u_m$. Indeed, let $f \in C_b(X)$ and $\epsilon > 0$. consider the equivalence relation $\sim$ on X, $x \sim y$ iff $|f(x) - f(y)| \leq \epsilon$. Let $(V_i)_{i \in I}$ be the clopen partition of X corresponding to $\sim$. Let $x_i \in V_i$, $\alpha_i = f(x_i)$. For each finite subset J of I, let $g_J = \sum_{i \in J} \alpha_i \chi_{V_i}$, $\quad h_J = \sum_{i \notin J} \alpha_i \chi_{V_i}$. Then (h_J) is equicontinuous and $h_J \downarrow 0$. By our hypothesis, $\phi(h_j) \to 0$. Also, $u_m(h_J) \to 0$. Hence there exists J such that $|u_m(h_J)| < \epsilon$, $|\phi(h_j)| < \epsilon$. Let $g = f - g_J - h_J$. Then $\|g\| \leq \epsilon$. Hence

$$|\phi(g)| \leq \|\phi\| \cdot \|g\| \leq \epsilon \|\phi\|, \quad |u_m(g)| \leq \epsilon \|m\|.$$

Since $\phi(g_J) = u_m(g_J)$, it follows that

$$|\phi(f) - u_m(f)| \leq \max\{\epsilon \|\phi\|, \quad \epsilon \|m\|\}.$$

As $\epsilon > 0$ was arbitrary, we conclude that $\phi(f) = u_m(f)$ and the proof is complete.

Lemma 12. *For d a bounded continuous ultrapseudometric on X the map*

$$T_d^\star : (M_s(X), e_u) \to (M_\tau(X_d), e_u)$$

is continuous.

Proof: It follows from the fact that, if $A \in \mathcal{E}_u(X_d)$, then $B = T_d(A) \in \mathcal{E}_u(X)$ and $T_d^\star(B^o) \subset A^o$.

Theorem 28. *$(M_s(X), e_u)$ is the projective limit of the spaces $(M_\tau(X_d), e_u)$, with respect to the maps $T_d^\star$, where d ranges over the family of all bounded continuous ultrapseudometrics on X.*

Proof: We need to show that the topology e_u is the weakest of all locally convex topologies τ on $M_s(X)$ for which each

$$T_d^\star : (M_s(X), \tau) \to (M_\tau(X_d), e_u)$$

is continuous. Let τ be such a topology and let $B \in \mathcal{E}_u(X)$. Define $d(x, y) = \sup_{f \in B} |f(x) - f(y)|$. Then d is a bounded continuous ultrapseudometric on X. For each $f \in B$, the function

$$\tilde{f} : X_d \to \mathbb{K}, \quad \tilde{f}(\tilde{x}_d) = f(x),$$

is well defined and continuous. Clearly the set $A = \{\tilde{f} : f \in B\}$ is uniformly bounded. Let $\tilde{x}_d \in X_d$ and $\epsilon > 0$. The set

$$V = \{\tilde{y}_d : \tilde{d}(\tilde{x}_d, \tilde{y}_d) \leq \epsilon\}$$

is a neighborhood of $\tilde{x}_d$ and, for $\tilde{y}_d \in V$ and $f \in B$, we have

$$|\tilde{f}(\tilde{y}_d) - \tilde{f}(\tilde{x}_d)| \leq \tilde{d}(\tilde{x}_d, \tilde{y}_d) \leq \epsilon.$$

Thus $A \in \mathcal{E}_u(X_d)$. Since $T_d^\star$ is τ-continuous, the set $M = (T_d^\star)^{-1}(A^o)$ is a τ-neighborhood of zero. But $M \subset B^o$. Thus B^o is a τ-neighborhood of zero, which proves that τ is finer than e_u. Hence the result follows.

6 $M_{s\upsilon_o}(X)$ as a Completion

For $X \subset Y \subset \beta_o X$, and $m \in M(X)$, we denote by m^Y the element of $M(Y)$ defined by $m^Y(V) = m(V \cap X)$. We denote by m^{υ_o} and m^{β_o} the m^Y for $Y = \upsilon_o X$ and $Y = \beta_o X$, respectively.

Theorem 29. *([17], Theorem 2.4) Let $m \in M_p(X, E')$ and $\mu = m^{\beta_o}$. The following are equivalent:*

(1) $supp(\mu) \subset \upsilon_o X$.

(2) *If* $V_n \downarrow \emptyset$, *then there exists an* n_o *such that* $m(V_n) = 0$ *for every* $n \geq n_o$.

(3) *If* $V_n \downarrow \emptyset$, *then there exists an* n *such that* $m(V) = 0$ *for every clopen set* V *contained in* V_n.

(4) *For every* $Z \in \Omega_1$ *there exists a clopen subset* A *on* $\beta_o X$ *disjoint from* Z *and such that* $supp(\mu) \subset A$.

(5) *If* $V_n \downarrow \emptyset$, *then there exists an* n *such that* $m_p(V_n) = 0$.

For each $x \in X$, δ_x may be considered as an element of the algebraic dual $C(X)^\star$ of the space $C(X)$. Let $L(X)$ be the subspace of $C(X)^\star$ spanned by the set $\{\delta_x : x \in X\}$. Let $\mathcal{E} = \mathcal{E}(X)$ be the family of all pointwise bounded equicontinuous subsets of $C(X)$.

Lemma 13. *The bidual* B^{oo}, *of a set* $B \in \mathcal{E}$, *with respect to the pair* $< C(X), L(X) >$, *is also in* $\mathcal{E}$.

Proof: The proof is analogous to the one of Lemma 10.

Consider on $L(X)$ the locally convex topology e of uniform convergence on the members of $\mathcal{E}$. As in Theorem 30, we have the following

Theorem 30. *If* $\Delta : X \to L(X), \quad x \mapsto \delta_x$, *then the map*

$$\Delta : X \to (\Delta(X), e|_{\Delta(X)})$$

is a homeomorphism.

In view of the preceding Theorem, we may consider X as a topological subspace of $(L(X), e)$.

Theorem 31. *e is the finest of all polar topologies on $L(X)$ which induce on X its topology.*

Proof: The proof is analogous to the one of Theorem 11.

The proof of the following Theorem is analogous to the one of Theorem 24.

Theorem 32. *The dual space of* $G = (L(X), e)$ *coincides with* $C(X)$.

Lemma 14. *A subset* B, *of the dual space* $C(X)$ *of* $G = (L(X), e)$, *is* e*-equicontinuous iff* $B \in \mathcal{E}$.

Proof: The proof is analogous to that of Lemma 11.

Next we will look at the completion of the space $G = (L(X), e)$. Since G is Hausdorff and polar, its completion $\hat{G}$ coincides with the space of all linear functionals on $G' = C(X)$ which are $\sigma(C(X), L(X))$-continuous (equivalently continuous with respect to the topology of simple convergence on e-equicontinuous subsets of $C(X)$, i.e. on the members of $\mathcal{E}$. The topology of $\hat{G}$ is that of uniform convergence on the members of $\mathcal{E}$. Let $M_{s\upsilon_o}(X)$ be the space of all $m \in M_s(X)$ for which $supp(m^{\beta_o}) \subset \upsilon_o X$. For $m \in M_{s\upsilon_o}(X)$, we will show that every $f \in C(X)$ is m-integrable . Thus m defines a linear functional u_m on $C(X)$, $u_m(f) = \int f\, dm$. We will prove that $M_{s\upsilon_o}(X)$ is algebraically isomorphic to $\hat{G}$ via the isomorphism $m \mapsto u_m$.

Theorem 33. *If* $m \in M_b(X)$, *then* $u_m \in \hat{G}$.

Proof: Let D be a bounding subset of X which is a support set for m. The set $Z = \bar{D}^{\beta_o X}$ is contained in $\theta_o X$. Let $B \in \mathcal{E}$ and let (f_δ) be a net in B which converges pointwise to the zero function. Since the set $B^{\theta_o} = \{f^{\theta_o} : f \in B\}$ is in $\mathcal{E}(\theta_o X)$ (by [17] Theorem 3.10), given $z \in Z$ and $\epsilon > 0$, there exists a clopen neighborhood W_z of z in $\theta_o X$ such that $|f^{\theta_o}(z) - f^{\theta_o}(y)| \leq \epsilon/\|m\|$ for all $f \in B$ and all $y \in W_z$. In view of the compactness of Z, there are $z_1, \ldots, z_n$ in Z such that $Z \subset \bigcup_{k=1}^n W_{z_k}$. Let $V_k = X \cap W_{z_k}$. If $a, b \in V_k$, then $|f(a) - f(b)| \leq \epsilon/\|m\|$ for all

$f \in B$. Let $A_1 = V_1, \quad A_{k+1} = V_{k+1} \setminus \bigcup_{i=1}^{k} V_k$, for $k = 1, \ldots, n-1$. Keeping those A_i which are not empty, we may assume that $A_i \neq \emptyset$ for all i. Choose $x_i \in A_i$. Clearly $|m|(X \setminus \bigcup_{k=1}^{n} A_k) = 0$. Since $f_\delta \to 0$ pointwise, there exists δ_o such that

$$\max\{|f_\delta(x_k)| : 1 \leq k \leq n\} \leq \epsilon / \|m\|$$

for all $\delta \geq \delta_o$. Let now $\delta \geq \delta_o$. Then

$$\left| \int_{A_k} f_\delta \, dm - m(A_k) f_\delta(x_k) \right| \leq \epsilon \quad \textit{and} \quad |m(A_k) f_\delta(x_k)| \leq \epsilon,$$

which implies that $|\int_{A_k} f_\delta \, dm| \leq \epsilon$. Thus, for $\delta \geq \delta_o$, we have

$$\left| \int f_\delta \, dm \right| = \left| \sum_{k=1}^{n} \int_{A_k} f_\delta \, dm \right| \leq \epsilon,$$

which completes the proof.

Theorem 34. *Let $m \in M_{s\upsilon_o}(X)$, $g \in C(X)$ and d a continuous ultrapseudometric on X be such that g is d-uniformly continuous. Then :*

(1) g is m-integrable.

(2) If $\mu = T_d^\star m \in M_\tau(X_d)$, then μ has compact support.

(3) The function

$$\tilde{g} : X_d \to \mathbb{K}, \quad \tilde{g}(\tilde{x}_d) = g(x),$$

is well defined and continuous. Moreover $\int \tilde{g} \, d\mu = \int g \, dm$.

(4) $u_m \in \hat{G}$.

Proof: (1). Let $V_n = \{x \in X : |g(x)| \leq n\}, \quad W_n = V_n^c$. Since $W_n \downarrow 0$ and $supp(m^{\beta_o}) \subset \upsilon_o X$, there exists n such that $|m|(W_n) = 0$ (by Theorem 29). Let $h = g \cdot \chi_{V_n}$. Then $f = h$ m.a.e. (see [14, Definition 2.4]), and so f is m-integrable since h is m-integrable. Moreover $\int g \, dm = \int h \, dm$.

(2) Since μ is τ-additive, we have

$$supp(\mu^{\beta_o}) = \overline{supp(\mu)}^{\beta_o X_d}.$$

Now it suffices to show that $supp(\mu)$ is bounding since X_d is a μ_o-space. So we need to prove that $supp(\mu^{\beta_o}) \subset \upsilon_o X_d$. To show this it is enough to prove that

$$supp(\mu^{\beta_o}) \subset \pi^{\beta_o}(supp(m^{\beta_o})) = D,$$

where $\pi : X \to X_d$ is the quotient map. So, let W be a clopen subset of $\beta_o X$ which is disjoint from D. Then $(\pi^{\beta_o})^{-1}(W)$ is disjoint from $supp(m^{\beta_o})$ and

$$\begin{aligned} \mu^{\beta_o}(W) &= \mu(W \cap X_d) = < T_d^\star m, \chi_{W \cap X_d} > \\ &= m(\pi^{-1}(W \cap X_d)) = m^{\beta_o}\left(\overline{\pi^{-1}(W \cap X_d)}^{\beta_o X}\right). \end{aligned}$$

But

$$\pi^{-1}(W \cap X_d) \subset (\pi^{\beta_o})^{-1}(W) \quad \textit{and so} \quad \overline{\pi^{-1}(W \cap X_d)}^{\beta_o X} \subset (\pi^{\beta_o})^{-1}(W)$$

which implies that $\mu^{\beta_o}(W) = 0$. It follows that the support of μ^{β_o} is contained in D and this proves (2).

(3). It is easy to see that $\tilde{g}$ is well defined and continuous. Let

$$A_n = \{x \in X : |g(x)| \le n\}.$$

There exists an n such that $|m|(A_n^c) = 0$. If $h = g \cdot \chi_{A_n}$, then $\pi(A_n)$ is d-clopen and $\tilde{h} = \tilde{g} \cdot \chi_{\pi(A_n)}$. If Y is a clopen subset of X_d disjoint from $\pi(A_n)$, then $\mu(Y) = m(\pi^{-1}(Y)) = 0$ since $\pi^{-1}(Y)$ is disjoint from A_n. Thus

$$\int g\,dm = \int h\,dm = \int \tilde{h}\,d\mu = \int \tilde{g}\,d\mu.$$

(4). Let $B \in \mathcal{E}$ and let (f_δ) be a net in B which converges pointwise to the zero function. Define $d(x,y) = \sup_{f\in B} |f(x) - f(y)|$. Now $\tilde{B} = \{\tilde{f} : f \in B\} \in \mathcal{E}(X_d)$ and $\tilde{f}_\delta \to 0$ pointwise. Since μ has a bounding support, we have that $\int f_\delta\,dm = \int \tilde{f}_\delta\,d\mu \to 0$ by the preceding Theorem. This proves that $u_m \in \tilde{G}$ and the result follows.

Theorem 35. *If $\phi \in \hat{G}$, then there exists an $m \in M_{sv_o}(X)$ such that $\phi = u_m$.*

Proof: Let $B \in \mathcal{E}_u$ and let (f_δ) be a net in B which converges pointwise to the zero function. Then $\phi(f_\delta) \to 0$, which proves that $\phi|_{C_b(X)}$ belongs to the completion of the space $F = (L(X), e_u)$. Thus, by Theorem 5.7, there exists $m \in M_s(X)$ such that $\phi(f) = \int f\,dm$ for all $f \in C_b(X)$. We will show first that $supp(m^{\beta_o}) \subset v_oX$. In fact, assume that there exists a $z \in supp(m^{\beta_o}) \setminus v_oX$. Let (V_n) be a sequence of clopen subsets of X, with $V_n \downarrow \emptyset$ and $z \in \overline{V_n}^{\beta_o X}$ for all n. Since $z \in supp(m^{\beta_o})$, there exists a clopen subset A_n of $\overline{V_n}^{\beta_o X}$ with $m^{\beta_o}(A_n) = \alpha_n \neq 0$. Let $B_n = A_n \cap X$ and $f_n = \alpha_n^{-1}\chi_{B_n}$. Given $x \in X$, there exists n_o such that $x \notin V_{n_o}$. For $y \notin V_{n_o}$, we have $f_n(y) = 0$ for all $n \ge n_o$. Hence $(f_n) \in \mathcal{E}$ and $f_n \to 0$ pointwise. Thus

$$1 = \alpha_n^{-1} m(B_n) = \int f_n\,dm \to 0,$$

a contradiction. This proves that $m \in M_{sv_o}(X)$. We will finish the proof by showing that $\phi(f) = \int f\,dm$ for all $f \in C(X)$. So, let $f \in C(X)$. For each positive integer n, let

$$A_n = \{x : |f(x)| \ge n\}, \quad f_n = f \cdot \chi_{A_n}, \quad g_n = f - f_n.$$

Then $(f_n) \in \mathcal{E}$ and $f_n \to 0$ pointwise. Thus $\phi(f_n) \to 0$ and $u_m(f_n) \to 0$. Also, $\phi(g_n) = u_m(g_n)$. It follows that $\phi(f) - u_m(f) = 0$, which completes the proof.

Combining Theorems 34 and 35, we get

Theorem 36. *The completion of the space $G = (L(X), e)$ coincides with the space $M_{sv_o}(X)$ equipped with the topology of uniform convergence on the members of $\mathcal{E}$.*

By Theorem 33, $M_b(X)$ is a subspace of $M_{sv_o}(X)$. We will denote also by e the topology on $M_b(X)$ of uniform convergence on the members of $\mathcal{E}$. For d a continuous ultrapseudometric on X, let $\pi_d : X \to X_d$ be the quotient map and let $S_d : C(X_d) \to C(X)$ be the induced linear map. As it is shown in Theorem 34, if $m \in M_{sv_o}(X)$, then $S_d^* m$ has compact support, i.e. $S_d^* m \in M_c(X_d)$.

Lemma 15. *For each continuous ultrapseudometric d on X, the map*

$$S_d^\star : (M_{sv_o}(X), e) \to (M_c(X_d), e)$$

is continuous.

Proof: Let $A \in \mathcal{E}(X_d)$, $\quad B = S_d(A)$. Then $B \in \mathcal{E}(X)$. If B^o is the polar of B in $M_{sv_o}(X)$ and A^o the polar of A in $M_b(X_d) = M_c(X_d)$, then $S_d^\star(B^o) \subset A^o$ and the result follows.

Theorem 37. *$(M_{sv_o}(X), e)$ is the projective limit of the spaces $(M_c(X_d), e)$, with respect to the maps $S_d^\star$, where d ranges over the family of all continuous ultrapseudometrics on X.*

Proof: We need to show that e is the weakest of all locally convex topologies τ on $M_{sv_o}(X)$ for which each of the maps

$$S_d^\star : (M_{sv_o}(X), \tau) \to (M_c(X_d), e)$$

is continuous. So, let τ be such a topology and let $B \in \mathcal{E}(X)$. Define

$$d(x,y) = \sup_{f\in B} |f(x) - f(y)|.$$

Then d is a continuous ultrapseudometric on X. For each $f \in B$, the function

$$\tilde{f} : X_d \to \mathbb{K}, \quad \tilde{f}(\tilde{x}_d) = f(x)$$

is well defined and continuous. Clearly the set $A = \{\tilde{f} : f \in B\}$ is in $\mathcal{E}(X_d)$. Since $S_d^\star$ is τ-continuous, the set $M = (S_d^\star)^{-1}(A^o)$ is a τ-neighborhood of zero. But $M \subset B^o$. Thus B^o is a τ-neighborhood of zero, which proves that τ is finer that e. Hence the result follows.

Theorem 38. *For an $m \in M(X)$, the following are equivalent:*

(1) $m \in M_{sv_o}(X)$.

(2) For each continuous ultrapseudometric d on X the measure

$$m_d : K(X_d) \to \mathbb{K}, \quad m_d(A) = m(\pi_d^{-1}(A))$$

has compact support.

(3) For each clopen partition $(A_i)_{i\in I}$ of X, there exists a finite subset J_o of I such that $m(\bigcup_{i\notin J} A_i) = 0$ for all finite subsets J of I which contain J_o.

Proof: (1) $\Rightarrow$ (2). It follows from the fact that $m_d = S_d^\star m$.

(2) $\Rightarrow$ (3). Let $(A_i)_{i\in I}$ be a clopen partition of X and take $f_i = \chi_{A_i}$. If $B_i = \pi_d(A_i)$, then $(B_i)_{i\in I}$ is a clopen partition of X_d. Let Z be a compact support of m_d. There exists a finite subset J_o of I such that $Z \subset \bigcup_{i\in J_o} B_i$. Let the finite subset J of I contain J_o. If $A = \bigcup_{i\notin J} A_i$ and $B = \pi_d(A)$, then $0 = m_d(B) = m(\pi_d^{-1}(B)) = m(A)$.

(3) $\Rightarrow$ (1). Let $(A_i)_{i\in I}$ be a clopen partition of X and let J_o be as in (3). Clearly $m(A_i) = 0$ for all $i \notin J_o$. Thus

$$m(X) = m\left(\bigcup_{i\in J_o} A_i\right) + m\left(\bigcup_{i\notin J_o} A_i\right) = \sum_{i\in J_o} m(A_i) = \sum_{i\in I} m(A_i),$$

and so $m \in M_s(X)$ by [12], Theorem 6.9. To show that

$$supp(m^{\beta_o}) \subset \upsilon_o X$$

it suffices, by Theorem 6.1, to show that if (W_n) is a sequence of clopen subsets of X, with $W_n \downarrow \emptyset$, then there exists n_o such that $m(W_n) = 0$ if $n \geq n_o$. Given such a sequence, let $D_1 = W_1^c$, $D_{n+1} = W_n \setminus W_{n+1}$ for $n \geq 1$. Then (D_n) is a clopen partition of X. By our hypothesis, there exists n_o such that $m(\bigcup_{n\geq n_1} D_n) = 0$ if $n_1 \geq n_o$. For each n, we have $W_n = \bigcup_{k>n} D_k$. Hence, for $n \geq n_o$, we have $m(W_n) = 0$, which completes the proof.

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