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The effect of a linear constraint on the small oscillations of a dynamical system with three degrees of freedom

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Abstract. We consider the effect of a linear constraint on the small oscillations about an equilibrium position of a conservative dynamical system which has just three degrees of freedom. Explicit expressions are presented for the eigenfrequencies and eigenvectors of the constrained system in terms of the eigenfrequencies and eigenvectors of the unconstrained system. The unconstrained system is described in terms of two positive definite 3×3 matrices with which two concentric ellipsoids may be associated. A 'plane' corresponds to the linear constraint. It is seen that in general it is possible to choose two constraints such that the constrained motion has a double eigenfrequency, or equivalently, two central planes may be chosen which cut the two concentric ellipsoids in a pair of similar and similarly situated ellipses.

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dedicated to Alan Day, fine scholar and good friend

1 Introduction

In this note we consider the small oscillations about an equilibrium position of a conservative dynamical system. It is assumed that the system has three degrees of freedom and that its eigenfrequencies and eigenvectors are known. It is then assumed that this system is subject to a linear constraint, described in terms of a vector **n**. The eigenfrequencies and eigenvectors for the constraint system are written down explicitly in terms of **n**. It is shown that in general there are two choices of the linear constraint, i.e. of **n**, which ensure that the constrained system has a double eigenfrequency and consequently the corresponding eigenvectors are such that the motion may be, loosely speaking, either linearly or circularly or elliptically polarized. We here choose the same approach as in (Synge [2, p. 183]). This approach, using an undetermined multiplier as in the case of a non-holonomic constraint, is appropriate for obtaining simply results in terms of the vector **n**, and in bringing out the role of the special directions **n** leading to a double eigenfrequency.

In the description of the motion of a system with three degrees of freedom two 3×3 positive definite symmetric matrices play a fundamental role. One may associate ellipsoids with these matrices. For the constrained motion two ellipses - the two coplanar central sections of the ellipsoids by the plane with normal **n** play the fundamental role. What determines whether or not there is a double eigenfrequency is whether these ellipses have an infinity of pairs of common conjugate directions or just one such pair, equivalently, whether these ellipses are similar and similarly situated or not. In general there are two planes which cut a pair of concentric ellipsoids in a pair of similar (same aspect ratio) and similarly situated (same direction of the major axis) ellipses [1].

There is a possible hint of these results in Synge [2]. After equation (102.10) p. 184, Synge says: "Degeneracy may be produced by constraint; in geometrical language an ellipsoid possesses circular sections". By 'degeneracy' he means here the coincidence of eigenfrequencies. But, we are not aware of any explicit results such as we present here.

In §2, the basic equations and orthogonality relations are set out. Then in §3 we give the 'Fresnel form' of the secular equation, using a result of Darboux for bordered determinants. Next (§4) explicit expressions are given for the eigenvectors and eigenfrequencies. Finally (§5) we present some examples.

2 Secular Equation and Orthogonality Relations

Consider the small oscillations about an equilibrium position of a conservative dynamical system with three degrees of freedom. Let the system be described by three generalized coordinates $x^i, (i = 1, 2, 3)$, and let $x^i = 0$ be the equilibrium position. For small oscillations about $x^i = 0$, the kinetic energy T and the potential energy V are written, at the quadratic approximation, as

$$T = \frac{1}{2} a_{ij} \dot{x}^i \dot{x}^j \ , \qquad V = \frac{1}{2} v_{ij} x^i x^j \ , \tag{1}$$

where $\mathbf{A} = (a_{ij})$ and $\mathbf{V} = (v_{ij})$ are constant positive definite symmetric 3×3 matrices. The Lagrange equations governing the small oscillations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \ , \quad (i = 1, 2, 3) \quad , \tag{2}$$

where $L = T - V$. These read

$$\mathbf{A}\ddot{\mathbf{x}} + \mathbf{V}\mathbf{x} = \mathbf{0} \ . \tag{3}$$

Seeking solutions in the form

$$\mathbf{x} = \mathrm{Re}\{\mathbf{r}e^{i\omega t}\} \ , \tag{4}$$

where ω is a real constant (the angular frequency), and $\mathbf{r}$ is a complex vector (a bivector), equations (2) give the eigenvalue problem

$$(\mathbf{V} - \omega^2\mathbf{A})\mathbf{r} = \mathbf{0} \ , \quad (v_{ij} - \omega^2 a_{ij})r^j = 0 \ , \tag{5}$$

and hence the frequency equation

$$\det(\mathbf{V} - \omega^2\mathbf{A}) = 0 \ . \tag{6}$$

This is a cubic in ω^2, with roots ω_i^2 (say) which we order $\omega_1^2 \geq \omega_2^2 \geq \omega_3^2 > 0$, and the corresponding eigenvectors are $\mathbf{r}_i$ (say), so that the general solution of (3) is of the form

$$\mathbf{x} = \mathrm{Re}\{\alpha_1\mathbf{r}_1 e^{i\omega_1 t} + \alpha_2\mathbf{r}_2 e^{i\omega_2 t} + \alpha_3\mathbf{r}_3 e^{i\omega_3 t}\} \ , \tag{7}$$

where α_i are three arbitrary complex constants.

Also, recall that

$$\mathbf{r}_i \cdot \mathbf{A}\mathbf{r}_j = \mathbf{r}_i \cdot \mathbf{V}\mathbf{r}_j = 0 \ , \quad (i \neq j) \ . \tag{8}$$

Suppose now that the system is subjected to a time independent constraint, compatible with the equilibrium position $x^i = 0$, so that, at the linear approximation used for small oscillations, this constraint may be written

$$\mathbf{n} \cdot \mathbf{x} = 0 \ , \quad n_i x^i = 0 \ , \tag{9}$$

where n_i are constants. We may assume $\mathbf{n} \cdot \mathbf{n} = 1$ without loss in generality. As stated in (Synge [2, p. 183]), (9) "may be thought of as arising from any constraint which is independent of time; it may even be non-holonomic, there being no distinction in a linear approximation between holonomic and non-holonomic". The equations describing the motion of the constrained system may be written

$$\mathbf{A}\ddot{\mathbf{x}} + \mathbf{V}\mathbf{x} = \mu\mathbf{n} \ , \quad a_{ij}\ddot{x}^j + v_{ij}x^j = \mu n_i \ , \tag{10}$$

where μ is an undetermined multiplier. To find the normal modes of vibration, we now write

$$\mathbf{x} = \mathrm{Re}\{\mathbf{r}e^{i\omega t}\} \ , \quad \mu = \mathrm{Re}\{\lambda e^{i\omega t}\} \ , \tag{11}$$

where $\mathbf{r}$ and λ are constant (possibly complex). Then, from (9) and (10), we obtain

$$(\mathbf{V} - \omega^2\mathbf{A})\mathbf{r} = \lambda\mathbf{n} \ , \quad \mathbf{n}\cdot\mathbf{r} = 0 \ . \tag{12}$$

Let

$$\mathbf{\Pi} = \mathbf{1} - \mathbf{n}\otimes\mathbf{n} \ , \quad \Pi_{ij} = \delta_{ij} - n_i n_j \ , \tag{13}$$

be the projection tensor onto the plane $\mathbf{n}\cdot\mathbf{x} = 0$ orthogonal to $\mathbf{n}$. Then, $\mathbf{\Pi r} = \mathbf{r}\ ,\quad \mathbf{\Pi n} = \mathbf{0}$ and equations (12) give

$$(\mathbf{\Pi V \Pi} - \omega^2\mathbf{\Pi A \Pi})\mathbf{r} = \mathbf{0} \ , \quad \mathbf{n}\cdot\mathbf{r} = 0 \ . \tag{14}$$

Thus, in the plane $\mathbf{n}\cdot\mathbf{x} = 0$, the solutions for $\mathbf{r}$ are the eigenvectors (or eigenbivectors) of $\mathbf{\Pi V \Pi}$ with respect to $\mathbf{\Pi A \Pi}$ corresponding to the eigenvalues ω^2.

Because $(\mathbf{\Pi V \Pi} - \omega^2\mathbf{\Pi A \Pi})\mathbf{n} = \mathbf{0}$, the condition for (14) to have non trivial solutions for $\mathbf{r}$ is that the matrix $(\mathbf{\Pi V \Pi} - \omega^2\mathbf{\Pi A \Pi})$ be of rank 1, i.e. that its adjugate (the cofactors matrix) be zero:

$$(\mathbf{\Pi V \Pi} - \omega^2\mathbf{\Pi A \Pi})^\star = \mathbf{\Pi}^\star(\mathbf{V} - \omega^2\mathbf{A})^\star\mathbf{\Pi}^\star = \mathbf{0} \ , \tag{15}$$

where $\star$ denotes the adjugate (see, for instance Boulanger & Hayes [3, p. 84], for properties of the adjugate). Because $\mathbf{\Pi}^\star = \mathbf{n}\otimes\mathbf{n}$, (15) also reads

$$\mathbf{n}\cdot(\mathbf{V} - \omega^2\mathbf{A})^\star\mathbf{n} = 0 \ , \tag{16}$$

and, because (see Eves [4] for the adjugate of the sum of two 3×3 matrices)

$$\begin{aligned}(\mathbf{V} - \omega^2\mathbf{A})^\star &= (\det\mathbf{V})\mathbf{V}^{-1} - \omega^2(\det\mathbf{A})\{\mathrm{tr}(\mathbf{A}^{-1}\mathbf{V})\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{V}\mathbf{A}^{-1}\} \\ &\qquad + \omega^4(\det\mathbf{A})\mathbf{A}^{-1} \\ &= (\det\mathbf{V})\mathbf{V}^{-1} - \omega^2(\det\mathbf{V})\{\mathrm{tr}(\mathbf{V}^{-1}\mathbf{A})\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{A}\mathbf{V}^{-1}\} \\ &\qquad + \omega^4(\det\mathbf{A})\mathbf{A}^{-1} \ ,\end{aligned} \tag{17}$$

equation (15) may be written in the two equivalent forms

$$\begin{aligned}&\omega^4(\det\mathbf{A})\mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{n} - \omega^2(\det\mathbf{A})\{\mathrm{tr}(\mathbf{A}^{-1}\mathbf{V})\mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{n} - \mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{V}\mathbf{A}^{-1}\mathbf{n}\} \\ &\qquad + (\det\mathbf{V})\mathbf{n}\cdot\mathbf{V}^{-1}\mathbf{n} = 0, \\ &\omega^4(\det\mathbf{A})\mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{n} - \omega^2(\det\mathbf{V})\{\mathrm{tr}(\mathbf{V}^{-1}\mathbf{A})\mathbf{n}\cdot\mathbf{V}^{-1}\mathbf{n} - \mathbf{n}\cdot\mathbf{V}^{-1}\mathbf{A}\mathbf{V}^{-1}\mathbf{n}\} \\ &\qquad + (\det\mathbf{V})\mathbf{n}\cdot\mathbf{V}^{-1}\mathbf{n} = 0.\end{aligned} \tag{18}$$

This is a quadratic in ω^2 whose roots ω'^2 and ω''^2 (say) are the eigenfrequencies of the constrained system.

Assuming ${\omega'}^2 \neq {\omega''}^2$ the corresponding solutions of (14) for $\mathbf{n}$ are $\mathbf{r}'$ and $\mathbf{r}''$ (say). They satisfy

$$\mathbf{r}' \cdot \mathbf{\Pi V \Pi r}'' = 0\ , \quad \mathbf{r}' \cdot \mathbf{\Pi A \Pi r}'' = 0\ , \quad \mathbf{n} \cdot \mathbf{r}' = \mathbf{n} \cdot \mathbf{r}'' = 0\ , \tag{19}$$

or, equivalently,

$$\mathbf{r}' \cdot \mathbf{V r}'' = 0\ , \quad \mathbf{r}' \cdot \mathbf{A r}'' = 0\ , \quad \mathbf{n} \cdot \mathbf{r}' = \mathbf{n} \cdot \mathbf{r}'' = 0\ . \tag{20}$$

But, in the plane $\mathbf{n} \cdot \mathbf{x} = 0$, orthogonal to $\mathbf{n}$, the equations $\mathbf{x} \cdot \mathbf{\Pi V \Pi x} = 1$ and $\mathbf{x} \cdot \mathbf{\Pi A \Pi x} = 1$ describe two ellipses $\mathcal{E}$ and $\mathcal{F}$ (say), so that (19) expresses the fact that $\mathbf{r}'$ and $\mathbf{r}''$ are conjugate with respect to both ellipses. Or, put another way, the eigenvectors $\mathbf{r}'$ and $\mathbf{r}''$ are along the common conjugate directions of the ellipses $\mathcal{E}$ and $\mathcal{F}$ in which the plane $\mathbf{n} \cdot \mathbf{x} = 0$ cuts the ellipsoids $\mathbf{x} \cdot \mathbf{V x} = 1$ and $\mathbf{x} \cdot \mathbf{A x} = 1$. In general there is just one pair of common conjugate directions. If, however, both ellipses are similar (same aspect ratio) and similarly situated (major axes parallel) there is an infinity of such pairs. In this case ${\omega'}^2 = {\omega''}^2$: the secular equation (15) has a double root. That this is so follows by noting that if the ellipses $\mathcal{E}$ and $\mathcal{F}$ are similar and similarly situated, then $\mathbf{\Pi V \Pi} = \delta \mathbf{\Pi A \Pi}$ for some scalar δ and so the secular equation (15) gives $(\delta - \omega^2)^2 = 0$, and thus a double root for ω^2. Note that in this case, the general solution for the constrained motion is

$$\mathbf{x} = \ \mathrm{Re}\{(\alpha' \mathbf{r}' + \alpha'' \mathbf{r}'') e^{i\omega t}\}\ , \tag{21}$$

where ω is the double eigenfrequency, $(\mathbf{r}', \mathbf{r}'')$ is any pair of directions which are conjugate with respect to the ellipses $\mathcal{E}$ and $\mathcal{F}$, and α' and α'' are arbitrary complex numbers. If x^i are interpreted as Cartesian coordinates, it follows from (21) that $\mathbf{x}$ describes an ellipse, the directional ellipse [3] associated with the bivector $\mathbf{R} = \alpha' \mathbf{r}' + \alpha'' \mathbf{r}''$. Because α' and α'' are arbitrary, this ellipse may be any ellipse or circle, or straight line in the plane $\mathbf{n} \cdot \mathbf{x} = 0$.

3 Fresnel Form of the Secular Equation

The secular equation (18) may also be written using the eigenvalues ω_1^2, ω_2^2, ω_3^2 and eigenvectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ of the unconstrained problem. This leads to a form of the secular equation which is similar to that presented by Fresnel in the context of crystal optics [5].

Regarding (12) as a system for the four unknowns $\mathbf{r}^i$, λ, we obtain the secular equation in the form

$$\det\Theta = 0\ , \qquad \text{with} \ \ \Theta = \begin{pmatrix} \mathbf{V} - \omega^2 \mathbf{A} & \mathbf{n} \\ \mathbf{n}^T & 0 \end{pmatrix}\ . \tag{22}$$

Note that $\det\Theta$ is a 4×4 bordered determinant. For such a determinant, a result attributed to Darboux by Bromwich [6] is the identity

$$(\det\Theta)/\det(\mathbf{V}-\omega^2\mathbf{A}) = \frac{L_1^2}{\omega_1^2-\omega^2} + \frac{L_2^2}{\omega_2^2-\omega^2} + \frac{L_3^2}{\omega_3^2-\omega^2} \ , \tag{23}$$

where L_1, L_2, L_3 are appropriate linear combinations of the components n^i of $\mathbf{n}$, and thus the secular equation takes the Fresnel form

$$\frac{L_1^2}{\omega_1^2-\omega^2} + \frac{L_2^2}{\omega_2^2-\omega^2} + \frac{L_3^2}{\omega_3^2-\omega^2} = 0 \ . \tag{24}$$

An explicit expression of this Fresnel form of the secular equation may be obtained using normal coordinates in writing (22). Indeed, consider the 4×4 matrix Σ defined by

$$\Sigma = \begin{pmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & 1 \end{pmatrix} \ , \quad \mathbf{S} = (\mathbf{r}_1|\mathbf{r}_2|\mathbf{r}_3) \ , \tag{25}$$

where $\mathbf{S}$, the non-singular 3×3 matrix whose columns are the eigenvectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, is such that $\mathbf{S}^T\mathbf{V}\mathbf{S}$ and $\mathbf{S}^T\mathbf{A}\mathbf{S}$ are both diagonal. Then, writing the secular equation (22) in the equivalent form $\det(\Sigma^T\Theta\Sigma)=0$ yields

$$\begin{aligned} &(\mathbf{r}_1\cdot\mathbf{n})^2(\mathbf{r}_2\cdot\mathbf{A}\mathbf{r}_2)(\mathbf{r}_3\cdot\mathbf{A}\mathbf{r}_3)(\omega_2^2-\omega^2)(\omega_3^2-\omega^2) \\ &+(\mathbf{r}_2\cdot\mathbf{n})^2(\mathbf{r}_3\cdot\mathbf{A}\mathbf{r}_3)(\mathbf{r}_1\cdot\mathbf{A}\mathbf{r}_1)(\omega_3^2-\omega^2)(\omega_1^2-\omega^2) \\ &+(\mathbf{r}_3\cdot\mathbf{n})^2(\mathbf{r}_1\cdot\mathbf{A}\mathbf{r}_1)(\mathbf{r}_2\cdot\mathbf{A}\mathbf{r}_2)(\omega_1^2-\omega^2)(\omega_2^2-\omega^2) = 0 \ , \end{aligned} \tag{26}$$

or, in Fresnel form

$$\frac{(\mathbf{r}_1\cdot\mathbf{n})^2}{(\mathbf{r}_1\cdot\mathbf{A}\mathbf{r}_1)(\omega_1^2-\omega^2)} + \frac{(\mathbf{r}_2\cdot\mathbf{n})^2}{(\mathbf{r}_2\cdot\mathbf{A}\mathbf{r}_2)(\omega_2^2-\omega^2)} + \frac{(\mathbf{r}_3\cdot\mathbf{n})^2}{(\mathbf{r}_3\cdot\mathbf{A}\mathbf{r}_3)(\omega_3^2-\omega^2)} = 0 \ , \tag{27}$$

or, equivalently,

$$\frac{(\mathbf{r}_1\cdot\mathbf{n})^2}{(\mathbf{r}_1\cdot\mathbf{V}\mathbf{r}_1)(\omega_1^{-2}-\omega^{-2})} + \frac{(\mathbf{r}_2\cdot\mathbf{n})^2}{(\mathbf{r}_2\cdot\mathbf{V}\mathbf{r}_2)(\omega_2^{-2}-\omega^{-2})} + \frac{(\mathbf{r}_3\cdot\mathbf{n})^2}{(\mathbf{r}_3\cdot\mathbf{V}\mathbf{r}_3)(\omega_3^{-2}-\omega^{-2})} = 0 \ . \tag{28}$$

In the Appendix, we also show that (26) may be directly derived from (18).

Of course, each root of the secular equation (26) must be such that the three terms in equation (26) are not all of the same sign, and hence the eigenvalues ω'^2, ω''^2 of the constrained system separate the eigenvalues $\omega_1^2, \omega_2^2, \omega_3^2$ of the unconstrained system [2] : with ω'^2, ω''^2 ordered $\omega'^2 \geq \omega''^2$, we have

$$\omega_3^2 \leq \omega''^2 \leq \omega_2^2 \leq \omega'^2 \leq \omega_1^2 \ . \tag{29}$$

We note that it follows from this that if the constrained system has a double eigenfrequency, $\omega'^2 = \omega''^2$, it must necessarily be equal to the intermediate eigenfrequency of the unconstrained system: $\omega'^2 = \omega''^2 = \omega_2^2$.

4 Explicit Expressions for the Eigenvectors and Eigenvalues in terms of n

Because the eigenvectors $\mathbf{r}'$ and $\mathbf{r}''$ of the constrained system are along the common conjugate directions of the ellipse $\mathcal{E}$ and $\mathcal{F}$ in which the plane $\mathbf{n}\cdot\mathbf{x}=0$ cuts the ellipsoids $\mathbf{x}\cdot\mathbf{V}\mathbf{x}=1$ and $\mathbf{x}\cdot\mathbf{A}\mathbf{x}=1$, we may now write down explicit expressions for $\mathbf{r}'$ and $\mathbf{r}''$, using results from a previous paper [1]. Explicit expressions for the eigenvalues ω'^2 and ω''^2 follow from these.

Assume now that the eigenvectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ of the unconstrained system are normalized by $\mathbf{r}_1\cdot\mathbf{A}\mathbf{r}_1 = 1, \ldots$, and let $\mathbf{r}_\star^1, \mathbf{r}_\star^2, \mathbf{r}_\star^3$ be defined by

$$\mathbf{r}_\star^1 = \mathbf{A}\mathbf{r}_1 \ , \quad \mathbf{r}_\star^2 = \mathbf{A}\mathbf{r}_2 \ , \quad \mathbf{r}_\star^3 = \mathbf{A}\mathbf{r}_3 \ . \tag{30}$$

Then, $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ and $(\mathbf{r}_\star^1, \mathbf{r}_\star^2, \mathbf{r}_\star^3)$ are reciprocal triads:

$$\mathbf{r}_\star^i \cdot \mathbf{r}_j = \delta_j^i \ , \tag{31}$$

and we have

$$\mathbf{r}_i \cdot \mathbf{A}\mathbf{r}_j = \delta_{ij} \ , \quad \mathbf{r}_\star^i \cdot \mathbf{A}^{-1}\mathbf{r}_\star^j = \delta^{ij} \ . \tag{32}$$

Moreover, $\mathbf{r}_\star^1, \mathbf{r}_\star^2, \mathbf{r}_\star^3$ are the eigenvectors of the eigenvalue problem

$$(\mathbf{V}^{-1} - \omega^{-2}\mathbf{A}^{-1})\mathbf{r}^\star = \mathbf{0} \ , \tag{33}$$

corresponding to the eigenvalues $\omega_1^{-2}, \omega_2^{-2}, \omega_3^{-2}$.

We now consider in turn the cases: (**A**) when the eigenvalues $\omega_1^2, \omega_2^2, \omega_3^2$ are all different, (**B**) when two of the eigenvalues are equal. The case when all three eigenvalues are equal need not be considered, because then the ellipsoids $\mathbf{x}\cdot\mathbf{V}\mathbf{x}=1$ and $\mathbf{x}\cdot\mathbf{A}\mathbf{x}=1$ are similar and similarly situated, and so the ellipses $\mathcal{E}$ and $\mathcal{F}$ are similar and similarly situated for all $\mathbf{n}$ and the constrained system has always a double eigenfrequency equal to the triple eigenfrequency of the unconstrained system.

A. Three different eigenfrequencies: $\omega_1^2 > \omega_2^2 > \omega_3^2$.

Then, we have [3, chap. 5, p. 106]

$$\mathbf{V} = \omega_2^2\mathbf{A} + \frac{\omega_1^2 - \omega_3^2}{2}(\mathbf{h}^+ \otimes \mathbf{h}^- + \mathbf{h}^- \otimes \mathbf{h}^+) \ , \tag{34}$$

where

$$\mathbf{h}^\pm = \left(\frac{\omega_1^2 - \omega_2^2}{\omega_1^2 - \omega_3^2}\right)^{\frac{1}{2}} \mathbf{r}_\star^1 \pm \left(\frac{\omega_2^2 - \omega_3^2}{\omega_1^2 - \omega_3^2}\right)^{\frac{1}{2}} \mathbf{r}_\star^3 \ . \tag{35}$$

The vectors $\mathbf{h}^{\pm}$ are normal to the planes that cut the two ellipsoids in similar and similarly situated ellipses.

There are three subcases: (i) $\mathbf{n}$ is not coplanar with $\mathbf{h}^+$ and $\mathbf{h}^-$; (ii) $\mathbf{n}$ is coplanar with $\mathbf{h}^+$ and $\mathbf{h}^-$ but not along either; (iii) $\mathbf{n}$ is along $\mathbf{n}^+$ or $\mathbf{n}^-$.

Case (i): $\mathbf{n}\cdot\mathbf{h}^+\times\mathbf{h}^-\neq 0$.

We assume here that $\mathbf{n}$ is not coplanar with $\mathbf{h}^+$ and $\mathbf{h}^-$, or, equivalently, not orthogonal to $\mathbf{r}_2$. Then, up to a scalar factor, the eigenvectors $\mathbf{r}'$ and $\mathbf{r}''$ of the constrained system are given in terms of $\mathbf{n}$ by [1]

$$\begin{aligned}\mathbf{r}' &= (\mathbf{n}\times\mathbf{h}^+)/s_+ - (\mathbf{n}\times\mathbf{h}^-)/s_- \ ,\\ \mathbf{r}'' &= (\mathbf{n}\times\mathbf{h}^+)/s_+ + (\mathbf{n}\times\mathbf{h}^-)/s_- \ ,\end{aligned} \tag{36}$$

where

$$s_{\pm}^2 = (\mathbf{n}\times\mathbf{h}^{\pm})\cdot\mathbf{A}(\mathbf{n}\times\mathbf{h}^{\pm}) = \omega_2^{-2}(\mathbf{n}\times\mathbf{h}^{\pm})\cdot\mathbf{V}(\mathbf{n}\times\mathbf{h}^{\pm}) \ . \tag{37}$$

The eigenfrequencies of the constrained system may now be obtained from

$$\omega'^2 = (\mathbf{r}'\cdot\mathbf{V}\mathbf{r}')/(\mathbf{r}'\cdot\mathbf{A}\mathbf{r}') \ , \quad \omega''^2 = (\mathbf{r}''\cdot\mathbf{V}\mathbf{r}'')/(\mathbf{r}''\cdot\mathbf{A}\mathbf{r}'') \ . \tag{38}$$

They are given in terms of $\mathbf{n}$ by [1]

$$\begin{aligned}\omega'^2 &= \frac{1}{2}(\omega_1^2+\omega_3^2) - \frac{1}{2}(\omega_1^2-\omega_3^2)\ \det\mathbf{A}^{-1}(\mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{n})^{-1}(c_+c_- - s_+s_-) \ ,\\ \omega''^2 &= \frac{1}{2}(\omega_1^2+\omega_3^2) - \frac{1}{2}(\omega_1^2-\omega_3^2)\ \det\mathbf{A}^{-1}(\mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{n})^{-1}(c_+c_- + s_+s_-) \ ,\end{aligned} \tag{39}$$

where

$$c_{\pm} = (\det\mathbf{A})^{\frac{1}{2}}\mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{h}^{\pm} \ . \tag{40}$$

Case (ii): $\mathbf{n}\cdot\mathbf{h}^+\times\mathbf{h}^- = 0, \quad \mathbf{n}\times\mathbf{h}^{\pm}\neq\mathbf{0}$.

Now we consider the case when $\mathbf{n}$ is in the plane of $\mathbf{h}^+$ and $\mathbf{h}^-$, thus orthogonal to $\mathbf{r}_2$, but not along $\mathbf{h}^+$ and $\mathbf{h}^-$. Then, up to a scalar factor, the eigenvectors $\mathbf{r}'$ and $\mathbf{r}''$ are given in terms of $\mathbf{n}$ by [1]

$$\mathbf{r}' = \mathbf{n}\times\mathbf{A}(\mathbf{n}\times\mathbf{h}^+) \ , \quad \mathbf{r}'' = \mathbf{n}\times\mathbf{h}^+ \ . \tag{41}$$

Note that in (41) $\mathbf{h}^+$ may be replaced by $\mathbf{h}^-$, because in this case $\mathbf{n}\times\mathbf{h}^+$ and $\mathbf{n}\times\mathbf{h}^-$ are along the same direction in the plane $\mathbf{n}\cdot\mathbf{x}=0$. Also, the expressions (39) for the eigenvalues ω'^2 and ω''^2 remain valid.

Case (iii): $\mathbf{n}$ along $\mathbf{h}^+$ or $\mathbf{h}^-$.

When $\mathbf{n}$ is along $\mathbf{h}^+$ or $\mathbf{h}^-$ given by (35), the ellipses $\mathcal{E}$ and $\mathcal{F}$ are similar and similarly situated, and the constrained system has a double frequency. In this case, $\mathbf{r}'$ and $\mathbf{r}''$ may be chosen arbitrarily in the plane $\mathbf{n}\cdot\mathbf{x} = 0$. For instance, for $\mathbf{n} = \mathbf{h}^+$, a simple choice is

$$\mathbf{r}' = \mathbf{r}_2 \ , \quad \mathbf{r}'' = (\omega_2^2 - \omega_3^2)^{\frac{1}{2}}\mathbf{r}_1 - (\omega_1^2 - \omega_2^2)^{\frac{1}{2}}\mathbf{r}_3 \ . \tag{42}$$

Moreover, (39) now yields $\omega'^2 = \omega''^2 = \omega_2^2$ as expected, because $s_+ = 0$ and

$$\begin{aligned} c_+ =&(\det\mathbf{A})^{\frac{1}{2}}\mathbf{h}^+ \cdot \mathbf{A}^{-1}\mathbf{h}^+ = (\det\mathbf{A})^{\frac{1}{2}} \ , \\ c_- =&(\det\mathbf{A})^{\frac{1}{2}}\mathbf{h}^+ \cdot \mathbf{A}^{-1}\mathbf{h}^- = \frac{(\det\mathbf{A})^{\frac{1}{2}}(\omega_1^2 + \omega_3^2 - 2\omega_2^2)}{\omega_1^2 - \omega_3^2} \ . \end{aligned} \tag{43}$$

B. Two eigenfrequencies equal.

Here, the eigenfrequencies are no longer assumed to be ordered: $\omega_1^2 \neq \omega_2^2 = \omega_3^2$.

In this case [3] (chap. 5, p. 107), we have

$$\mathbf{V} = \omega_2^2\mathbf{A} + (\omega_1^2 - \omega_3^2)\mathbf{r}_\star^1 \otimes \mathbf{r}_\star^1 \ . \tag{44}$$

The vector $\mathbf{r}_\star^1$ is normal to the plane that cuts the two ellipsoids in similar and similarly situated ellipses.

Here there are two subcases: (i) $\mathbf{n}$ is not along $\mathbf{r}_\star^1$; (ii) $\mathbf{n}$ is along $\mathbf{r}_\star^1$.

Case (i): $\mathbf{n} \times \mathbf{r}_\star^1 \neq \mathbf{0}$.

Now we consider the general case when $\mathbf{n}$ is not along $\mathbf{r}_\star^1$. Then, up to a scalar factor, we have [1]

$$\mathbf{r}' = \mathbf{n} \times \mathbf{A}(\mathbf{n} \times \mathbf{r}_\star^1) \ , \quad \mathbf{r}'' = \mathbf{n} \times \mathbf{r}_\star^1 \ . \tag{45}$$

Also, the eigenfrequencies of the constrained system are given by [1]

$$\begin{aligned} &\omega'^2 = \omega_2^2 + (\omega_1^2 - \omega_3^2)(\det\mathbf{A})^{-1}(\mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{n})^{-1}(\mathbf{n}\times\mathbf{r}_\star^1)\cdot\mathbf{A}(\mathbf{n}\times\mathbf{r}_\star^1) \ , \\ &\omega''^2 = \omega_2^2 \ . \end{aligned} \tag{46}$$

Thus, whatever the constraint may be, one eigenfrequency is equal to the double eigenfrequency of the unconstrained system. The other eigenfrequency depends on $\mathbf{n}$.

Case (ii): $\mathbf{n}$ along $\mathbf{r}_\star^1$.

When $\mathbf{n}$ is along $\mathbf{r}^1_\star$, the ellipses $\mathcal{E}$ and $\mathcal{F}$ are similar and similarly situated, and the constrained system has a double eigenfrequency as has the unconstrained system. In this case, $\mathbf{r}'$ and $\mathbf{r}''$ may be chosen arbitrarily in the plane $\mathbf{r}^1_\star \cdot \mathbf{x} = 0$. For instance, we may take

$$\mathbf{r}' = \mathbf{r}_2 \ , \quad \mathbf{r}'' = \mathbf{r}_3 \ . \tag{47}$$

Moreover, (46) now yields $\omega'^2 = \omega''^2 = \omega_2^2$ as expected.

5 Examples

Here we present examples.

1 Example. Let

$$\mathbf{V} = \begin{pmatrix} 2k^2\nu^2 & 0 & 0 \\ 0 & \nu^2 & 0 \\ 0 & 0 & \nu^2 \end{pmatrix} \ , \quad \mathbf{A} = \begin{pmatrix} 2k^2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \ , \tag{48}$$

where k and ν are two constants, with $k > 1$. In Pars [7], a mechanical system whose small oscillations are governed by the matrices (48), is described. It consists of two pendula connected by a horizontal hanging rod, and it is used to illustrate the transference of vibrations.

The eigenvalues and eigenvectors of this system are given by

$$\omega_1^2 = \nu^2 k/(k-1) \ ; \quad \omega_2^2 = \nu^2 \ ; \quad \omega_3^2 = \nu^2 k/(k+1) \ , \tag{49}$$

$$2[(k-1)/k]^{\frac{1}{2}}\mathbf{r}_1 = (-1/k, 1, 1) \ ; \quad \sqrt{2}\mathbf{r}_2 = (0, 1, -1) \ ; \quad 2[(k+1)/k]^{\frac{1}{2}}\mathbf{r}_3 = (1/k, 1, 1), \tag{50}$$

and hence,

$$2[k/(k-1)]^{\frac{1}{2}}\mathbf{r}^1_\star = (-2k, 1, 1) \ ; \quad \sqrt{2}\mathbf{r}^2_\star = (0, 1, -1) \ ; \quad 2[k/(k+1)]^{\frac{1}{2}}\mathbf{r}^3_\star = (2k, 1, 1) \ . \tag{51}$$

Then, (35) yields

$$[2k^2/(k^2-1)]^{\frac{1}{2}}\mathbf{h}^+ = (0, 1, 1) \ ; \ [2(k^2-1)]^{-\frac{1}{2}}\mathbf{h}^- = (-1, 0, 0) \ . \tag{52}$$

Using (48), (49) and (52) it is easily checked that (34) holds.

If the mechanical system is subjected to the constraint $x^2 + x^3 = 0$, or to the constraint $x^1 = 0$, then it has the double eigenfrequency $\omega' = \omega'' = \nu$.

For a general constraint $n_1x^1 + n_2x^2 + n_3x^3 = 0$, the Fresnel form (27) of the secular equation is

$$\frac{(-n_1 + kn_2 + kn_3)^2}{\nu^2 k - \omega^2(k-1)} + 2k\frac{(n_2 - n_3)^2}{\nu^2 - \omega^2} + \frac{(n_1 + kn_2 + kn_3)^2}{\nu^2 k - \omega^2(k+1)} = 0 \ . \tag{53}$$

Writing this as a quadratic for ω^2, it is easily seen that the secular equation has indeed the double root $\omega^2 = \nu^2$ when $\mathbf{n}$ is along $(0,1,1)$, or along $(-1,0,0)$. Also, from (36) and (37), we obtain the eigenvectors $\mathbf{r}'$ and $\mathbf{r}''$ for a general constraint $\mathbf{n}\cdot\mathbf{x} = 0$. Up to a scalar factor we have

$$\left.\begin{matrix}\mathbf{r}'\\ \mathbf{r}''\end{matrix}\right\} = \frac{(n_2 - n_3, -n_1, n_1)}{[2n_1^2 + 2k^2(n_2 - n_3)^2]^{\frac{1}{2}}} \pm \frac{(0, -n_3, n_2)}{[n_2^2 + n_3^2]^{\frac{1}{2}}} \ . \tag{54}$$

2 Example. For the transverse vibrations of a system consisting of three equal masses on a light stretched string, we have [8]

$$\mathbf{V} = \omega_0^2 \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \ , \quad \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \tag{55}$$

where ω_0^2 is a positive constant.

The eigenvalues and eigenvectors of this system are

$$\omega_1^2 = (2+\sqrt{2})\omega_0^2 \ , \quad \omega_2^2 = 2\omega_0^2 \ , \quad \omega_3^2 = (2-\sqrt{2})\omega_0^2 \ , \tag{56}$$

$$\mathbf{r}_1 = (1/2, -\sqrt{2}/2, 1/2) \ , \quad \mathbf{r}_2 = (\sqrt{2}/2, 0, -\sqrt{2}/2) \ , \quad \mathbf{r}_3 = (1/2, \sqrt{2}/2, 1/2) \ , \tag{57}$$

and here $\mathbf{r}_i = \mathbf{r}^i_\star$ because $\mathbf{A} = \mathbf{1}$. Then, (35) yields

$$\sqrt{2}\mathbf{h}^+ = (1,0,1) \ , \quad \sqrt{2}\mathbf{h}^- = (0,-1,0) \ . \tag{58}$$

Because $\mathbf{A} = \mathbf{1}$, $\mathbf{h}^+$ and $\mathbf{h}^-$ are the normals to the planes of central circular sections of the ellipsoid $\mathbf{x}\cdot\mathbf{V}\mathbf{x} = 1$.

If the mechanical system is subjected to the constraint $x^1 + x^3 = 0$, or to the constraint $x^2 = 0$, then it has the double eigenfrequency $\omega' = \omega'' = \sqrt{2}\omega_0$.

For a general constraint $n_1x^1 + n_2x^2 + n_3x^3 = 0$, the Fresnel form (27) of the secular equation is

$$\frac{(n_1 - \sqrt{2}n_2 + n_3)^2}{(2+\sqrt{2})\omega_0^2 - \omega^2} + \frac{2(n_1 - n_3)^2}{2\omega_0^2 - \omega^2} + \frac{(n_1 + \sqrt{2}n_2 + n_3)^2}{(2-\sqrt{2})\omega_0^2 - \omega^2} = 0 \ . \tag{59}$$

Again, writing this in the form of a quadratic for ω^2, it is easily seen that it has the double root $\omega^2 = 2\omega_0^2$ when $\mathbf{n}$ is along $(1,0,1)$, or along $(0,-1,0)$. Also, from (36) and (37), we obtain the eigenvectors $\mathbf{r}'$ and $\mathbf{r}''$ for a general constraint $\mathbf{n}\cdot\mathbf{x} = 0$. Up to a scalar factor we have

$$\left.\begin{matrix}\mathbf{r}'\\ \mathbf{r}''\end{matrix}\right\} = \frac{(n_2, n_3 - n_1, -n_2)}{[2n_2^2 + (n_3 - n_1)^2]^{\frac{1}{2}}} + \frac{(n_3, 0, -n_1)}{[n_3^2 + n_1^2]^{\frac{1}{2}}} \ . \tag{60}$$

3 Example. Finally, we give an example when the unconstrained system has a double eigenfrequency. For an appropriate system consisting of a swinging rod to which a mass is attached by a light elastic string (see details in [9]), we have

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 20/3 & 8/3 \\ 0 & 8/3 & 16/9 \end{pmatrix} \ , \quad \mathbf{V} = \omega_0^2 \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4/3 \end{pmatrix} \ , \tag{61}$$

where ω_0^2 is a positive constant. The eigenvalues and eigenvectors of the unconstrained problem are then given by

$$\omega_1^2 = (3/8)\omega_0^2 \quad , \qquad \omega_2^2 = \omega_3^2 = 3\omega_0^2 \ , \tag{62}$$

$$\mathbf{r}_1 = \sqrt{\frac{3}{14}}(0, \frac{1}{2}, \frac{3}{4}) \ , \quad \mathbf{r}_2 = (1, 0, 0) \ , \quad \mathbf{r}_3 = \frac{1}{\sqrt{7}}(0, \frac{3}{2}, -3) \ , \tag{63}$$

and

$$\mathbf{r}_\star^1 = \sqrt{\frac{3}{14}}(0, \frac{16}{3}, \frac{8}{3}), \quad \mathbf{r}_\star^2 = (1, 0, 0) \ , \quad \mathbf{r}_\star^3 = \frac{1}{\sqrt{7}}(0, 2, -\frac{4}{3}) \ . \tag{64}$$

Then using (26) the secular equation is

$$(\omega^2 - 3\omega_0^2)\{(\frac{3}{32})(2n_2 + 3n_3)^2(\omega^2 - 3\omega_0^2) + [7n_1^2 + \frac{9}{4}(n_2 - 2n_3)^2](\omega^2 - \frac{3\omega_0^2}{8})\} = 0 \ . \tag{65}$$

Irrespective of the choice of $\mathbf{n}$, one of the eigenvalues of the constrained system is thus equal to the double eigenvalue $3\omega_0^2$ of the unconstrained system.

When $\mathbf{n}$ is chosen along $\mathbf{r}_\star^1$, so that the constraint is $2x^2 + x^3 = 0$, then the constrained system has the double eigenfrequency $3\omega_0^2$.

For an arbitrary constraint, the eigenvectors are along $\mathbf{r}'$ and $\mathbf{r}''$ given by (45), thus along

$$\mathbf{r}' = (8n_1n_2 + 12n_1n_3, \ -8n_1^2 - 18n_3^2 + 9n_2n_3, \ -12n_1^2 - 9n_2^2 + 18n_2n_3) \ , \tag{66}$$

$$\mathbf{r}'' = (n_2 - 2n_3, \ -n_1, \ 2n_1) \ . \tag{67}$$

The eigenvector along $\mathbf{r}''$ corresponds to the eigenvalue $3\omega_0^2$. The eigenvalue corresponding to $\mathbf{r}'$ depends on $\mathbf{n}$.

6 Closure

Small oscillations about an equilibrium position of a conservative dynamical system with three degrees of freedom have been considered. When the system is subjected to an arbitrary linear constraint, explicit expressions for the eigenvectors and eigenfrequencies have been presented in terms of the eigenvectors and eigenfrequencies of the unconstrained system. It has been shown that, in general, there are two choices of the linear constraint such that when it is subject to the constraint the system has a double eigenfrequency. Illustrative examples have been presented.

Appendix

The secular equation $(18)_1$ divided by det$\mathbf{A}$ reads

$$\omega^4\mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{n}-\omega^2\{\mathrm{tr}(\mathbf{A}^{-1}\mathbf{V})\mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{n}-\mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{V}\mathbf{A}^{-1}\mathbf{n}\}+\det(\mathbf{A}^{-1}\mathbf{V})\mathbf{n}\cdot\mathbf{V}^{-1}\mathbf{n}=0\,. \tag{68}$$

As in §4, assume that the eigenvectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are normalized by $\mathbf{r}_1\cdot\mathbf{A}\mathbf{r}_1 = 1, \ldots$ and write

$$\mathbf{n} = n_1\mathbf{r}^1_\star + n_2\mathbf{r}^2_\star + n_3\mathbf{r}^3_\star\ , \quad n_i = \mathbf{n}\cdot\mathbf{r}_i\ , \tag{69}$$

where $(\mathbf{r}^1_\star, \mathbf{r}^2_\star, \mathbf{r}^3_\star)$ is the reciprocal triad of $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$, defined by (30) or (31).

Then, recalling (5) (32) (33), we note that the quadratic forms $\mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{n}$, $\mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{V}\mathbf{A}^{-1}\mathbf{n}$, $\mathbf{n}\cdot\mathbf{V}^{-1}\mathbf{n}$ entering (68) may be written as sums of squares as follows:

$$\begin{aligned}\mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{n} &= n_1^2 + n_2^2 + n_3^2\ ,\\ \mathbf{n}\cdot\mathbf{A}^{-1}\mathbf{V}\mathbf{A}^{-1}\mathbf{n} &= \omega_1^2 n_1^2 + \omega_2^2 n_2^2 + \omega_3^2 n_3^2\ ,\\ \mathbf{n}\cdot\mathbf{V}^{-1}\mathbf{n} &= \omega_1^{-2} n_1^2 + \omega_2^{-2} n_2^2 + \omega_3^{-2} n_3^2\ .\end{aligned} \tag{70}$$

Moreover,

$$\mathrm{tr}(\mathbf{A}^{-1}\mathbf{V}) = \omega_1^2 + \omega_2^2 + \omega_3^2\ , \quad \det(\mathbf{A}^{-1}\mathbf{V}) = \omega_1^2\omega_2^2\omega_3^2\ . \tag{71}$$

Inserting (70) and (71) into the secular equation (68) yields

$$\begin{aligned}\omega^4(n_1^2+n_2^2+n_3^2) - \omega^2\{n_1^2(\omega_2^2+\omega_3^2) + n_2^2(\omega_3^2+\omega_1^2) + n_3^2(\omega_1^2+\omega_2^2)\}\\ + (n_1^2\omega_2^2\omega_3^2 + n_2^2\omega_3^2\omega_1^2 + n_3^2\omega_1^2\omega_2^2) = 0\ ,\end{aligned} \tag{72}$$

or, collecting the terms in n_1^2, n_2^2, n_3^2,

$$n_1^2(\omega_2^2-\omega^2)(\omega_3^2-\omega^2)+n_2^2(\omega_3^2-\omega^2)(\omega_1^2-\omega^2)+n_3^2(\omega_1^2-\omega^2)(\omega_2^2-\omega^2)=0\ . \tag{73}$$

But $n_1 = \mathbf{n}\cdot\mathbf{r}_1$, $n_2 = \mathbf{n}\cdot\mathbf{r}_2$, $n_3 = \mathbf{n}\cdot\mathbf{r}_3$ when $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ are normalized by $\mathbf{r}_1\cdot\mathbf{A}\mathbf{r}_1 = \mathbf{r}_2\cdot\mathbf{A}\mathbf{r}_2 = \mathbf{r}_3\cdot\mathbf{A}\mathbf{r}_3 = 1$. Hence, if $\mathbf{r}_i$ are not normalized by these conditions, then $\mathbf{r}_i/(\mathbf{r}_i\cdot\mathbf{A}\mathbf{r}_i)^{\frac{1}{2}}$, (no sum), are normalized, and we have then

$$n_1 = \frac{\mathbf{n}\cdot\mathbf{r}_1}{(\mathbf{r}_1\cdot\mathbf{A}\mathbf{r}_1)^{\frac{1}{2}}}\ , \quad n_2 = \frac{\mathbf{n}\cdot\mathbf{r}_2}{(\mathbf{r}_2\cdot\mathbf{A}\mathbf{r}_2)^{\frac{1}{2}}}\ , \quad n_3 = \frac{\mathbf{n}\cdot\mathbf{r}_3}{(\mathbf{r}_3\cdot\mathbf{A}\mathbf{r}_3)^{\frac{1}{2}}}\ . \tag{74}$$

Substituting this into (73) and multiplying by the product $(\mathbf{r}_1\cdot\mathbf{A}\mathbf{r}_1)(\mathbf{r}_2\cdot\mathbf{A}\mathbf{r}_2)(\mathbf{r}_3\cdot\mathbf{A}\mathbf{r}_3)$ immediately gives (26).

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