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Simply connected two-step homogeneous nilmanifolds of dimension 5

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Abstract. The aim of this paper is to classify all simply connected two-step nilpotent Lie groups of dimension 5 equipped with left-invariant metrics ("two-step nilmanifolds") up to isometry. We also calculate the corresponding isometry groups.

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1 Introduction

Two-step nilpotent Lie groups endowed with a left-invariant metric, often called two-step homogeneous nilmanifolds are studied intensively in the last twenty years. A special subclass of two-step homogeneous nilmanifolds, the Heisenberg type groups, was introduced and studied by A. Kaplan (cf. [6], [7]) and others. We refer to [1] for a survey about the geometry of generalized Heisenberg groups. The Heisenberg type groups play an important role in geometric analysis, Lie groups and mathematical physics. J. Lauret (cf. [10]) essentially generalized this concept by introducing so-called *modified H-type groups* (See below).

In [9] J. Lauret classified, up to isometry, all homogeneous nilmanifolds of dimension 3 and 4 (not necessarily two-step nilpotent) and computed the corresponding isometry groups. He also studied, as example, the structure of specific 5-dimensional two-step nilmanifolds with 2-dimensional center. His results will be used in the present paper. Our purpose is to classify all simply connected two-step Riemannian nilmanifolds of dimension 5 and to determine their full isometry groups.

2 Two-step nilpotent Lie groups

A connected Riemannian manifold which admits a transitive nilpotent Lie group N of isometries is called a *nilmanifold*. E. Wilson proved in [12] that, when given a homogeneous nilmanifold M, there exists a unique nilpotent Lie subgroup N of $I(M)$ acting simply transitively on M, and N is normal in $I(M)$. Hence the Riemannian manifold M can be identified with the group N equipped with a left-invariant metric $\langle .,. \rangle$. A left-invariant metric $\langle .,. \rangle$ on N determines an inner product $\langle , \rangle$ on the corresponding Lie algebra $\mathfrak{n} = T_e N$ and conversely. According to [12], the full group of isometries of $(N, \langle .,. \rangle)$ can be expressed as a semi-direct product

$$I\,(N, \langle .,. \rangle) = K \ltimes N, \tag{1}$$

where $K = Aut\,(\mathfrak{n}) \cap O\,(\mathfrak{n}, \langle , \rangle)$ is the isotropy subgroup at the identity element e and N acts by left translations. K is the full group of automorphisms of the Lie algebra $\mathfrak{n}$ which preserve the inner product $\langle , \rangle$. Thus the structure of the full isometry group is completely determined by the isotropy subgroup K. Moreover, if N is simply connected, then the exponential mapping $\exp : \mathfrak{n} \to N$ is a diffeomorphism. We need not make distinction between automorphisms of $\mathfrak{n}$ and those of N.

A Lie algebra $\mathfrak{n}$ is said to be *two-step nilpotent* if $[\mathfrak{n}, \mathfrak{n}] \neq \{0\}$ but $[[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] = \{0\}$. In the following we shall work usually with a two-step nilpotent metric Lie algebra $(\mathfrak{n}, \langle , \rangle)$. We denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and by $\mathfrak{a} = \mathfrak{z}^{\perp}$ the orthogonal complement of $\mathfrak{z}$. Then we have the orthogonal direct sum decomposition $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{z}$. We denote by $so(\mathfrak{a})$ the Lie algebra of skew-symmetric endomorphisms of the Euclidean vector subspace $(\mathfrak{a}, \langle , \rangle_{\mathfrak{a}})$ of $(\mathfrak{n}, \langle , \rangle)$.

1 Definition. Let $(\mathfrak{n}, \langle , \rangle)$ be a two-step nilpotent metric Lie algebra, $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{z}$. For each element Z of $\mathfrak{z}$, define an endomorphism $j(Z) \in so(\mathfrak{a})$ by

$$\langle j(Z)X, Y \rangle = \langle [X, Y], Z \rangle \text{ for all } X, Y \in \mathfrak{a}. \tag{2}$$

The algebraic properties of $(\mathfrak{n}, \langle , \rangle)$ can be expressed in terms of the maps $j(Z)$ with $Z \in \mathfrak{z}$. Indeed, let two metric vector spaces $(\mathfrak{a}, \langle , \rangle_{\mathfrak{a}})$ and $(\mathfrak{z}, \langle , \rangle_{\mathfrak{z}})$ be given and let $(\mathfrak{n}, \langle , \rangle_{\mathfrak{n}})$ be their orthogonal direct sum. Let a linear map $j : \mathfrak{z} \to so(\mathfrak{a})$ be fixed. Define first a Lie bracket on $\mathfrak{a}$ by the condition $[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{z}$ and by (2). Then extend this Lie bracket to the whole of $\mathfrak{n}$ by putting $[\mathfrak{n}, \mathfrak{z}] = \{0\}$. Since $[\mathfrak{n}, \mathfrak{n}] = [\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{z}$ and $[\mathfrak{n}, \mathfrak{z}] = \{0\}$, the algebra $\mathfrak{n}$ (and hence the corresponding simply connected group N with the left-invariant metric $\langle .,. \rangle$) is two-step nilpotent. The mappings $\{ j(Z) : Z \in \mathfrak{z} \}$ contain the "geometry of $(N, \langle , \rangle)$" in the sense that the covariant derivative, curvature tensor and Ricci tensor can be formulated entirely using $j, \mathfrak{a}$ and $\mathfrak{z}$ (cf. [2]). If $\mathfrak{z}$ is equal to the commutator of $\mathfrak{n}$, i.e. $\mathfrak{z} = [\mathfrak{n}, \mathfrak{n}]$, then $j : \mathfrak{z} \to so(\mathfrak{a})$ is an injective linear map.

Among two-step nilpotent Lie groups with left-invariant metrics, the Heisenberg type Lie groups are of particular significance. A two-step nilpotent Lie group N with a left-invariant metric $\langle .,. \rangle$ is said to be an *Heisenberg type* (H-type) *Lie group* if $[j(Z)]^2 = -\langle Z, Z \rangle id_{\mathfrak{a}}$ for any $Z \in \mathfrak{z}$.

More generally, $(N, \langle .,. \rangle)$ is called a *modified H-type group* if $[j(Z)]^2 = -h(Z) id_{\mathfrak{a}}$ for any $Z \in \mathfrak{z}$, where $h(Z)$ is a positive definite quadratic form on $\mathfrak{z}$ (see [10]).

2 Example. *The Heisenberg group* is, up to isomorphism, the only two-step nilpotent Lie group with a 1-dimensional center. The $(2n+1)$-dimensional Heisenberg group H_{2n+1} is the group of all real $(n+2) \times (n+2)$ matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n & z \\ 0 & 1 & 0 & \cdots & 0 & y_1 \\ \vdots & 0 & 1 & 0 & \vdots & y_2 \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ \vdots & & & \ddots & 1 & y_n \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

for $x_i, y_i, z \in \mathbb{R}, i = 1, \ldots, n$. The Lie algebra $\mathfrak{h}_{2n+1}$ of H_{2n+1} is a $(2n+1)$-dimensional vector space with basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ and the only nonzero Lie brackets are $[X_i, Y_i] = -[Y_i, X_i] = Z$ for $1 \le i \le n$.

The center $\mathfrak{z}$ of $\mathfrak{h}_{2n+1}$ equals $\mathfrak{z} = \mathrm{span}_{\mathbb{R}}\{\, Z \,\}$. One determines a left-invariant metric on H_{2n+1} by specifying an inner product $\langle , \rangle$ on $\mathfrak{h}_{2n+1}$.

Now we choose the natural inner product making the basis above an orthonormal basis. Using the equation (2) and the basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z\}$ we obtain that

$$j(Z)X_i = Y_i \text{ and } j(Z)Y_i = -X_i \quad \text{for } 1 \le i \le n.$$

It follows that $[j(Z)]^2 = -id_{\mathfrak{a}}$, which means that the Heisenberg Lie group equipped with the corresponding natural left-invariant metric is an H-type Lie group.

3 5-dimensional two-step nilmanifolds

Now we give a classification of 5-dimensional simply connected two-step nilmanifolds up to isometry. This is equivalent to the classification of the corresponding metric Lie algebras. Clearly, the dimension of the center of a 5-dimensional two-step nilpotent Lie algebra is ≤ 3. We consider separately the cases where the dimension of the center is $1, 2$ or 3.

3.1 Metric Lie algebras with 1-dimensional center

Let $\mathfrak{h}_5$ denotes a 5-dimensional Lie algebra the center $\mathfrak{z}$ of which is one-dimensional. We assume that $\mathfrak{h}_5$ is equipped with an inner product $\langle , \rangle$. (Two-step nilmanifolds with one-dimensional center are usually called Heisenberg manifolds.) Let e_5 be a unit vector in $\mathfrak{z}$ and let $\mathfrak{a}$ be the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{h}_5$. We consider a 2-dimensional vector subspace $\mathfrak{a}_2$ such that $[\mathfrak{a}_2, \mathfrak{a}_2] = \mathfrak{z}$. Since the center of $\mathfrak{h}_5$ is one-dimensional there is a unique vector subspace $\mathfrak{b}_2$ of $\mathfrak{a}$ which is complementary to $\mathfrak{a}_2$ in $\mathfrak{a}$ and commutes with $\mathfrak{a}_2$, namely $\mathfrak{b}_2 = \operatorname{Ker}(ad(u)) \cap \operatorname{Ker}(ad(v)) \cap \mathfrak{a}$, where $\{u, v\}$ is any basis of $\mathfrak{a}_2$. Let us assume that the complementary subspaces $\mathfrak{a}_2$ and $\mathfrak{b}_2$ are not orthogonal. One can see easily that the angle of a variable vector in $\mathfrak{a}_2$ and of its orthogonal projection on $\mathfrak{b}_2$ achieves its maximum and minimum values in two mutually perpendicular directions in $\mathfrak{a}_2$. This angle can be also constant, then the perpendicular directions can be chosen in arbitrary way. (In such a case, $\mathfrak{a}_2$ and $\mathfrak{b}_2$ are called isoclinic.) Then one can choose an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for $\mathfrak{a}$ such that $\{e_1, e_2\}$ is an orthonormal basis of $\mathfrak{a}_2$ and $\{f_1 = \cos\alpha\, e_1 + \sin\alpha\, e_3,\ f_2 = \cos\beta\, e_2 + \sin\beta\, e_4\}$ forms an orthonormal basis for $\mathfrak{b}_2$, where $\alpha, \beta \neq 0$ denote the extremal values of the angles between the two-spaces $\mathfrak{a}_2$ and $\mathfrak{b}_2$. We have

$$[e_1, e_2] = \bar{\lambda}\, e_5 \quad \text{and} \quad [e_3, e_4] = \bar{\mu}\, e_5$$

where $\bar{\lambda} \neq 0$ and e_5 is a unit vector of the center. Then we can compute the remaining Lie brackets of the basis elements:

$$\begin{aligned} [e_1, e_3] &= 0, \qquad [e_1, e_4] = -\bar{\lambda}\, \cot\beta\, e_5 \\ [e_2, e_4] &= 0, \qquad [e_2, e_3] = \bar{\lambda}\, \cot\alpha\, e_5. \end{aligned}$$

Now, we can find a new orthonormal basis of the form:

$$e_1' = \cos t\, e_1 + \sin t\, e_3, \quad e_3' = -\sin t\, e_1 + \cos t\, e_3,$$

$$e_2' = \cos s\, e_2 + \sin s\, e_4 \quad e_4' = -\sin s\, e_2 + \cos s\, e_4,$$

such that $[e_2', e_3'] = 0$ and $[e_1', e_4'] = 0$. Such a basis is determined by the solution $\{\, t, s \in \mathbb{R}\,\}$ of the equations

$$\begin{aligned} \left(\bar{\lambda} + \bar{\mu}\right) \sin(t - s) &= \bar{\lambda}\, (\cot\beta - \cot\alpha) \cos(t - s)\,, \\ \left(\bar{\lambda} - \bar{\mu}\right) \sin(t + s) &= -\bar{\lambda}\, (\cot\beta + \cot\alpha) \cos(t + s)\,. \end{aligned}$$

Hence there exists an orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{a}$ such that

$$[e_1, e_2] = -\,[e_2, e_1] = \lambda\, e_5, \quad [e_3, e_4] = -\,[e_4, e_3] = \mu\, e_5, \tag{3}$$

$$[e_1, e_3] = [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = 0. \tag{4}$$

Moreover we can assume that $\lambda \geq \mu > 0$. A *metric Heisenberg algebra of type* (λ, μ) is defined as a 5-dimensional metric Lie algebra having an orthonormal basis $\{e_1, e_2, e_3, e_4, e_5\}$ satisfying the commutation relations (3) and (4), where $\lambda \geq \mu > 0$. We will denote it by $\mathfrak{h}_5(\lambda, \mu)$.

3 Proposition. *For any* 5*-dimensional* 2*-step nilpotent metric Lie algebra* $\mathfrak{n}$ *with* 1*-dimensional center there exist real numbers* $\lambda \geq \mu > 0$ *such that* $\mathfrak{n}$ *is isomorphic to the metric Heisenberg algebra* $\mathfrak{h}_5(\lambda, \mu)$*. The metric Heisenberg algebras* $\mathfrak{h}_5(\lambda, \mu)$ *and* $\mathfrak{h}_5(\lambda', \mu')$ *are isometrically isomorphic if and only if* $\lambda = \lambda'$ *and* $\mu = \mu'$*.*

PROOF. The first assertion follows from the previous discussion.

Let us consider the metric Heisenberg algebras $\mathfrak{h}_5(\lambda, \mu)$ and $\mathfrak{h}_5(\lambda', \mu')$ given by the orthonormal bases $\{e_1, e_2, e_3, e_4, e_5\}$ and $\{e'_1, e'_2, e'_3, e'_4, e'_5\}$, respectively. A linear isomorphism $\varphi : \mathfrak{h}_5(\lambda, \mu) \to \mathfrak{h}_5(\lambda', \mu')$ is an isometric Lie algebra isomorphism only if it maps the center $\mathrm{span}(e_5)$ onto the center $\mathrm{span}(e'_5)$ and the orthogonal complement $\mathfrak{a}$ onto the orthogonal complement $\mathfrak{a}'$. In particular $\varphi(e_5) = \pm e'_5$. Hence the maps $j(e_5)^2 : \mathfrak{a} \to \mathfrak{a}$ and $j(e'_5)^2 : \mathfrak{a}' \to \mathfrak{a}'$ are selfadjoint endomorphisms with eigenvalues $-\lambda^2$, $-\mu^2$ and $-\lambda'^2$, $-\mu'^2$, respectively. We have the relation

$$\varphi \circ j(e_5)^2 \circ \varphi^{-1} = j(e'_5)^2.$$

Hence any isometric isomorphism $\varphi : \mathfrak{h}_5(\lambda, \mu) \to \mathfrak{h}_5(\lambda', \mu')$ maps the eigenspaces of $j(e_5)^2$ into the eigenspaces of $j(e'_5)^2$ corresponding to the same eigenvalue. It follows $\lambda = \lambda'$ and $\mu = \mu'$. QED

From the above computations and also from [8] we get the following:

4 Corollary. *Each* 5*-dimensional Heisenberg group space* N *corresponding to a metric algebra* $\mathfrak{h}_5(\lambda, \mu)$ *is a modified H-type group in the sense of J. Lauret and it is naturally reductive. It is an H-type group if and only if* $\lambda = \mu$*.*

If now $\lambda \neq \mu$, then the group of isometric isomorphisms of $\mathfrak{h}_5(\lambda, \mu)$ can be represented by the group of matrices

$$\left\{ \begin{pmatrix} \varepsilon \cos t & -\sin t & 0 & 0 & 0 \\ \varepsilon \sin t & \cos t & 0 & 0 & 0 \\ 0 & 0 & \varepsilon \cos s & -\sin s & 0 \\ 0 & 0 & \varepsilon \sin s & \cos s & 0 \\ 0 & 0 & 0 & 0 & \varepsilon \end{pmatrix}, \quad \varepsilon = \pm 1, s, t \in \mathbb{R} \right\}.$$

In the case $\lambda = \mu$ an orthogonal transformation φ is an orthogonal automorphism if and only if its restriction to $\mathfrak{a}$ commutes with the complex structure $J : \mathfrak{a} \to \mathfrak{a}$ determined by

$$J(e_1) = \varepsilon e_2, \quad J(e_2) = -\varepsilon e_1, \quad J(e_3) = \varepsilon e_4, \quad J(e_4) = -\varepsilon e_3$$

and, moreover,

$$\varphi(e_5) = \varepsilon e_5, \text{ where } \varepsilon = \pm 1.$$

Hence one obtains:

5 Proposition. *The group of orthogonal automorphisms of the metric Lie algebra* $\mathfrak{h}_5(\lambda, \mu)$ *is isomorphic to the group* $O(2) \times SO(2)$ *for* $\lambda \neq \mu$*, and it is isomorphic to the group* $U(2) \times \mathbb{Z}_2$ *for* $\lambda = \mu$*.*

3.2 Metric Lie algebras with 2-dimensional center

Let $\mathfrak{n}_5$ denotes a 5-dimensional Lie algebra the center $\mathfrak{z}$ of which is two-dimensional and let N_5 be the corresponding simply connected Lie group. We assume that $\mathfrak{n}_5$ is equipped with an inner product $\langle , \rangle$. Let $\mathfrak{a}$ denote the orthogonal complement of the center $\mathfrak{z}$ in $\mathfrak{n}_5$. Since the 3-dimensional vector space $\mathfrak{a}$ is isomorphic to the exterior product $\mathfrak{a} \wedge \mathfrak{a}$, the linear map $[.,.] : \mathfrak{a} \wedge \mathfrak{a} \to \mathfrak{z}$ has a one-dimensional kernel spanned by a bivector $u \wedge v$. The two-dimensional subalgebra $\mathfrak{a}_2 = \mathbb{R}u + \mathbb{R}v$ is a unique two-dimensional commutative subalgebra in $\mathfrak{a}$. Let $e_1 \in \mathfrak{a}$ be a unit vector which is orthogonal to $\mathfrak{a}_2$ and let $\{e_2', e_3'\}$ be an orthonormal basis for $\mathfrak{a}_2$. If we consider a new orthonormal basis of $\mathfrak{a}_2$ in the form

$$e_2 = \cos t\, e_2' + \sin t\, e_3', \quad e_3 = -\sin t\, e_2' + \cos t\, e_3', \quad t \in \mathbb{R},$$

then we have

$$\langle [e_1, e_2], [e_1, e_3] \rangle = \frac{1}{2}(\|[e_1, e_3']\|^2 - \|[e_1, e_2']\|^2) \sin 2t + \big\langle [e_1, e_2'], [e_1, e_3'] \big\rangle \cos 2t. \tag{5}$$

Clearly, one can find $t \in \mathbb{R}$ such that

$$\langle [e_1, e_2], [e_1, e_3] \rangle = 0 \tag{6}$$

and hence the vectors $[e_1, e_2], [e_1, e_3] \in \mathfrak{z}$ are orthogonal (and we still have $[e_2, e_3] = 0$). Thus there is an orthonormal basis $\{e_4, e_5\}$ of $\mathfrak{z}$ such that

$$[e_1, e_2] = \lambda\, e_4, \quad [e_1, e_3] = \mu\, e_5, \quad \lambda \geq \mu > 0. \tag{7}$$

We denote a 5-dimensional metric Lie algebra described above by $\mathfrak{n}_5(\lambda, \mu)$. From the considerations in [9, p. 153], we obtain at once

6 Proposition. *For any* 5*-dimensional* 2*-step nilpotent metric Lie algebra* $\mathfrak{n}$ *having a* 2*-dimensional center there exist real numbers* $\lambda \geq \mu > 0$ *such that* $\mathfrak{n}$ *is isomorphic to the metric algebra* $\mathfrak{n}_5(\lambda, \mu)$*. Moreover the metric Heisenberg Lie algebras* $\mathfrak{n}_5(\lambda, \mu)$ *and* $\mathfrak{n}_5(\lambda', \mu')$ *are isometrically isomorphic if and only if* $\lambda = \lambda'$ *and* $\mu = \mu'$*.*

7 Proposition. *The group of orthogonal automorphisms of the metric Lie algebra $\mathfrak{n}_5(\lambda,\mu)$ is isomorphic to the group $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ for $\lambda\neq\mu$, and it is isomorphic to the group $O(2)\times\mathbb{Z}_2$ for $\lambda=\mu$.*

Using (2) and (7) we see that, for any $Z=\alpha e_4+\beta e_5\in\mathfrak{z}$, the map $j(Z)$ and its square $j^2(Z)$ can be expressed by

$$\begin{aligned} j(Z) &= \begin{pmatrix} 0 & -\alpha\lambda & -\beta\mu \\ \alpha\lambda & 0 & 0 \\ \beta\mu & 0 & 0 \end{pmatrix}; \\ j^2(Z) &= \begin{pmatrix} -\alpha^2\lambda^2-\beta^2\mu^2 & 0 & 0 \\ 0 & -\alpha^2\lambda^2 & -\alpha\beta\lambda\mu \\ 0 & -\alpha\beta\lambda\mu & -\beta^2\mu^2 \end{pmatrix}. \end{aligned}$$

We see that the 5-dimensional group spaces corresponding to the metric algebras $\mathfrak{n}_5(\lambda,\mu)$ are not modified H-type groups. From [8] we also see easily that these spaces are never naturally reductive.

3.3 Metric Lie algebras with 3-dimensional center

Let $\mathfrak{z}$ denote the center of a two-step nilpotent 5-dimensional metric Lie algebra $\mathfrak{n}$, where $\dim\{\mathfrak{z}\}=3$. Clearly $\dim\{[\mathfrak{n},\mathfrak{n}]\}=1$ for the commutator $[\mathfrak{n},\mathfrak{n}]$ of the Lie algebra $\mathfrak{n}$. Let $\mathfrak{a}$ denote the orthogonal complement of the center $\mathfrak{z}$ in $\mathfrak{n}$ and let $\mathfrak{b}$ denote the orthogonal complement of $[\mathfrak{n},\mathfrak{n}]$ in the center $\mathfrak{z}$. Then $[\mathfrak{n},\mathfrak{n}]=[\mathfrak{a},\mathfrak{a}]$ and the subalgebra $\mathfrak{h}_3=\mathfrak{a}\oplus[\mathfrak{a},\mathfrak{a}]$ is a 3-dimensional metric Heisenberg algebra. The metric Lie algebra $\mathfrak{n}$ decomposes into the orthogonal direct sum $\mathfrak{n}=\mathfrak{h}_3\oplus\mathfrak{b}$ of the metric Heisenberg subalgebra $\mathfrak{h}_3$ and of the abelian metric algebra $\mathfrak{b}$.

Let $\{e_1,e_2\}$ be an orthonormal basis for $\mathfrak{a}$ and $e_3\in[\mathfrak{a},\mathfrak{a}]$ a unit vector such that

$$[e_1,e_2]=-[e_2,e_1]=\lambda\, e_3 \tag{8}$$

with $\lambda>0$. Moreover we denote by $\{e_4,e_5\}$ an orthonormal basis for $\mathfrak{b}$. The corresponding Lie algebra will be denoted by $(\mathfrak{h}_3)(\lambda)\oplus\mathbb{R}^2$.

Using the results from [9, pp. 148–149], or by an easy calculation we get

8 Proposition. *For any 5-dimensional 2-step nilpotent metric Lie algebra $\mathfrak{n}$ having a 3-dimensional center there exist a real number $\lambda>0$ such that $\mathfrak{n}$ is isomorphic to the metric algebra $\mathfrak{h}_3(\lambda)\oplus\mathbb{R}^2$. The metric Heisenberg Lie algebras $\mathfrak{h}_3(\lambda)\oplus\mathbb{R}^2$ and $\mathfrak{h}_3(\lambda')\oplus\mathbb{R}^2$ are isometrically isomorphic if and only if $\lambda=\lambda'$.*

9 Proposition. *The group of orthogonal automorphisms of the metric Lie algebra $\mathfrak{h}_3\oplus\mathbb{R}^2$ is isomorphic to the group $O(2)\times O(2)$.*

Now we prove that the 5-dimensional group spaces corresponding to the metric algebras $\mathfrak{h}_3(\lambda) \oplus \mathbb{R}^2$ are not modified H-type groups. Using the formulas (2) and (8), we obtain the following matrix representations for $j(e_3), j(e_4)$ and $j(e_5)$:

$$j(e_3) = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}; \; j(e_4) = j(e_5) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We can see that, for any $Z = \alpha e_3 + \beta e_4 + \gamma e_5 \in \mathfrak{z}$, the map $j^2(Z)$ has the form

$$j^2(Z) = \begin{pmatrix} -\alpha^2\lambda^2 & 0 \\ 0 & -\alpha^2\lambda^2 \end{pmatrix}.$$

Hence $-j^2(Z)$ is only positive semidefinite. On the other hand, it is known that all these spaces are naturally reductive.

3.4 Classification up to isometry

We can summarize our results:

10 Theorem. *The simply connected two-step homogeneous nilmanifolds of dimension* 5 *are, up to isometry,*

$$\begin{aligned} (H_5, \langle .,. \rangle_{\lambda,\mu}) &\;:\; \lambda \geq \mu > 0, \\ (N_5, \langle .,. \rangle_{\lambda,\mu}) &\;:\; \lambda \geq \mu > 0, \\ \left(H_3 \times \mathbb{R}^2, \langle .,. \rangle_{\lambda}\right) &\;:\; \lambda > 0. \end{aligned}$$

Furthermore the full isometry groups of the corresponding nilmanifolds are expressed by:

$$\begin{aligned} I(H_5, \langle .,. \rangle_{\lambda,\mu}) &= \begin{cases} (U(2) \times \mathbb{Z}_2) \ltimes H_5, & \text{if } \lambda = \mu, \\ (O(2) \times SO(2)) \ltimes H_5, & \text{if } \lambda \neq \mu, \end{cases} \\ I(N_5, \langle .,. \rangle_{\lambda,\mu}) &= \begin{cases} (O(2) \times \mathbb{Z}_2) \ltimes N_5, & \text{if } \lambda = \mu, \\ (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes N_5, & \text{if } \lambda \neq \mu, \end{cases} \\ I\left(H_3 \times \mathbb{R}^2, \langle .,. \rangle_{\lambda}\right) &= (O(2) \times O(2)) \ltimes \left(H_3 \times \mathbb{R}^2\right). \end{aligned}$$

PROOF. The result follows at once from the one-to-one correspondence between simply connected two-step nilmanifolds and metric Lie algebras. Here we just use Propositions 3-9. QED

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