Triangle transitive translation planes

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Abstract. A classification is given of all translation planes of order q^n and kernel containing $GF(q)$ that admit 'triangle transitive' collineation groups (autotopism groups that act transitively on the non-vertex points of each side of the autotopism triangle), containing symmetric homology groups of order $(q^n - 1)/2$.

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1 Introduction

An 'autotopism group' of an affine translation plane π is a group of collineations G that fixes two points on the line of infinity of the projective extension π^+ of π and fixes an affine point P. The triangle so formed in π^+ is called the 'autotopism' triangle of G. Since the collineation group of π is a semidirect product of the stabilizer of a point by the translation group, the affine point fixed by an autotopism group may always be taken arbitrarily. When autotopism groups become important for the analysis of a translation plane, there is a natural choice for the two infinite points fixed by the group. For example, if π is a finite non-Desarguesian semifield plane of order p^u, every collineation fixes the center of the affine elation group E of order p^u, which we denote by (∞). Furthermore, the second fixed infinite point may be chosen arbitrarily since E is transitive on the set of remaining infinite points. More generally, we may coordinatize so that the two infinite points are (∞) and (0) and the affine fixed point is the zero vector $(0,0)$ of the associated underlying vector space.

As the title might suggest, we are interested in 'triangle transitive translation planes'. Since we will be dealing with autotopism groups, such planes are defined as translation planes that admit an autotopism group that acts transitively on the set of non-vertex points of each side of the autotopism triangle. (In other considerations, a triangle transitive plane may be required only to have a group leaving invariant the triangle and acting transitively on the set of non-vertex points of each side of the triangle.) When this occurs, there is essentially a fixed triangle to consider since Jha and Johnson [14] have shown that the plane may be completely determined if either of the vertices on the line at infinity are moved outside of vertex set of the given triangle.

But, of course, another definition might be that a 'triangle transitive translation plane' is a translation plane that admits a collineation group that acts transitively on all affine triangles. However, it is clear that the only possible finite translation planes of this type are Desarguesian, since there is a corresponding group acting doubly transitively on the affine points. So, we are content with our definition of triangle transitivity as it coincides with the autotopism groups with a certain maximal level of transitivity.

A major problem in translation planes is the following:

Completely determine the class of triangle transitive translation planes.

To indicate the difficulty of this problem, we mention some examples of triangle transitive translation planes.

Finite generalized twisted field planes are triangle transitive precisely when the right, middle and left nuclei of the associated semifield plane coincide (see Biliotti, Jha, Johnson [2]). If the order is p^u, there is a cyclic autotopism group of order $p^u - 1$ that acts on the plane. Furthermore, the Suetake planes [22] are triangle transitive under the less restrictive definition.

Perhaps the most standard examples of triangle transitive planes are the nearfield planes of order p^u. In this case, there is an autotopism group of order $(p^u - 1)^2$ that is triangle transitive, the group being the direct product of two affine homology groups of order $p^u - 1$ where the axis and co-axis of one group is the co-axis and axis of the second group, respectively.

There are a great variety of triangle transitive planes of order p^u in Williams [24] that admit a cyclic autotopism group of order $p^u - 1$. These planes are discussed further in Kantor [18], who explicates the connections with these and the construction of Blokhuis, Coulter, Henderson and O'Keefe [3]. In particular, the number of non-isomorphic planes is not bounded by any polynomial in $q = p^u$.

In this article, we shall construct a class of triangle transitive planes of order p^u that admit an autotopism group of order $(p^u - 1)^2/2$. These planes are

generalized André planes and also admit symmetric affine homology groups of orders $(p^u - 1)/2$ but fail to be nearfield planes.

If a triangle transitive group has order $(p^u - 1)$, it may be quite difficult to determine the general nature of the plane as noted above, given the great variety of triangle transitive planes with 'small' groups. So, the question is whether the existence of a 'large' triangle transitive autotopism group necessarily forces the plane to be a nearfield plane or at least belong to a known class of planes. We define 'large' to any group of order divisible by $(p^u - 1)_{2'}^2$. We consider this question in the last section and show that, with the exception of a few sporadic orders, such a plane is at least forced to be a generalized André plane.

To elaborate and digress slightly, a nearfield plane is a translation plane of order k that admits a group of affine homologies of order $k - 1$. In this case, it follows directly that there are two affine homology groups of order $k - 1$. The basic question is how large must a homology group be before it can be concluded that one obtains, in fact, the full group? For example, suppose there is a homology group of order $(k - 1)/2$ or perhaps two such groups. Must then the plane be a nearfield plane? Actually, the answer is 'no'; there are translation planes of order k that admit one or even two affine homology groups of order $(k - 1)/2$. The complete answer when there are two groups is due to Hiramine and Johnson.

1 Theorem (Hiramine and Johnson [12]). *Let π be a translation plane of order k that admits two distinct homology groups of order $(k-1)/2$ with distinct axes. Then one of the following occurs:*

(1) π is Desarguesian or

(2) the axis and co-axis of one homology group is the co-axis and axis, respectively, of the remaining homology group.

Furthermore, either the translation plane is

        (a) a generalized André plane or

        (b) the order is 7^2 and the plane is the irregular nearfield plane,

        (c) the order is 7^2 and the plane is the exceptional Lüneburg plane, or

        (d) the order is 23^2 and the plane is the irregular nearfield plane.

So, it might be said that such translation planes are 'known'. However, the question then remains to determine all generalized André planes of order k that admit two affine homology groups of order $(k - 1)/2$. This problem was considered by Hiramine and Johnson also. However, it was improperly argued that when $k = q^n$ then $\{q, n\}$ is always a Dickson pair. In fact, this need not be the case, although it is true that the prime divisors of n will divide $q - 1$, the problem is when this occurs and $n \equiv 0 \bmod 4$, as was pointed out and corrected in Draayer [5]. The following theorem combines Hiramine and Johnson and the work of Draayer to completely determine the generalized André planes of

order $k = q^n$ admitting symmetric homology groups of order $(q^n - 1)/2$, where it is assumed that the right and middle nuclei of the corresponding quasifield coordinatizing π are equal. We state the result below although, we immediately argue that one of the cases (case (c)) does not occur.

In the statement of the following theorem, we shall assume that the order of the plane p^w is q^n, there all prime divisors of n divide $q - 1$ and either $\{q, n\}$ is a Dickson pair or $q \equiv -1 \bmod 4$ and $n \equiv 0 \bmod 4$.

2 Theorem (See Hiramine and Johnson [11] and Draayer [5] (4.2) and (5.2)). *Let π be a generalized André plane of order q^n that admits symmetric homology groups of order $(q^n - 1)/2$ (the axis and co-axis of one group is the co-axis and axis, respectively, of the second group). Assume that the right and middle nuclei of the corresponding quasifield coordinatizing π are equal.*

Then the spread may be represented as follows:

$$x = 0, y = x^{q^i} w^{s(q^i-1)/(q-1)} \alpha, y = x^{p^t q^j} w^{vq^j} w^{s(q^j-1)/(q-1)} \beta$$

where $\alpha, \beta \in \mathcal{A}$ a subgroup of $GF(q^n)^*$ of order $(q^n - 1)/2n$, where $i, j = 0, 1, \ldots, n - 1$ and where t and v have the following restrictions:

for $q \equiv 1 \bmod 4$ or n odd,

we may choose $s = 2, \ v = 1, \ t \geq 0$,

for $q \equiv -1 \bmod 4$ and $n \equiv 2 \bmod 4$,

we may choose $s = 1, v = n, t \geq 0$ and even, or

for $q \equiv 3 \bmod 8$ and $n \equiv 0 \bmod 4$,

we may choose $s = 1, v = 2[n]_{2'}, \ t \geq 0$ and even.

Furthermore, we have the congruence:

$$\begin{aligned} s(p^t - 1) &\equiv v(q - 1) + 2nz \ \bmod (q^n - 1), \textit{ for some } z \textit{ so that} \\ s(p^t - 1) &\equiv v(q - 1) \bmod 2n. \end{aligned}$$

Conversely, generalized André planes with the above conditions have two symmetric homology groups corresponding to equal right and middle nuclei in the associated quasifield.

Draayer [5] has show that, in fact, there is a generalized André planes of order 3^4, which is not a nearfield planes and nevertheless has two affine homology groups of order $(3^4 - 1)/2$, where the corresponding right and middle nuclei are not equal. There is a faulty argument in Hiramine and Johnson [11, (5.1)] that reputes to show that the two nuclei are always equal or the plane must be a

nearfield plane. This argument infected the analysis of planes of order 7^2, so we repair both of these problems this in this article.

This brings up an interesting question: **What is the subclass of translation planes of order q^n, where $\{q, n\}$ is not a Dickson pair that admit two homology groups of order $(q^n - 1)/2$, where the associated middle and right nuclei are equal? And, what is the subclass when the associated middle and right nuclei are not equal?**

Moreover, it is of general interest to completely determine the generalized André planes of order q^n admitting two affine homology groups of order $(q^n - 1)/2$, where the associated nuclei are not equal. We have previously called planes admitting two 'half-order' homology groups, 'near nearfield planes'. When the associated nuclei are not equal, we show that such translation planes admit an autotopism group that acts transitively on the non-vertex points on the 'infinite line' and completely classify such planes. Furthermore, we show that any near nearfield plane always admits an autotopism group that acts transitively on the non-vertex points of each affine side of the autotopism triangle, thus providing additional examples of triangle transitivity planes.

Our main result is

3 Theorem. *Let π be a non-Desarguesian affine plane of order p^w, p a prime that admits two affine homology groups H_x and H_y of order $(p^w - 1)/2$ with axis (respectively, co-axis) $x = 0$ and co-axis (respectively, axis) $y = 0$.*

Then there is an autotopism group that is triangle transitive if and only if π is one of the following types of planes:

(1) the plane is a Dickson nearfield plane,

(2) $p^w = 7^2$ or 23^2 and the plane is an irregular nearfield plane,

(3) the plane is a generalized André plane of order $p^w = q^n$, where $q \equiv 3 \bmod 8$, $n \equiv 0 \bmod 4$, $n/4$ is odd.

Furthermore, the spread may be represented as follows:

$$\left\{ x = 0, y = 0, \{y = x, y = w^{n/2}\}H_y \right\},$$

where

$$H_y = \left\langle (x, y) \longmapsto (x, y^{q^i} w^{s(q^i - 1)/(q-1)} \alpha); \alpha \in \mathcal{A}, \text{ where } i = 0, 1, \dots, n - 1 \right\rangle,$$

and $\mathcal{A}$ the cyclic subgroup of $GF(q^n)^*$ of order $(q^n - 1)/2n$.

2   Transitive generalized André planes

We begin by answering the question discussed in the introduction; what are the translation planes of indicated type that have corresponding equal middle and right nuclei?

Since Theorem 2 gives the classification of all such translation planes, we may utilize the congruence given in part (c) to analyze such planes.

Thus, we have

$$q \equiv 3 \bmod 8 \text{ and } n \equiv 0 \bmod 4, s = 1, v = 2[n]_{2'}, t \geq 0, t \text{ even and}$$
$$s(p^t - 1) \equiv v(q - 1) + 2nz \bmod (q^n - 1), \text{ for some } z.$$

Since t is even, it follows that 8 divides $s(p^t - 1)$ and, of course, 8 divides $2nz$. Since 8 divides $q^n - 1$, it follows that

$$v(q - 1) \equiv 0 \bmod 8.$$

However, $(q - 1)/2$ is odd since 4 divides $(q - 1)/2 - 1$. Since $v/2$ is odd, we have a contradiction. That is, case (c) never occurs.

Hence, we have:

4 Theorem. *Let π be a generalized André plane of order q^n admitting symmetric homology groups of order $(q^n - 1)/2$. Assume that the right and middle nuclei of the corresponding quasifield coordinatizing π are equal.*

Then $\{q, n\}$ is a Dickson pair.

So, we now continue to determine the complete class of planes where the right and middle nuclei are not equal.

We require the following result for our general analysis.

5 Theorem (Draayer [5]). *Let π be a translation plane of order q^n that admits an affine homology group H_y of order $(q^n - 1)/2$ with axis $y = 0$ and co-axis $x = 0$. Then the spread for π may be represented as follows:*

$$x = 0, \; y = x^{q^i} w^{s(q^i - 1)/(q-1)} \alpha, \; y = x^{p^t q^j} w^{v q^j} w^{s(q^j - 1)/(q-1)} \beta$$

where $\alpha, \beta \in \mathcal{A}$ a subgroup of $GF(q^n)^*$ of order $(q^n - 1)/2n$, where $i, j = 0, 1, \ldots, n - 1$ and where t and v have the following restrictions:

for $q \equiv 1 \bmod 4$ or n odd, let $s = 2$, $v = 1$, $t \geq 0$, or

for $q \equiv -1 \bmod 4$ and $n \equiv 2 \bmod 4$, let $s = 1, v = n, t \geq 0$ and even, or

for $q \equiv 3 \bmod 8$ and $n \equiv 0 \bmod 4$, let $s = 1, v = 2[n]_{2'}$, $t \geq 0$ and even.

Furthermore, the group H_y has the following form:

$$\left\langle (x, y) \longmapsto (x, y^{q^i} w^{s(q^i - 1)/(q-1)} \alpha); \alpha \in \mathcal{A}, \text{ where } i = 0, 1, \ldots, n - 1 \right\rangle.$$

Now assume we have a generalized André plane π represented as above and assume that there is a homology group H_x of order $(q^n - 1)/2$ that has axis $x = 0$ and co-axis $y = 0$.

6 Lemma. *Each element* $(x, y) \longmapsto (x\delta, y\gamma)$ *for* $\delta, \gamma \in \mathcal{A}$ *is a collineation of* π.

PROOF.

$$y = x^{q^i} w^{s(q^i-1)/(q-1)} \alpha \text{ is mapped onto } y = x^{q^i} w^{s(q^i-1)/(q-1)} \alpha \delta^{-q^i} \gamma$$

and

$$y = x^{p^t q^j} w^{vq^j} w^{s(q^j-1)/(q-1)} \beta$$
$$\text{is mapped onto } y = x^{p^t q^j} w^{vq^j} w^{s(q^j-1)/(q-1)} \beta \delta^{-p^t q^j} \gamma.$$

Since $\mathcal{A}$ is a subgroup of $GF(q^n)^*$ of order $(q^n - 1)/2n$, it follows that $\alpha\delta^{-q^i}\gamma$ and $\beta\delta^{-p^t q^j}\gamma$ are back in $\mathcal{A}$. Hence, the lemma follows immediately.   QED

Thus, there is an affine homology group $\mathcal{A}_x$ of order $(q^n - 1)/2n$ with axis $x = 0$ and co-axis $y = 0$ corresponding to $\mathcal{A}$. Let H_x^+ denote the full homology group of π with axis $x = 0$ and co-axis $y = 0$ so that $\mathcal{A}_x$ is a subgroup of H_x^+.

There are two possibilities: $\mathcal{A}_x \subseteq H_x$ or H_x is a proper subgroup of H_x^+. Of course, in the latter case, we have a nearfield plane and since all of these are Dickson nearfield planes, they are completely determined. Hence, we may assume that $\mathcal{A}_x \subseteq H_x$.

7 Remark. In Hiramine and Johnson [11], there is an analysis of generalized André systems of order q^n that admit right sub-nuclei of order $(q^n - 1)/2$. Such right sub-nuclei correspond to the affine homology group with axis $y = 0$ and co-axis $x = 0$. This analysis applies equally, in theory, to such systems that admit middle sub-nuclei of order $(q^n - 1)/2$. Such middle sub-nuclei correspond to the affine homology group with axis $x = 0$ and co-axis $y = 0$.

In particular, the set of generalized André systems of order q^n with right sub-nuclei of order $(q^n - 1)/2$ are in 1–1 correspondence with the set of all generalized André systems of order q^n with middle sub-nuclei of order $(q^n - 1)/2$ under the mapping $(x, y) \longmapsto (y, x)$.

We may apply the analysis of generalized André systems of order q^n that admit right sub-nuclei of order $(q^n - 1)/2$ to generalized André systems of order h^{n^*} that admit middle sub-nuclei of order $(h^{n^*} - 1)/2$. Assume that we have both a right nucleus of order $(q^n - 1)/2$ and a middle nucleus of order $(h^{n^*} - 1)/2 = (q^n - 1)/2$.

Then we have a group $\mathcal{B}$ of order $(h^{n^*} - 1)/2n^*$ that takes the place of $\mathcal{A}$ but acting on the left; $(x, y) \longmapsto (x\delta, y)$ for $\delta \in \mathcal{B}$, is a collineation of the plane π, so we obtain a corresponding group $\mathcal{B}_x$ and the above lemma shows that we also would have a group $\mathcal{B}_y$ as a collineation group. So, either we have a nearfield plane or we may assume that $\mathcal{A}_x$ is a subgroup of H_x and $\mathcal{B}_y$ is a subgroup of

H_y. This implies that $\mathcal{A}$ and $\mathcal{B}$ are equal. Since $h^{n^*} = q^n$ and we now have that $2n^* = 2n$, which implies that $h = q$ and $n^* = n$.

8 Lemma. *Under the above assumptions, $H_x/\mathcal{A}_x$ may be represented cyclically with generator $(x, y) \longmapsto (x^q w^{s^*}, y)$.*

PROOF. Again, the analysis for the decomposition over the right nucleus applies to show that the decomposition over the middle nucleus is cyclic modulo $\mathcal{A}$. Without using the special form given in the previous theorem (that is, without specifying the nature of $\{s, t, v\}$), we would know that set of all generalized André planes with a right nucleus of order $(q^n - 1)/2$ produce under the mapping $\phi : (x, y) \longmapsto (y, x)$ the set of all generalized André planes with a middle nucleus of order $(q^n - 1)/2$. Hence, the basic generator on the right $(x, y) \longmapsto (x, y^q w^s)$, without yet specifying s, maps under ϕ to $(y, x) \longmapsto (y^q w^s, x)$. This means that the homology group H_x modulo $\mathcal{A}_x$ is also cyclic and generated for some w^{s^*} exactly as indicated. $\boxed{\text{QED}}$

9 Lemma. *Assume that the multiplicative groups of the right nucleus N_r^* and middle nucleus N_m^* are both of order $(q^n - 1)/2$ but are distinct.*

(1) Then $(N_r^*, \circ) \cap (N_m^*, \circ)$ defines a multiplicative group of order $(q^n - 1)/4$. Note that this also says that $b \in N_m^*$ implies that $b \circ b \in N_r^*$.

(2) Hence, if n is odd then $q \equiv 1 \bmod 4$.

PROOF. This result is clear if in the associated generalized André system $(Q - \{0\}, \circ)$ is a group, so we need to check that the standard counting argument works in this setting.

We consider $N_m^* \circ N_r^* \subseteq Q - \{0\} = Q^*$. Assume that

$$b_m \circ a_r = b_m^* \circ a_r^*,$$

where $b_m, b_m^* \in N_m^*$ and $a_r, a_r^* \in N_r^*$. Then

$$
\begin{aligned}
b_m^{-1} \circ (b_m \circ a_r) &= b_m^{-1} \circ (b_m^* \circ a_r^*) \iff \\
(b_m^{-1} \circ b_m) \circ a_r &= a_r = (b_m^{-1} \circ b_m^*) \circ a_r^*. \text{ This is valid } \iff \\
c &= a_r \circ a_r^{*-1} = ((b_m^{-1} \circ b_m^*) \circ a_r^*) \circ a_r^{*-1} = (b_m^{-1} \circ b_m^*),
\end{aligned}
$$

is in $N_m^* \cap N_r^*$, using the fact that the corresponding elements are in the middle and right nuclei.

Thus,

$$|N_m^* \circ N_r^*| \leq (q^n - 1) \text{ and } |N_m^* \circ N_r^*| = ((q^n - 1)^2/4)/I$$

where I is the order of $N_m^* \cap N_r^*$, since

$$b_m \circ a_r = (b_m \circ c) \circ (c^{-1} \circ a_r) \text{ and } c \in N_m^* \cap N_r^*.$$

But,

$$(q^n - 1)^2/4I \leq (q^n - 1) \iff$$
$$(q^n - 1)/4 \leq I.$$

Since $N_m^* \neq N_r^*$, it follows that $I = (q^n - 1)/4$.   QED

10 Lemma. *(1) The group generated by the associated right and left homology groups generate a group that is transitive on the components not equal to the axis and co-axis $x = 0, y = 0$.*

(2) The groups H_x and H_y commute and H_y permutes the two H_x-orbits of components. $N_r \neq N_m$ if and only if H_y inverts the two H_x-orbits of components.

PROOF. (2) implies (1) so it suffices to complete the proof to (2). Clearly, the groups H_x and H_y commute since they both leave invariant the axis and co-axis of the remaining group. Consider the image set of $y = x$ under $\rho : (x, y) \longmapsto (x^q w^{s^*}, y)$, which is $y = x^{q^{-1}} w^{-s^* q^{-1}}$. Assume that $x \circ b = x^q w^{s^*}$. Then

$$y = x \circ a \longmapsto y = x \circ (b^{-1} \circ a),$$

under ρ. Suppose that $y = x^{q^{-1}} w^{-s^* q^{-1}} = x \circ b^{-1}$ is in the H_y orbit of $y = x$ then b^{-1} would be forced into N_r^*, a contradiction. Hence, the image must be in the orbit of $y = x^{p^t} w^v$. Since this argument is symmetric, (2) is proved.   QED

11 Lemma. *Let H_y^- have elements*

$$(x, y) \longmapsto (x, y^{q^{2i}} w^{s(q^{2i}-1)/(q-1)} \alpha), \text{ for } \alpha \text{ of order dividing } (q^n - 1)/2n$$

and H_x^- have elements

$$(x, y) \longmapsto (x^{q^{2i}} w^{s^*(q^{2i}-1)/(q-1)} \beta, y)$$
$$\text{for } \beta \text{ of order dividing } (q^n - 1)/2n \text{ and } i = 1, 2, \ldots, n/2.$$

(1) Then,
$$w^{s^*(q+1)} = w^{s(q+1)} \alpha_o$$

for some α_o in $\mathcal{A}$.

(2) $2n$ divides $(s^* - s)(q + 1)$.

(3)

    (a) If $q \equiv 3 \bmod 8$ and $n \equiv 0 \bmod 4$ then we may assume that

$$s^* = s \pm n/2 \text{ or } s^* = s + n.$$

    (b) If n is even and $q \equiv 1 \bmod 4$ then

$$s^* = s + n.$$

(c) If $n \equiv 2 \bmod 4$ and $q \equiv -1 \bmod 4$ then

$$s^* = s \pm n/2 \ \text{or} \ s^* = s + n.$$

PROOF. We know that the stabilizer of $y = x$ in the group generated by H_x and H_y has order exactly $(q^n - 1)/4$. Moreover, we know that $(x, y) \longmapsto (x\alpha, y\alpha)$ is a subgroup of order $(q^n - 1)/2n$. Furthermore, modulo $\mathcal{A}_y$, we know that we have a cyclic subgroup of H_y of order n. Hence, the stabilizer of $y = x$ corresponds exactly to the cyclic subgroup of H_y of order $n/2$ module $\mathcal{A}_y$. It follows that for a given i, there exists α and β such that the product of the two listed elements in the statement of the lemma fix $y = x$. This proves (1).

(2) now follows since the order of α_o divides $(q^n - 1)/2n$.

If q is congruent to 3 mod 8 then $(q + 1)/4$ is odd. We may assume that every prime divisor of n divides $q - 1$, then it follows that since $2n$ divides $(s^* - s)(q + 1)$ then $n/2$ divides $(s^* - s)$, and since we may take everything modulo $2n$, (3) (a) then follows.

If $q \equiv 1 \bmod 4$ then $2n$ dividing $(s^* - s)(q + 1)$ implies that n divides $(s^* - s)$, since all prime divisors divide $q - 1$ and $(q + 1)/2$ is odd. This proves (b) for n even or odd.

If $n \equiv 2 \bmod 4$ and $q \equiv -1 \bmod 4$ then $2n$ dividing $(s^* - s)(q + 1)$ implies that $n/2$ divides $(s^* - s)$, as $n/2$ is odd and all prime divisors of n divide $q - 1$. $\boxed{QED}$

12 Lemma. *If $\theta_{s^*} : (x, y) \longmapsto (x^q w^{s^*}, y)$ generates H_x modulo $\mathcal{A}$ (the group corresponding to $\mathcal{A}$), then*

(1) $y = x$ maps onto $y = x^{q^{-1}} w^{-s^* q^{-1}}$.

(2) If θ_{s^*} is a generator then so is $(x, y) \longmapsto (x^q w^{s^*} w^{\pm 2n}, y)$.

(3) The spread is

$$\left\{ x = 0, y = 0, \{y = x, y = w^{s^* - s}\} H_y \right\},$$

where $s^* - s = \pm n/2$ or n (see above lemma).

PROOF. (1) is clear and (2) follows since $w^{\pm 2n}$ has order $(q^n - 1)/2n$. $\boxed{QED}$

13 Lemma. *Let $s^* - s = \pm n/k$, where $k = 1$ or 2. There exists an element j_0 such that*

$$\begin{aligned}
vq^{j_0} + s(q^{j_0} - 1)/(q - 1) &\equiv \pm n/k \bmod 2n, \\
\text{and } p^t &= q^{-j_0}.
\end{aligned}$$

PROOF. $y = xw^{s^* - s}$ is in $\{y = x^{p^t} w^v\} H_y$. $\boxed{QED}$

Now recall that

for $q \equiv 1 \mod 4$ or n odd, let $s = 2$, $v = 1$, $t \geq 0$, or

for $q \equiv -1 \mod 4$ and $n \equiv 2 \mod 4$, let $s = 1, v = n, t \geq 0$ and even, or

for $q \equiv 3 \mod 8$ and $n \equiv 0 \mod 4$, let $s = 1, v = 2[n]_{2'}$, $t \geq 0$ and even.

Note that for

$$q \equiv -1 \mod 4 \text{ and } n \equiv 2 \mod 4, \text{ let } s = 1, v = n, t \geq 0 \text{ and even,}$$

it follows that

$$vq^{j_0} + s(q^{j_0} - 1)/(q - 1) \equiv \pm n/k \mod 2n,$$

has a solution for $k = 1$ and $j_0 = 0$.

Furthermore, when

$$q \equiv 3 \mod 8 \text{ and } n \equiv 0 \mod 4, s = 1, v = 2[n]_{2'}, \ t \geq 0, \text{ even,}$$

then

$$2[n]_{2'} \equiv n/2 \mod 2n,$$

provided $n/4$ is odd.

14 Lemma. *If the spread is*

$$\left\{ x = 0, y = 0, \{y = x, y = w^n\}H_y \right\},$$

then the spread is a nearfield plane so that the case $q \equiv 3 \mod 8$ and $n \equiv 0 \mod 4$ does not occur.

PROOF. Simply note that w^{nq^j} is congruent to w^n modulo $\mathcal{A}$, implying that $\theta : (x, y) \longmapsto (x, yw^n)$ is a homology. However, θ is not in H_y, which implies that the plane is a nearfield plane. $\boxed{QED}$

15 Lemma. *If $q \equiv -1 \mod 4$ then there is not a solution to*

$$nq^{j_0} + (q^{j_0} - 1)/(q - 1) \equiv \pm n/2 \mod 2n.$$

PROOF. Let $q = p^r$, then r is odd. Since $p^t q^{j_0}$ is the identity and t is even, it follows that j_0 is even. But, $n/2$ is odd and $(q^{j_0} - 1)/(q - 1)$ is even, so we must have 2 dividing $n/2$, a contradiction. $\boxed{QED}$

16 Theorem. *If a transitive near nearfield plane is obtained and the spread is not a nearfield plane then $q \equiv 3 \mod 8, n \equiv 0 \mod 4$ and the spread is as follows:*

$$\left\{ x = 0, y = 0, \{y = x, y = w^{\pm n/2}\}H_y \right\},$$

which implies that $n/4$ is odd.

Furthermore, if $p^t = 0$, we obtain the spread for $\pm n/2 = n/2$ and when $p^t = q^{n/2}$ then we obtain the spread for $\pm n/2 = -n/2$.

PROOF. We must have

$$2[n]_{2'}q^{j_0} + (q^{j_0} - 1)/(q - 1) \equiv \pm n/2 \bmod 2n,$$
$$\text{and } p^t = q^{-j_0}.$$

By the above lemmas, we must have $q = p^r$, r odd and j_0 must be even. But, then $[n]_{2'}q^{j_0}$ is odd, $(q^{j_0} - 1)/2(q - 1)$, is even as 4 divides $q + 1$, and n is even. This forces $n/4$ to be odd.

Since we have $n/4$ odd, it follows that we have a solution when $p^t = 1$ and $\pm n/2 = n/2 = 2[n]_{2'}$. It remains to consider the case when

$$\frac{n}{2}q^{j_0} + (q^{j_0} - 1)/(q - 1) \equiv -n/2 \bmod 2n.$$

Clearly this is valid if and only if

$$n \text{ divides } (q^{j_0} - 1)/(q - 1),$$

where j_0 is even. If $n = 4$ then this works for any even j_0 since 4 divides $q + 1$. If u divides $q - 1$ (see [5, lemma (2.3)]) then u divides $(q^u - 1)/(q - 1)$. So, for any even j_0, 4 divides $(q^{j_0} - 1)/(q - 1)$ and $n/4$ divides $(q^{n/4} - 1)/(q - 1)$. Since $(q^{n/4} - 1)/(q - 1)$ divides $(q^{n/2} - 1)/(q - 1)$, it follows that n always divides $(q^{n/2} - 1)/(q - 1)$ provided n divides $(q - 1)$. Furthermore, by Corollary (2.5) (a) of [5], if all prime divisors of n divide $(q - 1)$ then n divides $(q^{n/2} - 1)/(q - 1)$. Hence, we may always choose $p^t = 1$ or $q^{n/2}$. This completes the proof. $\boxed{QED}$

17 Lemma. *A spread is obtained when $q \equiv 1 \bmod 4$ if and only if $n = 1$. In this case, the spread is Desarguesian.*

PROOF. By Draayer (4.2)(2), we must have that $\{q, n\}$ is a Dickson pair. Hence, the prime divisors of n divide $q - 1$. However, n divides $(q + 1)$ and n is odd, so we have a contradiction unless $n = 1$. But, in this case, we have that the kernel contains $GF(q)$ so the plane is Desarguesian of order q. $\boxed{QED}$

18 Theorem. *Let π be a translation plane of order $q^n = p^{nr}$, for p a prime, that admits two homology groups of order $(q^n - 1)/2$. Then one of the following occurs:*

(1) π is Desarguesian,

(2) $q^n = 7^2$ and the plane is the irregular nearfield plane,

(3) $q^n = 7^2$ and the plane is the exceptional Lüneburg plane,

(4) $q^n = 23^2$ and the plane is the irregular nearfield plane,

(5) π is a generalized André plane and the associated right and middle nuclei of the (sub) homology groups are equal. In this case, the representation of the plane is as follows: Then the spread may be represented as follows:

$$x = 0, y = x^{q^i} w^{s(q^i - 1)/(q - 1)}\alpha, y = x^{p^t q^j} w^{v q^j} w^{s(q^j - 1)/(q - 1)}\beta$$

where $\alpha, \beta \in \mathcal{A}$ a subgroup of $GF(q^n)^*$ of order $(q^n - 1)/2n$, where $i, j = 0, 1, \ldots, n - 1$ and where t and v have the following restrictions:

for $q \equiv 1 \bmod 4$ or n odd, let $s = 2$, $v = 1$, $t \geq 0$, or

for $q \equiv -1 \bmod 4$ and $n \equiv 2 \bmod 4$, let $s = 1, v = n, t \geq 0$ and even.

Furthermore, we have the congruence:

$$s(p^t - 1) \equiv v(q - 1) + 2nz \bmod (q^n - 1), \text{ for some } z.$$

(6) π is a generalized André plane such that there are subnuclei of the associated (sub) homology groups of index two and are not equal. Then one of two situations occurs:

(a) π is a Dickson nearfield plane and the orbit lengths of components under the full group is 2, $q^n - 1$ or

(b) π is not a nearfield plane but nevertheless, there is a transitive autotopism group and the spread may be represented as follows:

$$x = 0, y = 0, y = x^{q^i} w^{s(q^i-1)/(q-1)} \alpha,$$
$$y = x^{p^t q^j} w^{v q^j} w^{s(q^j-1)/(q-1)} \beta,$$

where $\alpha, \beta \in Z^-$, the cyclic subgroup of $GF(q^n)^*$ of order $(q^n - 1)/2n$, $i, j = 1, \ldots, n$, w a primitive element of $GF(q^n)^*$ and

$q \equiv 3 \bmod 8$ and $[n]_2 = 4$, $s = 1$, and $v = n/2$ and $t \in (0, nr/2)$.

PROOF. The argument given in [12, (5.2)] or in [11, (5.1)] incorrectly concludes that if the right and middle nuclei are not equal then the plane is a nearfield plane. This result was used to classify the planes of order 7^2, as the basic analysis shows these orders to be somewhat sporadic. Hence, in order to complete the theorem, we need to assume that the right and middle subnuclei of orders 24 are not equal and the group generated by the two associated affine homology groups acts as a transitive autotopism group. If there are two cyclic homology groups of order 8 then the plane is either Desarguesian or André or constructed from a Desarguesian affine plane Σ by $(q + 1)$-nest replacement, for $q = 7$, using the results of Johnson and Pomareda [17, Theorem (2.1)]. In the latter case, there is an associated homology group of order 8 and a kernel homology group of order 8 both collineation groups of Σ. That is, the two homology groups with different axes must define equal subnuclei; there is a right subnucleus of order 8 that is equal to a middle subnucleus of order 8. This says that the orbit of $y = x$ is the same for both groups of order 8. However, we have a right subnucleus homology group of order 24 that has two orbits of components of length 24 and these two orbits are permuted by the middle subnucleus

homology group also of order 24. We have assumed that the two orbits under the right subnucleus homology group are inverted by the middle subnucleus group. A Sylow 2-subgroup of order 8 of a right subnucleus homology group therefore has 6 orbits of length 8 and none of these can be fixed by the full Sylow 2-subgroup of a middle subnucleus homology group.

Thus, if there are two cyclic groups of order 8 then the plane is either Desarguesian or an André plane.

So, assume that at least one of the homology groups has non-cyclic Sylow 2-subgroups of order 8. In a translation plane, there is a unique involutory homology for a fixed axis and co-axis. Hence, there is a quaternion Sylow 2-subgroup of order 8 of one of the homology groups. Planes of order 49 with quaternion groups of homologies are classified by Heimbeck [10]. In particular, see the second table on page 1211 of Charnes and Dempwolff [4]. However, it is not necessary to appeal to this work as the following argument shows.

Now note that the intersection of the right and middle nuclei has order 12 so that there is a normal subgroup of order 12 in the right and middle nuclei. Assume that the right nucleus H has a Sylow 2-subgroup of order 8 and a normal subgroup of order 12. The center of a Sylow 2-subgroup is in the center Z of $GL(2,7)$. If Z is in H then we have a regulus-inducing homology group of order 6. However, by the work of Johnson [16], there is a corresponding flock of a hyperbolic quadric in $PG(3,7)$. However, by the classification theorem of such flocks, all associated planes are nearfield planes (see Thas [23] and Bader, Lunardon [1]). Thus, the quotient of H by $Z \cap H$ is A_4 so that H is isomorphic to $SL(2,3)$. However, there is not a subgroup of $SL(2,3)$ of order 12. (That is, if so then there is either a normal Sylow 3-subgroup or a normal 2-subgroup of order 4, neither of which can hold in $SL(2,3)$.)

Hence, we may always assume that either the plane is a nearfield plane or the right and middle reguli may be assumed equal.

This completes the proof to the theorem.   QED

It is always of interest to determine those translation planes that admit an autotopism group that acts transitively on one of the sides of the autotopism triangle. Any 'near nearfield' plane of order q^n has an autotopism group that has two orbits of lengths $(q^n - 1)/2$ on each side of the autotopism triangle.

19 Lemma. (1) $g_n : (x,y) \longmapsto (xw^n, yw^n)$ is a collineation of any near nearfield plane.

(2) $\langle g_n, H_y \rangle$ is transitive on the non-zero points of the co-axis $x = 0$ of H_y. Similarly, $\langle g_n, H_x \rangle$ is transitive on the non-zero points of the co-axis $y = 0$ of H_x.

PROOF. Note that g_n fixes $y = x$, and if $\sigma : (x,y) \longmapsto (x, y^q w^s)$, then $\sigma^{g_n} : (x,y) \longmapsto (x, y^q w^s w^{n(q-1)})$, since $2n$ divides $n(q-1)$, it follows that g_n

normalizes H_y. Also, $y = x^{p^t} w^v$ maps to $y = x^{p^t} w^v w^{n(p^t-1)}$ so is in $(y = x^{p^t} w^v) H_y$. Thus, g_n is a collineation. It then follows from Foulser [7, (15.3), p. 467], that $\langle g_n, H_y \rangle$ is transitive on the non-zero points of the co-axis $x = 0$ of H_y. Similarly, $\langle g_n, H_x \rangle$ is transitive on the non-zero points of the co-axis $y = 0$ of H_x. QED

20 Definition. We shall say that an autotopism group is 'side transitive' if it is transitive on one side of the autotopism triangle and 'affine side transitive' if it is transitive on an affine side of the autotopism triangle. The group is 'affine transitive' if it is transitive on both affine sides of the autotopism triangle and 'triangle transitive' if it is transitive on all three sides of the autotopism triangle. We shall also use the same terminology to describe the plane that admits such collineation groups.

21 Corollary. *Any near nearfield plane is affine transitive.*

22 Corollary. *A near nearfield plane of order q^n has an affine transitive autotopism group G of order $(q^n - 1)^2/2$. When $\{q, n\}$ is not a Dickson prime, G is a triangle transitive group.*

PROOF. $\langle g_n, H_x, H_y \rangle$ has order $(q^n - 1)^2/2$. QED

In one of the following sections, we shall show that the only near nearfield planes that are triangle transitive are the nearfield planes and what we have called 'transitive' near nearfield planes.

3   Isomorphisms

We have not yet dealt with the possibility that the two spreads of orders q^n where $q \equiv 3 \bmod 8$ and $n \equiv 0 \bmod 4$, with $n/4$ odd may be isomorphic. Recall that the spreads are

$$\left\{ x = 0, y = 0, \{y = x, y = w^{\pm n/2}\} H_y \right\}.$$

Since these planes are generalized André planes, we may assume that a collineation σ fixes or interchanges $x = 0$ and $y = 0$, and since we have a transitive autotopism group on each plane, we assume that σ fixes $y = x$. Say the 'plus'-plane is mapped onto the 'minus'-plane by σ.

Consider

$$\sigma : (x, y) \longmapsto (x^q w, y^q w).$$

Note that $y = x w^{n/2}$ maps onto $y = w^{qn/2}$. So, the question is whether

$$w^{qn/2} = w^{-n/2} \alpha_o,$$

where α_o has order dividing $(q^n - 1)/2n$. This is equivalent to

$$(q + 1)n/2 \equiv 0 \bmod 2n,$$

which is valid since 4 divides $q + 1$. It only remains to show that σ normalizes H_y.

But, we need only check that σ normalizes the generators of H_y. Let $\rho^* :$ $(x, y) \longmapsto (x, y\rho)$. Then $\rho^{*\sigma} = \rho^{*q}$, for $\rho \in \mathcal{A}$. Furthermore, if g is $(x, y) \longmapsto (x, y^q w)$, then $g^\sigma = g$.

Hence, we obtain the following theorem.

23 Theorem. *Let π be a non-nearfield translation plane of order q^n that admits two distinct homology groups of order $(q^n - 1)/2$. Then the axis and co-axis of one group is the co-axis and axis, respectively, of the second group.*

If the associated right and middle nuclei are not equal then

(1) $q \equiv 3 \bmod 8, n \equiv 0 \bmod 4, n/4$ is odd.

(2) The spread may be represented as follows:

$$\left\{ x = 0, y = 0, \{y = x, y = w^{n/2}\}H_y \right\},$$

where

$$H_y = \left\langle (x, y) \longmapsto (x, y^{q^i} w^{(q^i - 1)/(q-1)}\alpha); \alpha \in \mathcal{A}, \text{ where } i = 0, 1, \ldots, n - 1 \right\rangle,$$

and, where $\mathcal{A}$ is a cyclic group of order $(q^n - 1)/2n$ in $GF(q^n)^*$.

4 Transitive autotopism groups

In the previous sections, we have completely determined the set of translation planes of orders q^n that admit two homology groups of orders $(q^n - 1)/2$, say H_y and H_x such that $\langle H_y, H_x \rangle$ is a transitive autotopism group (fixes two infinite points, and an affine point and acts transitively on the remaining points of the infinite side of the autotopism triangle). However, it might be possible that there is a transitive autotopism group and the two homology groups have the same two orbits on the line at infinity. In this case, there would be an element g that interchanges the two orbits of H_y and fixes $x = 0$ and $y = 0$.

Specifically, suppose that we have a non-Desarguesian translation plane of order $q^n = p^{nr}$ with two homology groups of order $(q^n - 1)/2$. Since we have a generalized André plane (except for the indicated few special orders), there are two components $x = 0$ and $y = 0$ that are fixed or interchanged. Suppose that there is a group G containing the homology groups such that G is a transitive autotopism group (i.e. fixes $x = 0$ and $y = 0$ and is transitive on the

remaining components). If the two homology groups lead to different right and middle nuclei, we have a complete classification. However, it may be possible such transitive autotopism groups G (infinite side transitive) where the two right and middle nuclei are actually equal. We explore this situation in the current section.

Recall that we have the spreads

$$x = 0, y = x^{q^i} w^{s(q^i-1)/(q-1)} \alpha, y = x^{p^t q^j} w^{vq^j} w^{s(q^j-1)/(q-1)} \beta$$

where $\alpha, \beta \in \mathcal{A}$ a subgroup of $GF(q^n)^*$ of order $(q^n - 1)/2n$, where $i, j = 0, 1, \ldots, n-1$ and where t and v have the following restrictions:

for $q \equiv 1 \bmod 4$ or n odd, let $s = 2$, $v = 1$, $t \geq 0$, or

for $q \equiv -1 \bmod 4$ and $n \equiv 2 \bmod 4$, let $s = 1, v = n, t \geq 0$ and even.

Furthermore, we have the congruence:

$$\begin{aligned} s(p^t - 1) &\equiv v(q-1) + 2nz \ \bmod (q^n - 1), \text{ for some } z, \text{ I} \iff \\ s(p^t - 1) &\equiv v(q-1) \bmod 2n. \end{aligned}$$

If we assume that $q \equiv -1 \bmod 4$, so that $s = 1$ and $v = n$ then n divides $p^t - 1$.

The other possibility is that assume $q \equiv 1 \bmod 4$ (n may be even or odd), let $s = 2$, $v = 1$, $t \geq 0$,

$$2(p^t - 1) \equiv (q-1) \bmod 2n.$$

In any case, since G is an autotopism group, by our previous results, either the plane is a Dickson nearfield plane or the homology group H_y is normal in G. Hence, we may assume that there exists an element g in G that interchanges the two H_y-orbits of components. Furthermore, we know that there is an affine transitive autotopism group acting transitively on each of the affine axes of the autotopism triangle. Since the group element $g_n : (x, y) \longmapsto (xw^n, yw^n)$ normalizes both H_y and H_x, and fixes $y = x$, we may assume that g_n is in G. Let $G^- = \langle g_n, H_y, H_x \rangle$ and note that G^- normalizes H_y. Let g interchange the two H_y-orbits of components, we claim that g may be assumed to fix some point $(1, z)$. Assume that g maps 1 to s_0. Then, there exists an element of G^- that maps $(s_0, g(z))$ to $(1, d)$ for some element d. So, we may assume, without loss of generality, that g fixes 1 on $y = 0$.

24 Remark. Foulser [8] determines the full collineation group of a generalized André plane. In particular, if the plane is not Desarguesian or a Hall plane then the collineation group fixes the autotopism triangle and fixes or interchanges the two infinite vertices. Furthermore, the collineation group may be

represented using the group $\Gamma L(2, q^n)$ and the automorphism group of $GF(q^n)$ that fixes the kernel elementwise. (In Foulser, the case of order 3^4 is left open, but this has been taken care of in Rao's work [21].)

If the plane is not Desarguesian or Hall, has order $q^n = p^w$, then the collineation group is a subgroup of

$$\left\langle \Gamma L(2, q^n)_{\{x=0, y=0\}}, (x, y) \longmapsto (x, y^p) \right\rangle.$$

In the following lemmas, we assume the conditions given above.

25 Lemma. *Let g fix $x = 0, y = 0$ and interchange the two component orbits of H_y. (1) We we may assume that g has the following form:*

$$g : (x, y) \longmapsto (x^{p^a}, y^{p^c} w^{in}), i = 0 \text{ or } 1.$$

(2) $s(p^a - 1) \equiv s(p^c - 1) \equiv 0 \bmod 2n$.
(3) If $y = x$ maps to

$$y = x^{p^z} w^{in} = x^{p^t q^j} w^{vq^j} w^{s(q^j-1)/(q-1)} \alpha$$

under g then

$$vq^j + s(q^j - 1)/(q - 1) + in \equiv 0 \bmod 2n.$$

PROOF. By Foulser [7, (15.4)], if the group on $y = 0$ is $\left\langle w^{d^*}, w^{e^*} \alpha^{s^*} \right\rangle$, where w^{d^*} is $x \longmapsto x w^{d^*}$ and $\alpha : x \longmapsto x^p$, for $q = p^r$, then G_1 acting on $y = 0$ is $\left\langle \alpha^{d^* s^*} \right\rangle$. In this case, the order of w^{d^*} is $(q^n - 1)/d^*$ and the order of α^{s^*} is nr/s^*. Furthermore, we know that $(d^*, e^*) = 1$. We know that $\langle w^n \rangle$ is a subgroup of order $(q^n - 1)/n$ dividing $(q^n - 1)/d^*$, so that d^* divides n. We also have from Foulser that $nr \equiv 0 \bmod s^* d^*$. Now follow by applying $\langle w^n, H_x$, on $x = 0 \rangle$. Hence, we then have without loss of generality that $g : (x, y) \longmapsto (x^{p^{js^* d^*}}, y^{p^{ks_1 d_1}})$ or $(x, y) \longmapsto (x^{p^{js^* d^*}}, y^{p^{ks_1 d_1}} w^n)$, where the group on $x = 0$ is $\left\langle w^{d_1}, w^{e_1} \alpha^{s_1} \right\rangle$, for some integers j and k. Hence, $y = x$ maps to $y = x^{p^z}$ or to $y = x^{p^z} w^n$, for some integer z. Hence, we may assume that g has the form $g : (x, y) \longmapsto (x^{p^a}, y^{p^c} w^{in})$. Now g must normalize both H_x and H_y. Let $h : (x, y) \longrightarrow (x^{q^k} w^{s(q^k-1)/(q-1)} \alpha, y^{q^j} w^{s(q^j-1)/(q-1)} \beta)$, for $\alpha, \beta \in \mathcal{A}$. Noting that $g^{-1} : (x, y) \longmapsto (x^{p^{-a}}, y^{p^{-c}} w^{-p^{-c} in})$ then

$$ghg^{-1} : (x, y) \longmapsto (x^{q^k} w^{s(q^k-1)/(q-1)p^a} \alpha^{p^a}, y^{q^j} w^{s(q^j-1)p^c} \beta^{p^c} w^{-in(q^j-1)}).$$

Hence, we must have

$$w^{s(q^k-1)/(q-1)p^a} = w^{s(q^k-1)/(q-1)} \delta, \text{ for each } k \text{ and some } \delta \in \mathcal{A}.$$

Thus, we require that

$$w^{(s(p^a-1))(q^k-1)/(q-1)} \in \mathcal{A}.$$

This is true if and only if

$$s(p^a - 1) \equiv 0 \bmod 2n.$$

Note that

$$w^{-in(q^j - 1)} \in \mathcal{A},$$

since 2 divides $q^j - 1$, hence we also require that

$$s(p^c - 1) \equiv 0 \bmod 2n.$$

Since $y = x^{p^z} w^{in} = x^{p^t q^j} w^{vq^j} w^{s(q^j - 1)/(q-1)} \alpha$, it follows that

$$vq^j + s(q^j - 1)/(q - 1) + in \equiv 0 \bmod 2n.$$

This proves all parts of the lemma. $\boxed{QED}$

26 Lemma. *If $q \equiv 1 \bmod 4$ then the plane is a nearfield plane.*

PROOF. From the previous lemma and our initial set up, we have

$$q^j + 2(q^j - 1)/(q - 1) + in \equiv 0 \bmod 2n.$$

If n is even then q^j is forced to be even, a contradiction.

Hence, n is odd.

In addition,

$$\begin{aligned}
2(p^a - 1) &\equiv& 0 \bmod 2n, \ 2(p^c - 1) \equiv 0 \bmod 2n, \\
p^t q^j &\equiv& p^z \bmod (q^n - 1), \ z = c - a.
\end{aligned}$$

Moreover, g^2 is $(x, y) \longmapsto (x^{p^{2a}}, y^{p^{2c}} w^{in(p^a + 1)})$ produces an element:

$$g_1 : (x, y) \longmapsto (x^{p^{2a}}, y^{p^{2c}})$$

that maps $y = x$ into $y = x^{p^{2(c-a)}}$. Hence, $p^{2(c-a)} = q^k$, where

$$w^{(q^k - 1)/(q-1)} \in \mathcal{A}.$$

Thus, we have:

$$2n \text{ divides } (q^k - 1)/(q - 1).$$

By Draayer [5, (2.5) (b)], it follows that n divides k, so we may assume that $n = k$. If $a = c$ then $y = x$ is fixed by g, a contradiction, or $(x, y) \longmapsto (x^{p^a}, y^{p^a} w^n)$ is a collineation that maps $y = x$ onto $y = xw^n$. Hence, the spread is a nearfield spread by Lemma 14. If a or c is 0 then we have an 'extra' homology, implying that the plane is a nearfield plane. Thus, $x^{p^{2a}} = x^{p^{2c}}$, and since n is odd, it

follows that since $2(c - a) = rn$, it follows that either $a = c$, a contradiction as above or r is even and $c = a + rn/2$ so that $p^c = p^a \sqrt{q}^n$. So, the image of $y = x$ under g is $y = x^{\sqrt{q}^n} w^{in}$, n odd. It also follows that n divides $(\sqrt{q}^n - 1)$ since n divides $p^a - 1$ and $p^c - 1$. In any case, when we do satisfy all of this, is this plane a nearfield plane? Note that the spread is as follows:

$$\left\{ x = 0, y = 0, (y = x, y = x^{\sqrt{q}^n} w^{in}) H_y \right\}.$$

The question then is whether $h : (x, y) \longmapsto (x, y^{\sqrt{q}^n} w^{in})$ is a collineation group. But, in fact, this is a collineation group: h will map $y = x$ onto $y = x^{\sqrt{q}^n} w^{in}$ and map $y = x^{\sqrt{q}^n} w^{in}$ onto $(y = x^{\sqrt{q}^{2n}} w^{in(\sqrt{q}^n + 1)}) = (y = x\alpha_0)$, for $\alpha_0 \in \mathcal{A}$. Hence, we have shown that whenever, $q \equiv 1 \bmod 4$ and there is a triangle transitive autotopism group then the plane is a nearfield plane.   QED

27 Lemma. *If $q \equiv -1 \bmod 4$ then the plane is a nearfield plane.*

PROOF. So, we have $s = 1, v = n$ and n divides $p^t - 1$. Furthermore,

$$vq^j + s(q^j - 1)/(q - 1) + in \equiv 0 \bmod 2n$$

becomes

$$nq^j + (q^j - 1)/(q - 1) + in \equiv 0 \bmod 2n.$$

In this setting, n divides $(q^j - 1)/(q - 1)$, by Draayer [5, (2.5) (b)], either j is odd and n divides j or j is even and $[n]_{2'}$ divides j. In the latter case, since $n/2 = [n]_{2'}$, it follows that $n/2$ divides j so that n still divides j. Hence, in either case, $j = n$. This implies that $p^t q^j = p^t$, i.e. that $z = t$. We require that

$$(p^a - 1) \equiv 0 \bmod 2n, \ (p^c - 1) \equiv 0 \bmod 2n,$$

implying that n divides $p^a - 1$ and $p^c - 1$ and now $y = x$ maps onto $y = x^{p^t} w^{in}$, implying that $i = 1$. Thus, $p^{a+t} = p^c$. Then we would require that $p^{a+t} - p^a$ would be divisible by $2n$, so that $2n$ must divide $p^t - 1$. And, we must have

$$(p^t - 1) \equiv n(q - 1) \bmod 2n,$$

which is equivalent to the previous statement. So, the spread is represented exactly as before with the additional requirement that $g : (x, y) \longmapsto (x^{p^a}, y^{p^{a+t}} w^n)$ is a collineation. Note that g^2 maps $y = x^{p^t} w^n$ onto $y = x^{p^{2t}} w^{n(p^a + 1)}$ so that $(y = x^{p^{2t}}) = (y = x^{q^j} w^{(q^j - 1)/(q - 1)} \alpha)$, hence, we require that $p^{2t} = q^j$ and $2n$ divides $(q^j - 1)/(q - 1)$. Since $q \equiv -1 \bmod 4$, if j is odd then n divides j and if j is even then $[n]_{2'} = n/2$ divides j.

So, in any case, n divides j, so we may assume that $p^{2t} = q^n$, so that $p^t = 1$ or $q^{n/2}$. Now $(q^{n/2} - 1)/(q - 1)$ is odd and $(q - 1)/2$ is odd. Hence, $(q^{n/2} - 1)_2 = 2$.

But, we require that $2n$ divides $p^t - 1 = q^{n/2} - 1$, so that 4 must divide $q^{n/2} - 1$, a contradiction. Therefore, it can only be that $p^t = q^n = 1$. Hence, the spread is

$$\left\{\, x = 0, y = 0, \{y = x, y = xw^n\}H_y \,\right\}.$$

However, by Lemma 14, the plane is a nearfield plane. This completes the proof of the lemma. <div style="text-align:right">$\boxed{QED}$</div>

Hence, we have the following general result:

28 Theorem. *Let π be a non-Desarguesian affine plane of order p^w that admits two affine homology groups H_x and H_y of order $(p^w - 1)/2$ with axis (respectively, co-axis) $x = 0$ and co-axis (respectively, axis) $y = 0$.*

Then there is an autotopism group that is triangle transitive if and only if either

(1) the plane is a Dickson nearfield plane,

(2) $p^w = 7^2$ or 23^2 and the plane is an irregular nearfield plane,

(3) $p^w = q^n$ and the plane is a generalized André plane of order q^n, where

$$q \equiv 3 \bmod 8, n \equiv 0 \bmod 4, n/4 \text{ is odd.}$$

Furthermore, the spread may be represented as follows:

$$\left\{\, x = 0, y = 0, \{y = x, y = w^{n/2}\}H_y \,\right\}.$$

PROOF. From the classification theorem of the introduction, Theorem 5, it remains to consider the exceptional Lüneburg plane of order 7^2. In this case, there is a collineation group $SL(2,3) \times SL(2,3)$, where the $SL(2,3)$ groups are homology groups. However, in this case, Lüneburg [19] has shown that the full translation complement has rank 5. The five point orbits of G_0, are the sets of points on $x = 0$, on $y = 0$, $\{\,0\,\}$ and the points on lines of any $SL(2,3)$-component orbit of length 24. Hence, the plane cannot be triangle transitive. <div style="text-align:right">$\boxed{QED}$</div>

5   Large triangle transitive groups

29 Remark. The triangle transitive planes that we have constructed are generalized André planes of order q^n that have autotopism groups of order $(q^n - 1)^2/2$.

The question is how large an autotopism group must be in order to ensure that the plane is a generalized André plane?

30 Theorem. *Let π be a triangle transitive translation plane of order p^r.*

(1) If π is non-solvable (admits a non-solvable) then π is the irregular nearfield plane of order $11^2, 29^2$ or 59^2 with non-solvable group.

(2) If π is the Hall plane of order 9, then π is a triangle transitive translation plane.

If π is solvable assume that q^2 is not in $\left\{\, 5^2, 7^2, 11^2, 23^2, 3^4 \,\right\}$.

(1) If $p^r - 1$ admits a p-primitive divisor u and if π admits a group of order divisible by $(p^r - 1)^2_{2'}$ then π is a generalized André plane.

(2) If $p^r - 1$ does not admit a p-primitive divisor and $r = 2$, assume that the order of the group is divisible by $(p^2 - 1)^2/2$.

Then π is a generalized André plane.

PROOF. This proof is a modification of ideas in the proof of Kallaher and Ostrom [20, Theorem (2.2)] and Hiramine and Johnson [12, Theorem (3.1)]. Let G be a non-solvable triangle transitive autotopism group. By Johnson [15], the plane is an irregular nearfield plane with non-solvable group of order $11^2, 29^2$ or 59^2.

Thus, the group G is non-solvable and induces a solvable group on each affine axis. Except for the special orders of Huppert [13]—$p^2 = 3^2, 5^2, 7^2, 11^2, 23^2$ and 3^4—we have that G on each affine axis is a subgroup of $\Gamma L(1, p^r)$. If we have a prime p-primitive divisor u, let u^a be the highest u-power dividing $p^r - 1$, then we obtain an Abelian group of order u^{2a}, which is the direct product of homology groups of order u^a. It follows exactly as in Kallaher and Ostrom [20, Theorem (2.2)], there exist homology groups H_x and H_y of orders u^a. We claim that each homology group H_x or H_y of order u^a is normal in G. Since H_x acts on its co-axis $y = 0$, as a subgroup of $GL(1, p^r)$, H_x is characteristic in the restriction of G to $y = 0$. Hence, H_x is normal in $G/G_{[y=0]}$, where $G_{[y=0]}$ is the group fixing $y = 0$ pointwise. If h is in H_x and g in G then $g^{-1}hg$ is in $H_x G_{[y=0]}$, but $g^{-1}hg$ is an affine homology with axis $x = 0$ so $g^{-1}hg$ is in H_x, if it is in $H_x G_{[y=0]}$.

Therefore H_x and H_y are normal in G so that $H_x \times H_y$ is Abelian and normal in G. It now follows exactly as in the proof of Hiramine and Johnson [12, Theorem (3.1)] that π is a generalized André plane.

However, if π is non-Desarguesian and non-Hall then the automorphism group of a generalized André plane is solvable.

Hence, G is solvable to begin with and our above analysis still applies to show that the plane is a generalized André plane.

Let $G_2 = 2^{2(a+1)+b-1}$.

If $p^r - 1$ does not admit a p-primitive divisor, let $r = 2$ and $p + 1 = 2^a$. We still have by Ganley, Jha, Johnson [9], that we may assume that G induces a solvable group on each affine axis. Hence, G induced on $x = 0$ is a subgroup of $T(p^r)$.

Let S_2 be Sylow 2-subgroup of order 2^{2a+1+b}. Since the group induced on any side $x = 0$ is a subgroup of $\Gamma L(1, p^2)$, and $\Gamma L(1, p^2)_2 = 2^{2+a}$, we must have a homology group H_y of order divisible by 2^{a-1+b} with axis $x = 0$. Furthermore, we have a characteristic normal subgroup H_y^- of order divisible by 2^{a-2+b}. Hence, arguing as above, we have an Abelian normal subgroup $H_x^- \times H_y^-$ of order by $2^{2(a-2+b)}$. Now the stabilizer of a component ℓ distinct from $x = 0$ and $y = 0$ has Sylow 2-subgroups of order 2^{a+b} and let S_ℓ denote any such subgroup. Consider $S_\ell H_x^- \times H_y^-$ of order $2^{a+b+2(a-2+b)}/I$, where I denotes the order of the intersection. Thus,

$$2^{a+b+2(a-2+b)}/I \leq 2^{2a+1+b},$$

implying that

$$I \geq 2^{a+2b-5}.$$

We wish to show that $(H_x^- \times H_y^-)_\ell$ acts irreducibly. So, assume that there is a proper subspace W left invariant by the group. Since ℓ is a 2-dimensional $GF(p)$-subspace, W is a 1-dimensional $GF(p)$-subspace. Since $(p-1)_2 = 2$, it follows that we have a subgroup R of order at least 2^{a+2b-6} that fixes W pointwise. Let M be a Maschke complement of W fixed by R. Then, there is a subgroup R^- of R of order at least 2^{a+2b-7} that fixes M pointwise, implying that R^- fixes ℓ pointwise. But, R^- fixes $x = 0$ and $y = 0$ so $R^- = \langle 1 \rangle$. Hence,

$$a + 2b - 7 \leq 1.$$

In particular, $a \leq 8$. So, $p + 1 = 2^a \leq 2^8$.

$a = 2$, $p = 3$, is Hall or Desarguesian. $a = 3$ then $p = 7$, excluded. $a = 4$, $p = 15$, a contradiction. $a = 5$, $p = 31$. $a = 6$, $p = 63$, a contradiction. $a = 7, p = 127$. $a = 8$, $p = 255$, a contradiction. Thus, $p = 31$ or 127 are on the only possibilities. $\boxed{QED}$

6   Final remarks

With the exception of a few possible exceptional orders, we have shown that if a triangle transitive autotopism group is large enough then the corresponding translation plane is generalized André. We have considered those translation planes of order q^n and kernel containing $GF(q)$ that admit symmetric affine homology groups of order $(q^n - 1)/2$ and have completely classified the possible triangle transitive planes.

There is a related study of translation planes of order q^n that admit symmetric affine homology groups of order $(q^n - 1)/3$ by Draayer [6] and, except for a few sporadic orders, the planes must be generalized André planes. Since each

homology group of order $(q^n - 1)/3$ must permute the three orbits of the other homology group, it is possible that there are triangle transitive planes among the class of generalized André planes with such homology groups or perhaps merely 'large' homology groups.

We therefore close with the following problem:

31 Problem. Determine the triangle transitive generalized André planes that admit 'large' symmetric homology groups.

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