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On Armendariz and quasi-Armendariz modules

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Abstract. Let M_R be a module and let $M[x]$ denote the module of polynomials over $R[x]$. We study relations between the set of annihilators in M and the set of annihilators in $M[x]$.

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1 Introduction

Throughout this paper all rings R are associative with unity and all modules M are unital right R-modules. For a module M_R, let $M[x]$ be the set of all formal polynomials in indeterminate x with coefficients from M (i.e., $M[x] = \{ \sum_{i=0}^{s} m_i x^i : s \geq 0, m_i \in M \}$). Then $M[x]$ becomes a right $R[x]$-module under usual addition and multiplication of polynomials. For a subset X of a module M_R, let $r_R(X) = \{ r \in R \mid Xr = 0 \}$. Consider the module $M[x]$ over $R[x]$. Let

$$\mathrm{rAnn}_R(2^M) = \{ r_R(U) \mid U \subseteq M \}$$

and

$$\mathrm{rAnn}_{R[x]}(2^{M[x]}) = \{ r_{R[x]}(V) \mid V \subseteq M[x] \}.$$

For a polynomial $m(x) = m_0 + m_1 x + \cdots + m_s x^s \in M[x]$, $C_{m(x)} = \{ m_0, m_1, \ldots, m_s \}$ and for a subset V of $M[x]$, C_V denotes the set $\bigcup_{m(x)\in V} C_{m(x)}$. Then $r_{R[x]}(V) \cap R = r_R(V) = r_R(C_V)$. Hence we have a map

$$\Psi : \mathrm{rAnn}_{R[x]}(2^{M[x]}) \longrightarrow \mathrm{rAnn}_R(2^M)$$

defined by $\Psi(r_{R[x]}(V)) = r_{R[x]}(V) \cap R$ for each $r_{R[x]}(V) \in \mathrm{rAnn}_{R[x]}(2^{M[x]})$. Now, we are going to show that Ψ is surjective. Let $r_R(U) \in \mathrm{rAnn}_R(2^M)$ for some $U \subseteq M$. If we chose $V = \{ \sum_{i=0}^{t} m_i x^i : t \geq 0, m_i \in U \} \subseteq M[x]$ then $r_{R[x]}(V) \in \mathrm{rAnn}_{R[x]}(2^{M[x]})$ and moreover,

$$\Psi(r_{R[x]}(V)) = r_{R[x]}(V) \cap R = r_R(V) = r_R(C_V) = r_R(U).$$

Therefore Ψ is surjective.

If U is a subset of M_R, then $r_{R[x]}(U) = r_R(U)[x]$. Hence we also have a map

$$\Phi : \mathrm{rAnn}_R(2^M) \longrightarrow \mathrm{rAnn}_{R[x]}(2^{M[x]})$$

defined by $\Phi(r_R(U)) = r_{R[x]}(U) = r_R(U)[x]$ for each $r_R(U) \in \mathrm{rAnn}_R(2^M)$. The map Φ is injective. To show this, let $r_{R[x]}(U) = r_{R[x]}(U^{'})$ for $r_R(U), r_R(U^{'}) \in \mathrm{rAnn}_R(2^M)$. Then $r_R(U)[x] = r_R(U^{'})[x]$ and hence $r_R(U) = r_R(U^{'})$. Consequently, Φ is injective. If Φ is bijective, then its inverse is Ψ. In fact, for all $r_R(U) \in \mathrm{rAnn}_R(2^M)$:

$$(\Psi \circ \Phi)(r_R(U)) = \Psi(\Phi(r_R(U))) = \Psi(r_{R[x]}(U)) = r_{R[x]}(U) \cap R = r_R(U).$$

So $\Psi \circ \Phi = 1_{\mathrm{rAnn}_R(2^M)}$. For each $r_{R[x]}(V) \in \mathrm{rAnn}_{R[x]}(2^{M[x]})$ there exists $r_R(U) \in \mathrm{rAnn}_R(2^M)$ such that $\Phi(r_R(U)) = r_{R[x]}(V)$ since Φ is surjective. So $(\Phi \circ \Psi)(r_{R[x]}(V)) = \Phi(\Psi(r_{R[x]}(V))) = \Phi(\Psi\Phi(r_R(U))) = \Phi(1_{\mathrm{rAnn}_R(2^M)}(r_R(U))) = \Phi(r_R(U)) = r_{R[x]}(V)$ and hence $\Phi \circ \Psi = 1_{\mathrm{rAnn}_{R[x]}(2^{M[x]})}$. Consequently, the inverse of Φ is Ψ.

Following Anderson and Camillo [1] a module M_R is called an *Armendariz module* if whenever $m(x)f(x) = 0$ where $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{t} m_j x^j \in R[x]$, we have $m_i a_j = 0$ for all i and j. We show that Φ is bijective if and only if M_R is Armendariz.

In [6], a module M_R is called a *quasi-Armendariz module* if whenever $m(x)R[x]f(x) = 0$ where $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{t} m_j x^j \in R[x]$, we have $m_i R a_j = 0$ for all i and j.

Let

$$\mathrm{rAnn}_R(\mathrm{sub}(M)) = \{\, r_R(U) \mid U \text{ is a submodule of } M \,\}$$

and

$$\mathrm{rAnn}_{R[x]}(\mathrm{sub}(M[x])) = \{\, r_{R[x]}(V) \mid V \text{ is a submodule of } M[x] \,\}.$$

Consider the map

$$\Phi^{'} : \mathrm{rAnn}_R(\mathrm{sub}(M)) \longrightarrow \mathrm{rAnn}_{R[x]}(\mathrm{sub}(M[x]))$$

the restriction of Φ to $\mathrm{rAnn}_R(\mathrm{sub}(M))$. We show that $\Phi^{'}$ is bijective if and only if M_R is quasi-Armendariz. According to [7] the module M_R is called *quasi-Baer* if, for any submodule N of M, $r_R(N) = eR$ where $e^2 = e \in R$. We give a sufficient condition for a module to be quasi-Armendariz.

2 Armendariz and quasi-Armendariz modules

In this section, we give relations between the set of annihilators in M and the set of annihilators in $M[x]$. The following theorem shows that Φ is bijective if and only if M_R is Armendariz.

1 Theorem. *Let M_R be a module. Then the following statements are equivalent:*

(1) *M_R is an Armendariz module.*

(2) *The map $\Phi : \mathrm{rAnn}_R(2^M) \longrightarrow \mathrm{rAnn}_{R[x]}(2^{M[x]})$ defined by $\Phi(r_R(U)) = r_{R[x]}(U) = r_R(U)[x]$ for every $r_R(U) \in \mathrm{rAnn}_R(2^M)$, is bijective.*

Proof. (1) $\Rightarrow$ (2) Assume M is an Armendariz. Obviously Φ is injective. So it is enough to show Φ is surjective. Let $r_{R[x]}(V) \in \mathrm{rAnn}_{R[x]}(2^{M[x]})$ for some $V \subseteq M[x]$. Then for $r_R(C_V) \in \mathrm{rAnn}_R(2^M)$, $\Phi(r_R(C_V)) = r_{R[x]}(C_V) = r_{R[x]}(V)$. In fact, let $f(x) \in r_{R[x]}(C_V)$ where $f(x) = a_0 + a_1x + \cdots + a_nx^n$. Then $C_V f(x) = 0$. Thus for all $m \in C_V$, $mf(x) = ma_0 + ma_1x + \cdots + ma_nx^n = 0$ and hence $ma_j = 0$ for all j. Let $n(x) = n_0 + n_1x + \cdots + n_tx^t \in V$ be arbitrary. Then $n(x)f(x) = 0$ since $n_i \in C_V$ for all i. Hence $f(x) \in r_{R[x]}(V)$. Conversely, let $g(x) = b_0 + b_1x + \cdots + b_kx^k \in r_{R[x]}(V)$. Then for all $m(x) \in V$, $m(x)g(x) = 0$ where $m(x) = m_0 + m_1x + \cdots + m_lx^l \in V$. Since M_R is Armendariz, $m_ib_j = 0$ for all i and j. Hence $m_ig(x) = 0$ for all i. So $g(x) \in r_{R[x]}(C_V)$ since $m(x) \in V$ is arbitrary. Consequently for each $r_{R[x]}(V) \in \mathrm{rAnn}_{R[x]}(2^{M[x]})$ for some $V \subseteq M[x]$ there exists $r_R(C_V) \in \mathrm{rAnn}_R(2^M)$ such that $\Phi(r_R(C_V)) = r_{R[x]}(V)$ and therefore Φ is surjective.

(2) $\Rightarrow$ (1) Assume $m(x)f(x) = 0$ where $m(x) = m_0 + m_1x + \cdots + m_tx^t \in M[x]$ and $f(x) = a_0 + a_1x + \cdots + a_kx^k \in R[x]$. By hypothesis, $r_{R[x]}(m(x)) = r_R(U)[x]$ for some $U \subseteq M$. Then $f(x) \in r_R(U)[x]$ and hence $a_j \in r_R(U)$ for all j. So $a_j \in r_R(U) \subseteq r_R(U)[x] = r_{R[x]}(m(x))$ then $m(x)a_j = 0$. Consequently, $m_ia_j = 0$ for all i and j. Therefore M_R is an Armendariz. QED

Following Kaplansky [4], a ring R is a *Baer ring* if the left annihilator of each subset is generated by an idempotent. We note that the definition of Baer rings is left-right symmetric. A ring R is called a *left* (resp. *right*) *p.p. ring* if the left (resp. right) annihilator of each element of R is generated by an idempotent. A left and right *p.p.* ring is called a *p.p.* ring.

For a subset X of a module M_R, let $r_R(X) = \{ r \in R : Xr = 0 \}$. In [7] Lee and Zhou introduced Baer modules, quasi-Baer modules and *p.p.*-modules as follows.

(1) M_R is called *Baer* if, for any subset X of M, $r_R(X) = eR$ where $e^2 = e \in R$;

(2) M_R is called *quasi-Baer* if, for any submodule N of M, $r_R(N) = eR$ where $e^2 = e \in R$;

(3) M_R is called *principally projective* (or simply *p.p.*) if, for any $m \in M$, $r_R(m) = eR$ where $e^2 = e \in R$.

We obtain [7, Corollary 2.7 (1) and Corollary 2.12 (1)] as a corollary of Theorem 1.

2 Corollary. *Let M_R be an Armendariz module. Then M_R is a Baer module if and only if $M[x]_{R[x]}$ is a Baer module.*

PROOF. Assume M_R is a Baer module and let V be a subset of $M[x]$. Then by Theorem 1, there exists $U \subseteq M$ such that $\Phi(r_R(U)) = r_{R[x]}(V)$ since M_R is an Armendariz. So $r_R(U)[x] = r_{R[x]}(V)$. Since M_R is a Baer module, there exists $e^2 = e \in R$ such that $r_R(U) = eR$. Thus $r_{R[x]}(V) = eR[x]$ and hence $M[x]_{R[x]}$ is a Baer module. Conversely, the proof can be done by using the same method in the proof of [7, Theorem 2.5. (1)(a)]. QED

3 Corollary ([5], Theorem 10)**.** *Let R be an Armendariz ring. Then R is a Baer ring if and only if $R[x]$ is a Baer ring.*

4 Corollary. *Let M_R be Armendariz module. Then M_R is a p.p. module if and only if $M[x]_{R[x]}$ is a p.p. module.*

PROOF. Similar to the proof of Corollary 2. QED

If we take R instead of M in Corollary 4, then we have

5 Corollary ([5], Theorem 9)**.** *Let R be Armendariz ring. Then R is a p.p. ring if and only if $R[x]$ is a p.p. ring.*

In [6], a module M_R is called a *quasi-Armendariz module* if whenever $m(x)R[x]f(x) = 0$ where $m(x) = \sum_{i=0}^{s} m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^{t} m_j x^j \in R[x]$, we have $m_i R a_j = 0$ for all i and j. Put

$$\mathrm{rAnn}_R(\mathrm{sub}(M)) = \{\, r_R(N) \mid N \text{ is a submodule of } M \,\},$$

$$\mathrm{rAnn}_{R[x]}(\mathrm{sub}(M[x])) = \{\, r_{R[x]}(V) \mid V \text{ is a submodule of } M[x] \,\}.$$

6 Theorem. *Let M_R be a module. The following statements are equivalent:*

(1) *M_R is quasi-Armendariz.*

(2) *The map $\Phi' : \mathrm{rAnn}_R(\mathrm{sub}(M)) \longrightarrow \mathrm{rAnn}_{R[x]}(\mathrm{sub}(M[x]))$ defined by $\Phi'(r_R(N)) = r_{R[x]}(N) = r_{R[x]}(N[x])$ for every $r_R(N) \in \mathrm{rAnn}_R(\mathrm{sub}(M))$, is bijective.*

PROOF. (1) $\Rightarrow$ (2) Assume M_R is quasi-Armendariz. Obviously Φ' is injective. Therefore, it is enough to show Φ' is surjective.

Let $r_{R[x]}(V) \in \mathrm{rAnn}_{R[x]}(\mathrm{sub}(M[x]))$ for some submodule V of $M[x]$. Then for $r_R(C_V R) \in \mathrm{rAnn}_R(\mathrm{sub}(M))$, $\Phi'(r_R(C_V R)) = r_{R[x]}(C_V R) = r_{R[x]}(V)$. In fact, let $f(x) \in r_{R[x]}(C_V R)$. Then $C_V Rf(x) = 0$. In particular, $C_V f(x) = 0$ and hence $Vf(x) = 0$. So $f(x) \in r_{R[x]}(V)$. Conversely, let $g(x) = b_0 + b_1x + \cdots + b_kx^k \in r_{R[x]}(V)$. Then $Vg(x) = 0$. Since V is a submodule of $M[x]$, $VRg(x) = 0$. So $v(x)Rg(x) = 0$ for all $v(x) = v_0 + v_1x + \cdots + v_lx^l \in V$. Since M_R is quasi-Armendariz, $v_iRb_j = 0$ for all i and j. Hence $C_V Rg(x) = 0$ and therefore $g(x) \in r_{R[x]}(C_R V)$. Consequently Φ' is surjective.

(2) $\Rightarrow$ (1) Assume $m(x)R[x]f(x) = 0$ where $m(x) = m_0 + m_1x + \cdots + m_tx^t \in M[x]$ and $f(x) = a_0 + a_1x + \cdots + a_kx^k \in R[x]$. By hypothesis, $r_{R[x]}(m(x)R[x]) = r_R(N)[x]$ for some submodule N of M. Then $f(x) \in r_R(N)[x]$ and hence $a_j \in r_R(N)$ for all j. So $a_j \in r_R(N) \subseteq r_R(N)[x] = r_{R[x]}(m(x)R[x])$ and then $m(x)R[x]a_j = 0$. In particular $m(x)Ra_j = 0$ and hence $m_iRa_j = 0$ for all i and j. Therefore M_R is a quasi-Armendariz. QED

Following [2] a module M_R is called a *semi-commutative module* if it satisfies the following condition: whenever elements $a \in R$ and $m \in M$ satisfy $ma = 0$ then $mRa = 0$.

7 Corollary. *Let M_R be a semi-commutative module. Then M_R is Armendariz if and only if M_R is quasi-Armendariz.*

8 Corollary ([3], Corollary 3.5)**.** *Let R be a semi-commutative ring. Then R is Armendariz if and only if R is quasi-Armendariz.*

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