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Translation structures with a principal line

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Abstract. Here we study affine parallel translation structures, both finite and infinite, with a principal line, that is a line which intersects every line not in its parallel class. These structures can be regarded also as (finite or infinite) translation transversal divisible designs. An algebraic characterization of these structures in terms of semidirect product of groups is provided and the main properties related to their group of automorphisms are inspected. The particular case of kinematic spaces is also taken into consideration.

Keywords: Translation structures, kinematic spaces, group divisible designs, linear spaces with parallelism

MSC 2000 classification: primary 51A15, secondary 51J15

Introduction

An *affine parallel translation structure* (also called *translation structure*, see e.g. [2]), or briefly *apt-structure*, is an incidence structure $(\mathcal{P}, \mathcal{L})$ endowed with an equivalence relation "$\parallel$" on $\mathcal{L}$ called *parallelism* which fulfills the *euclidean parallel axiom*, and a group T of translations which acts regularly on the point-set $\mathcal{P}$. As in the case of affine spaces (that are, in particular, apt-structures) two distinct lines which are parallel do not intersect, but in general there can exist *skew lines*, that is distinct lines which do not intersect but are not parallel. A *principal line* is a line R such that, for any other line S, either $R \parallel S$ or $R \cap S \neq \emptyset$. The study of ap(t)-structures with a principal line was started in [13] and further continued in [15].

In the literature one can find different structures, both finite and infinite, which can be related to the translation structures with a principal line requiring eventually the structure itself to fulfill additional properties. In particular from this construction one can obtain the split extension of kinematic spaces of Marchi, Pianta and Zizioli [16, 18, 19, 22] and, in the finite case, the transversal translation group divisible designs [3, 8, 21].

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In this second situation, the link between the two points of view is quite deep, and many of the results from one branch can be rephrased (or even reproved) in the language of the other branch. This also justifies the idea to consider infinite apt-structures with a principal line as a natural generalization to the infinite case of the aforementioned designs. This point of view is further justified since it is possible to check that infinite translation structures fulfill the axioms of *infinite designs* as defined in [4], but here this aspect is simply touched, since we prefer the point of view of translation structures in our exposition.

This paper is devoted to deepen the analysis of finite and infinite apt-structures which admit a principal line.

In particular in **Section 1** we recall some classical definitions and results concerning apt-structures and groups with a partition in subgroups.

In **Section 2** we take into consideration finite apt-structures with a principal line. Here we underline explicitely the links with *transversal group divisible designs* (see [3, 20, 21]) and characterize apt-structures with “many points” on every line. We provide also a characterization of finite affine translation planes.

Starting from **Section 3** we drop the hypothesis on finiteness and we obtain an algebraic characterization of apt-structures which admit a principal line as semidirect product of groups with an additional property ($\sharp$) (Theorem 17).

Section 4 is devoted to the study of a particular subclass of the apt-structures with a principal line, namely those that fulfill an additional property ($*$) which involves some automorphisms of the algebraic structure. In particular here we obtain again a characterization of affine translation planes among those apt-structures (Theorem 22) and we try to answer the question whether some automorphisms of the algebraic structure are also collineations of the geometric one. Sufficient conditions for the property ($*$) to be fulfilled are proved here (Proposition 27) and in Section 5.

In **Section 5** the special case of *kinematic spaces* with a principal line is taken into consideration. Here we also show explicitely that our construction is a generalization of that of *split extension kinematic spaces* of Marchi, Zizioli and Pianta (see [16, 18, 22]).

Section 6, at the end, provides some examples of apt-structures (in fact kinematic spaces) with a principal line.

1 Setting and known results

For the following definitions and classical results on apt-structures and kinematic spaces we refer, for example, to [1, 2, 9, 10, 14].

1 Definition. Let $\mathcal{P}$ be a nonempty set whose elements we call *points*, $\mathcal{L}$ a family of subsets of $\mathcal{P}$ we call *lines* and “$\parallel$” a binary relation on $\mathcal{L}$ called *parallelism.* The triple $(\mathcal{P}, \mathcal{L}, /\!\!/)$ is an *affine parallel structure (ap-structure)* if it fulfills the following properties:

AP1) for any pair of distinct points p and q there exists exactly one line $\overline{p,q} \in \mathcal{L}$ such that $p, q \in \overline{p,q}$ (hence $(\mathcal{P}, \mathcal{L})$ is a *linear space*);

AP2) "$/\!\!/$" is an equivalence relation on $\mathcal{L}$ such that, for any line $R \in \mathcal{L}$ and for any point $p \in \mathcal{P}$ there exists exactly one line $S \in \mathcal{L}$ such that $p \in S$ and $S /\!\!/ R$ (*euclidean parallel axiom*);

AP3) in $\mathcal{P}$ there exist at least three non collinear points and every line contains at least two points.

2 Definition. Two ap-structures $(\mathcal{P}, \mathcal{L}, /\!\!/)$ and $(\mathcal{P}', \mathcal{L}', /\!\!/')$ are *isomorphic* if there exists a bijection $\varphi : \mathcal{P} \longrightarrow \mathcal{P}'$ which maps lines onto lines and preserves the parallelism. If $(\mathcal{P}, \mathcal{L}, /\!\!/)$ and $(\mathcal{P}', \mathcal{L}', /\!\!/')$ coincide, the isomorphism φ is called an *automorphism* or a *collineation.* A *dilatation* of $(\mathcal{P}, \mathcal{L}, /\!\!/)$ is a collineation φ such that, for any $R \in \mathcal{L}$, $R /\!\!/ \varphi(R)$; a *translation* is the identity or a dilatation without fixed points.

3 Definition. An ap-structure $(\mathcal{P}, \mathcal{L}, /\!\!/)$ is an *affine parallel translation structure* (*apt-structure* or *André structure*) if there exists a group T of translations of $(\mathcal{P}, \mathcal{L}, /\!\!/)$ which acts transitively (and hence regularly) on $\mathcal{P}$.

In the following we will always deal with translation structures.

Recall that a *partition* of a group $(G, \cdot, 1_G)$ is a family $\mathcal{F}$ of non-trivial subgroups of G such that for all $X, Y \in \mathcal{F}$, if $X \neq Y$ then $X \cap Y = \{1_G\}$ and for all $g \in G$ there exists $X \in F$ such that $g \in X$. The family $\mathcal{F} = \{G\}$ is always a partition of the group G and it is called the *trivial partition* of G; in the following we will always assume that a partition of a group G is not the trivial one. It is well known that apt-structures are equivalent to groups with a partition:

4 Theorem (André, [1])**.** *Let G be a group and $\mathcal{F}$ a partition of G. The triple $(G, \mathscr{L}(\mathcal{F}), /\!\!/)$ made up of:*

(1) the set of elements of G;

(2) the set $\mathscr{L}(\mathcal{F})$ of the left cosets of the elements of $\mathcal{F}$;

(3) the binary relation "$/\!\!/$" defined as follows:

$$\forall\ a, b \in G,\ \forall\ F_1, F_2 \in \mathcal{F} : \quad aF_1 /\!\!/ bF_2 \iff F_1 = F_2$$

is an apt-structure, denoted by $[G, \mathcal{F}]$, with transitive translation group isomorphic to G.

Vice versa if $(\mathcal{P}, \mathcal{L}, /\!\!/)$ *is an apt-structure,* $o \in \mathcal{P}$ *and* T *the group of translations regular on the point-set* $\mathcal{P}$*, then the set* $\mathcal{P}$ *can be endowed with an operation which makes it into a group isomorphic to* T *and such that the set* $\mathcal{L}_o := \{L \in \mathcal{L} \mid o \in L\}$ *is a partition of* $\mathcal{P}$ *and, moreover,* $[\mathcal{P}, \mathcal{L}_o] \cong (\mathcal{P}, \mathcal{L}, /\!\!/)$.

In the following we will denote, for any line $L \in \mathcal{L}$, by $[L]$ the *parallel class* of L, that is the equivalence class of L with respect to the equivalence relation "$/\!\!/$".

5 Definition. Let $(\mathcal{P}, \mathcal{L}, /\!\!/)$ be an apt-structure and let $N \in \mathcal{L}$ be a line. N is called a *principal line* if it meets any line in $\mathcal{L}$ which is not in $[N]$.

Note that, by the assumption of transitivity of the translation group, if N is a principal line, then every line in $[N]$ is again principal, so we could restate the previous definition as a definition concerning the parallelism classes of lines.

In [13] and [15] for ap-structures with a principal line the following is proved.

6 Theorem ([13, Prop. 1 and Thm. 4])**.** *Let* $\mathcal{A} = (\mathcal{P}, \mathcal{L}, /\!\!/)$ *be a finite ap-structure with principal lines. Then the following holds:*

(1) if every line of $\mathcal{L}$ *is principal, then* $\mathcal{A}$ *is an affine plane.*

(2) if all the lines of $\mathcal{L}$ *have the same cardinality (i.e.* $\mathcal{A}$ *is a* Sperner space*), then* $\mathcal{A}$ *is an affine plane.*

Recall that, if $(\mathcal{P}, \mathcal{L})$ is a linear space, a *linear subspace* of $(\mathcal{P}, \mathcal{L})$ is a linear space $(\mathcal{P}', \mathcal{L}')$ such that $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{L}' \subseteq \mathcal{L}$. If $X \subseteq \mathcal{P}$ is a set of points of $(\mathcal{P}, \mathcal{L})$, then the *closure* of X, denoted by $\mathscr{C}(X)$, is the intersection of all the linear subspaces of $(\mathcal{P}, \mathcal{L})$ which contain X; the points of the set X are called *generators* of $\mathscr{C}(X)$. If $X = \{p_1, p_2, \ldots\} \subseteq \mathcal{P}$ we say that the points $p_1, p_2, \ldots$ are *independent* if none of the $p_i \in X$ belongs to the closure of the remaining ones. A linear space $(\mathcal{P}, \mathcal{L})$ is said to be an *exchange space* if it fulfills the following axiom, namely the *exchange axiom*:

(E) for any pair of points $p, q \in \mathcal{P}$ and for any subset X of $\mathcal{P}$ such that $p \notin \mathscr{C}(X)$, if $p \in \mathscr{C}\big(\{q\} \cup X\big)$, then $q \in \mathscr{C}\big(\{p\} \cup X\big)$.

If $(\mathcal{P}, \mathcal{L})$ is an exchange space, the *dimension* of a linear subspace $(\mathcal{P}', \mathcal{L}')$ of $(\mathcal{P}, \mathcal{L})$, denoted by $\dim(\mathcal{P}', \mathcal{L}')$, is the number of independent generators of $(\mathcal{P}', \mathcal{L}')$ diminished by 1. In an analogous way, if $(\mathcal{P}, \mathcal{L}, /\!\!/)$ is an ap-structure, an *ap-substructure* of $(\mathcal{P}, \mathcal{L}, /\!\!/)$ is an ap-structure $(\mathcal{P}', \mathcal{L}', /\!\!/')$ such that $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$ and "$/\!\!/'$" is the restriction of "$/\!\!/$" to the lines of $\mathcal{L}'$. This gives rise to the notions of $/\!\!/$*-closure* $\mathscr{C}_{/\!\!/}(X)$ of a subset X of $\mathcal{P}$, $/\!\!/$*-generators*, $/\!\!/$*-independence* and $/\!\!/$*-dimension*.

7 Theorem ([15, Prop. 1]). *If $(\mathcal{P}, \mathcal{L}, /\!\!/)$ is an ap-structure with a principal line, then it is an exchange space with respect to the $/\!\!/$-closure and its $/\!\!/$-dimension is 2.*

2 Finite apt-structures with a principal line

All the apt-structures considered in this section are finite. In the finite case the notion of apt-structure and that of transversal group-divisible design are equivalent; let us recall the following definition (see [8, 20, 21]).

8 Definition. A *group-divisible design* or *GDD* is a finite incidence structure $(\mathscr{P}, \mathscr{B})$ such that

(1) for all $p, q \in \mathscr{P}$ there exists at most one $B \in \mathscr{B}$ such that $p, q \in \mathscr{B}$;

(2) in the set $\mathscr{P}$ an equivalence relation "$\sim$" is defined in the following way:

$$p \sim q \qquad \Longleftrightarrow \qquad p = q \text{ or } [p, q] = 0,$$

where we denote by $[p, q]$ the number of blocks of $\mathscr{B}$ which contain both p and q.

The set $\mathscr{P}/\sim$ is the set of *point classes*. A GDD is a *transversal design* if the following two conditions hold:

3. every block meets every point class;

4. there exist at least two different point classes.

A *translation group-divisible design* is a GDD such that there exists a group $T \leqslant \mathrm{Aut}(\mathscr{P}, \mathscr{B})$ regular on $\mathscr{P}$ and such that for any $B \in \mathscr{B}$ and $\tau \in T$, $\tau(B) = B$ or $\tau(B) \cap B = \emptyset$.

As a straightforward consequence of this Definition we can state the following Proposition.

9 Proposition. *Let $(\mathcal{P}, \mathcal{L}, /\!\!/)$ be a finite apt-structure with a principal line N. Then the incidence structure $(\mathscr{P} := \mathcal{P},\ \mathscr{B} := \mathcal{L} \setminus [N])$ is a translation transversal design, and vice versa.*

All the results on finite apt-structures with a principal line can be rephrased for translation transversal GDD, and vice versa. In particular in [20] a classification of finite translation group-divisible designs is provided. This translates in our context to an exhausting classification of finite apt-structures with a principal line. Recall that a *Frobenius group* is a group G (finite or infinite) such that G contains a normal subgroup N and a proper subgroup $H \neq \{1\}$ such that, for any $n \in N^*$, $H \cap H^n = \{1\}$ and $\mathcal{F} = \{H^n \mid n \in N\} \cup \{N\}$ is a partition of the group[1], called the *principal Frobenius partition.* A *Hughes-Thompson group* is a finite group G such that, for a prime p, G is not a p-group and the Hughes subgroup $H_p(G)$, defined as the group generated by all elements of G of order different from p, is proper and non-trivial.

10 Theorem. *Let $(\mathcal{P}, \mathcal{L}, /\!\!/) = [G, \mathcal{F}]$ be a non-trivial finite apt-structure with a principal line N through the origin. Then one of the following conditions holds.*

(1) G is a p-group and $H_p(G) \leqslant N$.

(2) G is a Hughes-Thompson group.

(3) G is a Frobenius group and $\mathcal{F}$ is the principal Frobenius partition of G.

(4) G is a Frobenius group with a p-group as Frobenius kernel and the partition $\mathcal{F}$ is built as in [3].

Proof. By Proposition 9 above this is simply [20, Thm. 2.1]. QED

11 Proposition. *Let $[G, \mathcal{F}] = (\mathcal{P}, \mathcal{L}, /\!\!/)$ be a finite apt-structure with two non parallel principal lines N_1 and N_2. Then $[G, \mathcal{F}]$ is an affine translation plane, and vice versa.*

Proof. Let L be a line parallel neither to N_1 nor to N_2 and consider the parallel class $[N_1]$. Every line in this set meets both L and N_2 because N_1 is a principal line and no two of these lines can intersect L or N_2 in the same point because this would contradict the euclidean parallel axiom. This shows that $|L| = |N_2|$. The same reasoning shows also that $|L| = |N_1|$, and so all the lines of $\mathcal{L}$ have the same cardinality. We can now conclude by applying Theorem 6. The opposite implication is obvious. QED

In the following, once a principal line N is fixed, we will also denote by $\mathcal{F}^*$ the set $\mathcal{F} \setminus \{N\}$. The following result, conveniently restated, is probably already known from the theory of translation transversal designs, however we cannot find a reference in the literature, thus we propose here a proof using the point of view of translation structures.

12 Proposition. *Let $[G, \mathcal{F}]$ be a finite apt-structure with a principal line N. Then, if n is the cardinality of the line N, the cardinality of any other line $H \in \mathcal{F}^*$ is at most n, and any two lines of $\mathcal{F}^*$ have the same cardinality.*

[1] In the finite case, in fact, it suffices to require the existence of such an H to deduce the existence of N. In the infinite case, instead, this last requirement alone is not enough to have a partition of the group: this is the reason why we choose this more restrictive definition of Frobenius group in this paper.

PROOF. Let $p \in \mathcal{P}$ be a point neither on H nor on N and let $\{q\} := \mathrm{Ln}(p, /\!\!/ N) \cap H$. For every point $x \in H \setminus \{q\}$ the line $\overline{p, x}$ intersects the line N and, when x varies on the line $H \setminus \{q\}$, all these intersection points are distinct, hence $|H| - 1 \leqslant |N|$, which shows that $|H|$ is at most $n+1$. If now $|H| = n+1$, then all the lines through p intersect H, in fact $|\mathcal{L}_p| = |N| + 1 = |H|$, but this is in contradiction with the existence of the parallel line to H through p.

Let now H' be a line of $\mathcal{F}^*$ distinct from H. Then, since N and all the lines in its parallel class are principal lines, the map $h \in H \mapsto H' \cap \mathrm{Ln}(h, /\!\!/ N) \in H'$ is a bijection. QED

It is now easy to characterize those apt-structures which achieve or approach the bound.

13 Proposition. *Let $[G, \mathcal{F}]$ be a finite apt-structure with a principal line N and denote by $n = |N|$, and let $H \in \mathcal{F}^*$.*

(1) If $|H| = n$ then $[G, \mathcal{F}]$ is an affine translation plane.

(2) If $|H| = n - 1$ then there exists an affine translation plane Π such that $[G, \mathcal{F}]$ is obtained from Π by removing a line ($[G, \mathcal{F}]$ is called a stripe plane*) and, moreover, G is a Frobenius group of kernel N.*

PROOF. Claim 1 follows easily from the fact that, by Proposition 12 above, all the lines of $\mathcal{L}$ have the same cardinality and from Theorem 6.

Claim 2 follows from the classification provided by Theorem 10 and from [10, § 6]. A direct construction for the stripe plane is also possible: see [17, Rmk 4.7] for details. QED

14 Remark. Note that, a posteriori, a finite apt-structure with a principal line N and such that every other line has cardinality $|N| - 1$ is a *kinematic space* (see Section 5) and, moreover, in this case N is a normal subgroup of G.

15 Proposition. *Let $[G, \mathcal{F}]$ be a finite apt-structure which is not an affine plane with a principal line N and assume moreover that H acts transitively on the points of N^*. Then for any $H \in \mathcal{F}^*$, $|H| = |N| - 1$, H acts regularly on N^* and G is a Frobenius group.*

PROOF. Denote by n the cardinality of N. If H acts transitively on N^*, then $|H| \geqslant |N| - 1$; this, combined with Propositions 12 and 13 above shows that $|H| = n - 1$, hence necessarily H acts regularly on N^*. QED

3 Characterization of apt-structures with a principal line

Starting from this Section we drop the hypothesis of finiteness of G. First of all let us start by proving the following Theorem which generalizes a result proved by L. Giuzzi in [6] and which deals with the linear dimension of an apt-structure with a principal line N, completing Theorem 7 above.

16 Theorem. *Let $(\mathcal{P}, \mathcal{L}, /\!\!/) = [G, \mathcal{F}]$ be an apt-structure with a principal line $N \in \mathcal{F}$. Then $(\mathcal{P}, \mathcal{L})$ is an exchange space and one of the following is fulfilled:*

(1) if there exists $F \in \mathcal{F} \setminus \{N\}$ such that $|F| = 2$, then $\dim(\mathcal{P}, \mathcal{L}) = 3$*;*

(2) if there exists $F \in \mathcal{F} \setminus \{N\}$, $|F| > 2$, then $\dim(\mathcal{P}, \mathcal{L}) = 2$.

Proof. Assume that $F \in \mathcal{F} \setminus \{N\}$ is a line such that $|F| = 2$. From the fact that every line in $[N]$ meets F, follows that the parallel class of N is made up of two distinct lines, and so $\mathcal{P} \setminus N \in \mathcal{L}$ and every other line in $\mathcal{L} \setminus [N]$ has cardinality two. This shows that, for any four triplewise non collinear points $a, b, c, d \in \mathcal{P}$, we may assume that $N = \overline{a, b}$ and $\mathcal{P} \setminus N = \overline{c, d}$, so

$$\mathcal{P} = N \cup (\mathcal{P} \setminus N) = \overline{a, b} \cup \overline{c, d} \subseteq \mathscr{C}\big(\{a, b, c, d\}\big) \subseteq \mathcal{P},$$

and hence $\dim(\mathcal{P}, \mathcal{L}) \leqslant 3$.

Let now $a, b, c \in \mathcal{P}$ be any three non collinear points; again we can assume without loss of generality that $\overline{c, b} \in [N]$. Hence $\mathscr{C}(a, b, c) = N \cup \{a\} \neq \mathcal{P}$, and this shows that $\dim(\mathcal{P}, \mathcal{L}) \geqslant 3$.

Assume now that for one, and hence for all $F \in \mathcal{F} \setminus \{N\}$, $|F| > 2$. Let a, b, c be three non collinear points and $\mathcal{A} = (\mathcal{P}', \mathcal{L}') := \mathscr{C}(\{a, b, c\})$. Without loss of generality we can assume that $\overline{a, b}, \overline{a, c} \notin [N]$. Let $L := \mathrm{Ln}(c, /\!\!/ \, N)$ and $M := \mathrm{Ln}(a, /\!\!/ \, N)$. If we denote by d the point $\{d\} := L \cap \overline{a, b}$, then d is a point of $\mathcal{A}$ and $L = \overline{c, d} \in \mathcal{L}'$. From $|F| > 2$ follows that there exists a point e in $\mathcal{A}$ such that $e \in \overline{a, b} \setminus \{a, d\}$, and so the line $\overline{c, e}$ is a line of $\mathcal{A}$. Denote by f the point of $\mathcal{P}'$ such that $\{f\} = M \cap \overline{c, e}$, then $M = \overline{a, f}$ is a line of $\mathcal{A}$. Of course every line R through c and different from L meets the line M, and so is entirely contained in $\mathcal{A}$, thus

$$\mathcal{P} = \bigcup_{x \in \mathcal{P}} \overline{c, x} = \bigcup_{\substack{R \in \mathcal{L} \\ c \in R}} R \subseteq \mathcal{P}' \subseteq \mathcal{P}$$

hence $\mathcal{A} = (\mathcal{P}, \mathcal{L})$, which shows that $\dim(\mathcal{P}, \mathcal{L}) = 2$. QED

In the next Theorem we obtain an algebraic characterization of apt-structures admitting a principal line.

17 Theorem. *Let $(\mathcal{P}, \mathcal{L}, /\!\!/) = [G, \mathcal{F}]$ be an apt-structure and $N \in \mathcal{F}$ a principal line. Then, the following property holds:*

$$\textit{for any line } H \in \mathcal{F} \setminus \{N\},\ G = N \rtimes H \qquad (\sharp)$$

Conversely let $G = N \rtimes H$ be a group with a partition $\mathcal{F}$ such that $N, H \in \mathcal{F}$ and the property ($\sharp$) holds. Then N is a principal line.

PROOF. Let $H \in \mathcal{F}$ be a line distinct from N. First we show that $G = N \cdot H$. Let $g \in G$ and consider the line $gH = \mathrm{Ln}(g, /\!\!/ \ H)$. By definition of principal line there exists $n \in N$ such that $\emptyset \neq gH \cap N = \{n\}$, thus

$$gH = \mathrm{Ln}(g, /\!\!/ \ H) = \mathrm{Ln}(n, /\!\!/ \ H) = nH,$$

indeed there exists $h \in H$ such that $g = nh$. The proof of the fact that N is normal is essentially the same as in the finite case, see [7, Prop. 1.6]. We recall it here for completeness. Let $g \in G$ and $n \in N$, assume that $f = g^{-1}ng \notin N$ and write F for the component of $\mathcal{F}$ containing f. Then, by the first part of this proof, there exist $n' \in N$ and $f' \in F$ such that $g = n'f'$, thus

$$f \in \widetilde{g}(N) \cap F = \widetilde{f'}(N) \cap F = \widetilde{f'}(N \cap F) = 1 \in N,$$

a contradiction.

To prove the converse let $a, b \in G$ and $F \in \mathcal{F}^*$, and consider the lines aF and bN. By property ($\sharp$) $G = N \rtimes F$, so there exist $n, n' \in N$ and $f, f' \in F$ such that $a = nf$, $b = n'f'$, $aF = nfF = nF$ and $bN = Nb = Nnf' = Nf'$, and so $nf' \in aF \cap bN$. QED

18 Remark. Note that by property ($\sharp$) all the subgroups of $\mathcal{F}^*$ are isomorphic. An isomorphism between two subgroups $H, H' \in \mathcal{F}$ is, for example, the following:

$$\begin{array}{ccccc} H & \longrightarrow & G/N & \longrightarrow & H' \\ h & \longmapsto & Nh = Nnh' & \longmapsto & h', \end{array}$$

where $h = nh' \in N \rtimes H' = G$. In the following we will refer to these isomorphisms as the *canonical* ones.

In some interesting cases it is also possible to extend such isomorphisms between a fixed line H and every other line of $\mathcal{F}^*$ to automorphisms of the whole group G obtaining a family $\Psi \subseteq \mathrm{Aut}(G, \cdot)$ which fulfills the following property:

$$\forall\ F \in \mathcal{F}^*\ \exists\ \psi \in \Psi \text{ such that } \psi(H) = F \text{ and } \psi(N) = N. \qquad (*)$$

In general for a group $G = N \rtimes H$ the existence of a family Ψ fulfilling property ($*$) (not necessarily obtained extending the canonical isomorphisms above) is stronger than property ($\sharp$) (see Section 4). Note also that the existence of such a family Ψ with respect to a given line H entails, for any other line $F \in \mathcal{F}^* \setminus \{H\}$, the existence of family Ψ' which fulfills property ($*$). In the case a family Ψ fulfilling property ($*$) exists it could be interesting to find out if, under suitable choices of the extensions, the family Ψ is also a family of collineations of the geometric structure (later this will become an assumption in this work). A partial answer to this question is provided by Proposition 21. Example 35 shows a case in which the family Ψ is in fact a set of collineations which do not fix N pointwise.

For any $F, H \in \mathcal{F}$ denote by $N_G(F)$ and $C_G(F)$ the normalizer and the centralizer respectively of the line F in G and by $N_H(F)$ and $C_H(F)$ the groups $N_G(F) \cap H$ and $C_G(F) \cap H$ respectively. The next Proposition gives a geometric description of the normalizer in G of a line $F \in \mathcal{F}$ which will be useful in section 5.

19 Proposition. *Let $(\mathcal{P}, \mathcal{L}, /\!\!/) = [G, \mathcal{F}]$ be an apt-structure. Then, for any $F \in \mathcal{F}$ the normalizer $N_G(F)$ of F in G is a union of lines parallel to F. Moreover, if $N \in \mathcal{L}$ is a principal line and $H \in \mathcal{F}^*$, then $N_G(H) = N_N(H) \times H$ and $N_G(N) = N_H(N) \times N$.*

Proof. If $g \in G$ belongs to $N_G(F)$, then, for all $f \in F$, $\widetilde{gf}(F) = \widetilde{g}(F) = F$, so $gf \in N_G(F)$, which shows that $gF = \mathrm{Ln}(g, /\!\!/ \, F) \subseteq N_G(F)$. The second part is immediate. QED

4 Apt-structures which fulfill property $(*)$

Let us start by observing that if for $G = N \rtimes H$ there exists a family $\Psi \subseteq \mathrm{Aut}(G, \cdot)$ fulfilling the property $(*)$, then for any $F \in \mathcal{F}^*$, $G = N \rtimes F$, that is property $(\sharp)$.

In order to achieve our results let us now point out some properties of a family Ψ which fulfills property $(*)$. Note that the family Ψ is not required to be a group. The Corollaries 32 and 33 of section 5 show two cases in which the family Ψ is a group (equal to the group of inner automorphisms of N); in the general case the situation is made clear by the following Proposition.

20 Proposition. *Let $[G = N \rtimes H, \mathcal{F}]$ be a non abelian apt-structure such that $N, H \in \mathcal{F}$. Then there exists a family Ψ which fulfills the property $(*)$ if and only if there exists a subgroup S of $\mathrm{Aut}(G, \cdot)$ which acts transitively on $\mathcal{F}^*$ and such that, for any φ in S, $\varphi(N) \in \mathcal{L}$.*

Proof. If Ψ is a family which fulfills property $(*)$, then the group $S := \langle \Psi \rangle$ achieves the required properties. The converse is trivial. QED

According to the previous Proposition we can assume that a set Ψ which fulfills property $(*)$ is in fact a group. Note however that, of course, a group Ψ does not need to fulfill the property that any automorphism of Ψ maps the line H to another line of $\mathcal{F}^*$. In fact it turns out that requiring that for all $\psi \in \Psi$,

$\psi(H)$ is still a line of $\mathcal{F}^*$ is equivalent to the fact that the group Ψ is a group of collineations of the geometric structure, as shown by the following Proposition.

21 Proposition. *Let $[G = N \rtimes H, \mathcal{F}]$ be an apt-structure such that $N, H \in \mathcal{F}$ and Ψ a group of automorphisms of G which fulfills property $(*)$. Then Ψ is such that for all $\psi \in \Psi$, $\psi(H)$ is still a line of $\mathcal{F}^*$ if and only if any automorphism of Ψ is a collineation.*

Proof. Assume that any automorphism of Ψ maps the line H to another line of $\mathcal{F}^*$. According to property $(*)$ for any $F \in \mathcal{F}^*$ there exists $\varphi \in \Psi$ such that $F = \varphi(H)$, and so, for any $\psi \in \Psi$, $\psi(F) = \psi \circ \phi(H)$, which is still a line, because $\psi \circ \phi \in \Psi$. The converse is obvious. QED

In the remainder of this section we will always assume that the group Ψ which fulfills property $(*)$ is in fact a group of *collineations* of the geometric structure $(\mathcal{P}, \mathcal{L}, /\!/)$, fixing N and acting transitively on $\mathcal{F}^*$ (by Proposition 20).

The next result shows that, the existence of two distinct and non parallel principal lines is in fact equivalent to the fact that every line of $\mathcal{L}$ is principal and so gives a characterization of affine translation planes among apt-structures (see also Proposition 11 where the assumption of finiteness replaces property $(*)$).

22 Theorem. *Let $(\mathcal{P}, \mathcal{L}, /\!/)$ be an apt-structure with a principal line N and a family Ψ of automorphisms which fulfills property $(*)$. Then $(\mathcal{P}, \mathcal{L}, /\!/)$ admits another normal line $N_1 \not\parallel N$ if and only if $\mathcal{P}$ is an abelian group. Moreover the line N_1 is principal if and only if $(\mathcal{P}, \mathcal{L}, /\!/)$ is an affine translation plane.*

Proof. Let F be any line of $\mathcal{F} \setminus \{N, N_1\}$. Then, $G = N \times N_1 = N \rtimes F$ by Theorem 17, thus, by property $(*)$, F is a normal subgroup of G, and hence $\mathcal{P} = G$ is abelian by a well known Theorem of Kontorovič (see [12]). If we assume moreover that N_1 is a principal line, $G = F \times N$ by property $(\sharp)$ referred to the principal line N and, by property $(\sharp)$ referred to the principal line N_1, $N \cong F$. If φ is such an isomorphism, then, for any line $H \in \mathcal{F}^* \setminus \{F\}$, the map $\overline{\varphi}$ defined as follows

$$\overline{\varphi} : \begin{cases} N \times H \longrightarrow F \times H \\ nh \longmapsto \varphi(n)h \end{cases}$$

is an automorphism of G since the group is abelian, and hence F is a principal line as well. By Theorem 6.1, $(\mathcal{P}, \mathcal{L}, /\!/)$ is an affine translation plane. The opposite implication is obvious. QED

The following result is already known for a class of kinematic spaces (see [19]), but it extends also to our situation.

23 Proposition. *Let $[G = N \rtimes H, \mathcal{F}]$ be a non abelian apt-structure such that $N, H \in \mathcal{F}$ and Ψ a family of automorphisms of G which fulfills property $(*)$.*

Then every automorphism φ of G which preserves the partition fulfills $\varphi(N) = N$.

PROOF. Let $\varphi \in \mathrm{Aut}(G, \cdot, \mathcal{F})$, then $\varphi(N) \in \mathcal{F}$ and $\varphi(N) \trianglelefteq G$. If $\varphi(N) \neq N$, then $G = N \times \varphi(N)$ would be an apt-structure with two distinct principal lines, and so, according to Theorem 22, G would be an abelian group, which contradicts our assumption. QED

24 Theorem. *Let $(\mathcal{P}, \mathcal{L}, /\!\!/)$ be an apt-structure with a principal line N and Ψ a group of automorphisms of G which fulfills property $(*)$. Then, for any line H in $\mathcal{F}^*$,*

$$\mathrm{Aut}(G, \cdot, \mathcal{F}) = \Psi \circ \Gamma_H$$

where Γ_H is the group $\Gamma_H := \{\gamma \in \mathrm{Aut}\,(G, \cdot, \mathcal{F}) \mid \gamma(H) = H\}$. Moreover if Ψ acts regularly on $\mathcal{F}^*$, then $\Psi \cap \Gamma_H = \{id\}$.

PROOF. Let $\varphi \in \mathrm{Aut}(G, \cdot, \mathcal{F})$ and consider $H \in \mathcal{F}^*$. Then, if we consider the automorphism ψ of Ψ sending H to $H' := \varphi(H)$, we have that

$$\psi^{-1} \circ \varphi(H) = H,$$

that is $\psi^{-1} \circ \phi \in \Gamma_H$. The second part is trivial. QED

25 Remark. Let us observe that if we succeed in extending the "canonical" isomorphisms of Remark 18 to a family Ψ of automorphisms of the whole group G, then any $\psi \in \Psi$ acts between H and $\psi(H)$ as a sort of "projection" in the direction of the principal line N, that is, from the geometrical viewpoint,

$$\psi(h) = \mathrm{Ln}(h, /\!\!/ \; N) \cap H'.$$

Note also that, for any line $L = hN /\!\!/ N$ and for any $\psi \in \Psi$, $\psi(L) = L$, in fact

$$\psi(hN) = \psi(h)\psi(N) = \psi(h)N = hN$$

because $\psi(h) \in \mathrm{Ln}(h, /\!\!/ \; N)$.

Moreover note that any automorphism $\psi \in \Psi$ restricted to a line $H \in \mathcal{F}^*$ acts as a projection in the sense above if and only if every line of $[N]$ is fixed by every automorphism of Ψ. A sufficient condition for this to hold is that, for all $\psi_1, \psi_2 \in \Psi$ and for all $L \in [N]$, $\psi_1(L) = \psi_2(L)$. Under the assumption that any line of $[N]$ is fixed by any automorphism ψ of Ψ, all those automorphisms act as "projections" not only between the lines H and $H' = \psi(H)$ as shown above, but between any pair of corresponding lines in $\mathcal{F}^*$. For we have that, if F and F' are such that $\psi(F) = F'$ and $f \in F$, if we denote by h the intersection point of H and fN and by h' the point $\psi(h)$, then

$$\psi\left(\overline{h, h'}\right) = \mathrm{Ln}(h', /\!\!/ \; N) = \overline{h, h'},$$

and so

$$\psi(f) = \psi(F) \cap \psi\left(\overline{h, h'}\right) = F' \cap \overline{h, h'} = F' \cap \mathrm{Ln}(f, /\!\!/\, N).$$

If we strengthen our hypothesis and assume also that all the automorphisms of Ψ are obtained extending the isomorphisms between the line H and every other line of $\mathcal{F}^*$ by the identity of N (this happens, for instance, in Example 35), then we can prove the following Proposition, which shows a sufficient condition for the existence of a family Ψ which fulfills property $(*)$ and which acts regularly on $\mathcal{F}^*$, and at the same time characterizes those groups with an abelian principal line.

26 Lemma. *Let $[G = N \rtimes H, \mathcal{F}]$ be an apt-structure such that $N, H \in \mathcal{F}$ and Ψ be a group of automorphisms of G which fulfills property $(*)$. Assume, moreover, that, for any $\psi, \varphi \in \Psi$, $\psi_{|_N} = \phi_{|_N}$ and ψ fixes any line parallel to N. Then $\forall\ \psi \in \Psi :\ \psi_{|_N} = \mathrm{id}_N$ and Ψ acts regularly on the set $\mathcal{F}^*$ and hence on the points of any line $L \neq N$ parallel to N.*

Proof. From Proposition 20 it follows that the group Ψ is transitive on the set $\mathcal{F}^*$. Assume now that ψ and φ are automorphisms of Ψ such that $\psi(F) = \varphi(F)$ for a line $F \in \mathcal{F}^*$. Then, for any $f \in F$, according to Remark 25,

$$\psi(f) = \mathrm{Ln}(f, /\!\!/\, N) \cap \psi(F) = \mathrm{Ln}(f, /\!\!/\, N) \cap \phi(F) = \{\varphi(f)\},$$

which shows that $\psi_{|_F} = \phi_{|_F}$. If for any $\psi, \varphi \in \Psi$ $\psi_{|N} = \varphi_{|N}$, then, by the fact that Ψ is a group, necessarily $\psi_{|N} = \phi_{|_N} = \mathrm{id}_N$. It is now enough to apply $(\sharp)$ to conclude. QED

27 Proposition. *Let $[G, \mathcal{F}]$ be an apt-structure with a principal line N. Then property $(*)$ is fulfilled for a family Ψ of automorphisms which act as projections in the direction of the principal line N and such that for any $\psi \in \Psi$ $\psi_{|N} = id_N$ if and only if the group N is abelian.*

Proof. Assume first that property $(*)$ holds. As an easy computation shows the fact that Ψ is made up of automorphisms of the whole group G is equivalent to the fact that, for any $n \in N$, $h \in H$ and $\psi \in \Psi$, $hnh^{-1} = \psi(h)n\phi(h)^{-1}$, or equivalentely

$$\widetilde{h}(n) = \widetilde{\psi(h)}(n). \tag{1}$$

Note that, since the automorphisms of Ψ act as projections, for any $h \in H$,

$$\Psi(h) := \{\psi(h) \mid \psi \in \Psi\} = hN = Nh,$$

hence, since equality (1) is true for any $\psi \in \Psi$ and the first part of this equation is independent of the choice of ψ, for any $n_1, n_2 \in N$, $\widetilde{n_1 h} = \widetilde{n_2 h}$, that is $\widetilde{n_1} = \widetilde{n_2}$, which shows that N is abelian.

Conversely assume now that N is abelian and, for any $H' \in \mathcal{F}^*$ write $\phi_{H'}$ for the "canonical" isomorphism between the two lines H and H'. Note that, as observed in Remark 25, these isomorphisms act as projections in the direction of the principal line N. Consider now the maps

$$\psi_{H'} : \begin{cases} N \rtimes H & \longrightarrow \; N \rtimes H' \\ nh & \longmapsto \; n\, \varphi_{H'}(h) \end{cases}$$

and write $\Psi := \{\psi_{H'} \mid H' \in \mathcal{F}^*\}$. Again (see (1)) those maps are automorphisms of G if and only if, for any $h \in H$, $n \in N$, $\psi \in \Psi$,

$$h^{-1}\psi(h) \in C_G(n) \qquad \Leftrightarrow \qquad \psi(h) \in h\; C_G(n).$$

Since N is abelian, $N \subseteq C_G(N)$, hence for any $h \in H$, $hN = \Psi(h) \subseteq h\; C_G(N)$ which concludes the proof. QED

28 Corollary. *Let $[G, \mathcal{F}]$ be an apt-structure with a principal line N and assume that N is an abelian group. Then property $(\sharp)$ is equivalent to the existence of a family $\Psi \subseteq \mathrm{Aut}(G, \cdot)$ fulfilling property $(*)$.*

5 Kinematic spaces with a principal line

In the following we will always assume that $[G, \mathcal{F}]$ is a *kinematic space*, i.e. the partition $\mathcal{F}$ of G is *normal*.

First of all let us remark that for a group $G = N \rtimes H$ all the conjugates of H are those made by elements of N, that is $\widetilde{G}(H) = \widetilde{N}(H)$, since for any $g = nh \in G$,

$$\widetilde{g}(H) = gHg^{-1} = nhHh^{-1}n^{-1} = \widetilde{n}(H).$$

The following Proposition once again gives a geometric description of the inner automorphisms of G as projections between a line $H \in \mathcal{F}^*$ and its image in the direction of the principal line N in the same sense as in Remark 25. This has a non trivial consequence, as shown in Corollary 30.

29 Proposition. *Let $[G, \mathcal{F}]$ be a kinematic space with a principal line N and let $n \in N$. Then, for any line $H \in \mathcal{F}^*$, the inner automorphism $\widetilde{n}$ acts on the line H as the projection of this line on $H' := \widetilde{n}(H)$ in the direction of N, i.e., for all $h \in H$,*

$$\widetilde{n}(h) = H' \cap \mathrm{Ln}(h, /\!/\, N).$$

Proof. Of course $\widetilde{n}(N) = N$. Fix $H \in \mathcal{F}^*$ and let $h \in H$. Then $\{h\} = H \cap hN$, thus

$$\begin{aligned} \widetilde{n}(h) &= \widetilde{n}(H) \cap \widetilde{n}(hN) = H' \cap \widetilde{n}(h)N = H' \cap hh^{-1}(nhn^{-1})N \\ &= H' \cap hN = H' \cap \mathrm{Ln}(h, /\!/\, N). \end{aligned}$$

QED

30 Corollary. *Let $[G, \mathcal{F}]$ be a kinematic space with a principal line N. Then, for any line $H \in \mathcal{F}^*$, $N_N(H) = C_N(H)$.*

Proof. Of course $C_N(H) \subseteq N_N(H)$. If $n \in N_N(H)$, then, by Proposition 29,

$$\widetilde{n}(h) = H \cap \mathrm{Ln}(h, /\!\!/ \; N) = \{h\}. \qquad \boxed{QED}$$

In Proposition 19 we gave a geometric description to the normalizer of a line $F \in \mathcal{F}$ in G. We are now in position to give both a geometric and an algebraic description also of the centralizer of a line H in G.

31 Proposition. *Let $[G, \mathcal{F}]$ be a kinematic space with a principal line N. Then, for any line $H \in \mathcal{F}^*$, if we denote by $Z(H)$ the centre of H, the centralizer of H in G is*

$$C_G(H) = N_G(H) \cap \bigcup_{z \in Z(H)} \mathrm{Ln}(z, /\!\!/ \; N),$$

or, equivalentely, $C_G(H) = N_N(H) \times Z(H)$.

Proof. Let us start by proving the first equality. Let $x \in N_G(H) \cap zN$ for a suitable $z \in Z(H)$. By 19 there exists $n \in N_N(H)$ such that $\{x\} = nH \cap zN$, and so $x = nz$. For any $h \in H$ we have

$$\widetilde{x}(h) = \widetilde{nz}(h) = nzhz^{-1}n^{-1} = nhn^{-1} = h$$

where in the last equality we are aware of Corollary 30. This shows that

$$N_G(H) \cap \bigcup_{z \in Z(H)} \mathrm{Ln}(z, /\!\!/ \; N) \subseteq C_G(H).$$

To show the opposite inclusion let now $x \in C_G(H) \subseteq N_G(H)$ and, since $G = N \rtimes H$, let $n \in N_N(H)$ and $h \in H$ be such that $x = nh$. Then for any $h' \in H$,

$$\widetilde{h}(h') = hh'h^{-1} = n^{-1}nhh'h^{-1}n^{-1}n = \widetilde{n^{-1}}\big(\widetilde{x}(h')\big) = \widetilde{n^{-1}}(h') = h'$$

by Corollary 30. This shows that $h \in Z(H)$, and so

$$x = nh \in hN \subseteq \bigcup_{z \in Z(H)} \mathrm{Ln}(z, /\!\!/ \; N).$$

Let us now show the second claim. The proof of the second inclusion above entails $C_G(H) = N_N(H) \cdot Z(H)$. Moreover $N_N(H) \cap Z(H) = \{1\}$ because $N_N(H) \subseteq N$, $Z(H) \subseteq H$ and $N \cap H = \{1\}$. Let us now show that $N_N(H) \trianglelefteq C_G(H)$. For, let $n \in N_N(H)$ and $x \in C_G(H)$. Then

$$\widetilde{\widetilde{x}(n)}(H) = \widetilde{xnx^{-1}}(H) = xnx^{-1}Hxn^{-1}x^{-1} = xnHn^{-1}x^{-1} = xHx^{-1} = H$$

which shows that $\widetilde{x}(n) \in N_N(H)$. The relation $Z(H) \trianglelefteq C_G(H)$ is trivial, and this complete the proof. QED

Let us denote, for any $H \in \mathcal{F}^*$, by $\mathcal{N}_H$ the set

$$\mathcal{N}_H := \frac{G}{N_G(H)},$$

which we will identify with the set $\{g_i \mid i \in I\}$ made up of a representative for each of the cosets of $N_G(H)$ in G. The conjugacy class of H is exactly the set $\mathcal{N}_H(H) := \{\widetilde{g_i}(H) \mid i \in I\}$, and every conjugate of H appears exactly once in this set. It is easy to see that we can choose the g_i in N, so we can identify the set $\mathcal{N}_H$ with a subset of N.

Consider the index $|G : N_G(H)|$. If $|G : N_G(H)| = 1$, then $G = N_G(H)$, and so $H \trianglelefteq G$, $G = N \times H$, $\mathcal{N}_H(H) = \{H\}$ and, if the group Ψ fulfills property $(*)$, G is an abelian group, according to Theorem 22. Two cases can occur:

(1) $\mathcal{N}_H(H) \subset \mathcal{F}^* \quad (\mathcal{N}_H(H) \neq \mathcal{F}^*)$;

(2) $\mathcal{N}_H(H) = \mathcal{F}^*$.

Let us consider more in detail the case $\mathcal{N}_H(H) = \mathcal{F}^*$. In the following two Corollaries of Theorem 17 we characterize those kinematic spaces with a single conjugacy class in terms of the group Ψ of automorphisms.

32 Corollary. *Let $G = N \rtimes H$ be a group with a partition $\mathcal{F}$ such that $N, H \in \mathcal{F}$ and the property $(*)$ holds for a family Ψ of automorphisms. If we assume, moreover, that $\Psi = \widetilde{N}$, then G is a kinematic space and the lines of the partition $\mathcal{F}$ different from N are all conjugate to H. Vice versa if the partition $\mathcal{F}$ is normal and the set $\mathcal{F}^*$ is made up of lines all conjugate to H, then the group Ψ can be chosen equal to $\widetilde{N}$.*

Proof. We have to check that, for any $g = nh \in G$ and $F \in \mathcal{F}$, the conjugate $\widetilde{g}(F)$ belongs to $\mathcal{F}$. This is obvious if $F = N$. If $F = H$ then

$$\widetilde{g}(H) = nhHh^{-1}n^{-1} = \widetilde{n}(H)$$

and this belongs to $\mathcal{F}$ because $\widetilde{N} = \Psi$. If $F \in \mathcal{F}$ is distinct from N and H, then, by property $(*)$, there exists $n' \in N$ such that $\widetilde{n'}(H) = F$, and so

$$\widetilde{g}(F) = hnn'Hn'^{-1}n^{-1}h^{-1} = n''hHh^{-1}n''^{-1} = \widetilde{n''}(H)$$

which still belongs to $\mathcal{F}$ for the same reason.

Vice versa it is easy to check that the conjugacy class of H is $\widetilde{N}(H)$ and this equals $\mathcal{F}^*$, because $\mathcal{F}$ is normal and all the lines of $\mathcal{F}^*$ belong to the same conjugacy class. Then, for all $F \in \mathcal{F}^*$ there exists $n_F \in N$ such that $F = \widetilde{n_F}(H)$. QED

33 Corollary. *Let $G = N \rtimes H$ be a group with a partition $\mathcal{F}$ such that $N, H \in \mathcal{F}$ and the property $(*)$ holds for a family Ψ of automorphisms. Then Ψ can be chosen equal to $\widetilde{N}$ and such that it acts regularly on the set $\mathcal{F}^*$ if and only if G is a split extension.*

Proof. Assume $\Psi = \widetilde{N}$ regular on $\mathcal{F}^*$. According to [22, Thm. 2] it is enough to prove that the conditions C1) and C2) hold. In particular C1) follows easily from the fact that N is a normal subgroup of G. $\Psi = \widetilde{N}$ implies that all the groups of the partition $\mathcal{F}$ distinct from N are conjugate to H and, moreover, under our assumptions, for any $H' \in \mathcal{F}^*$ there exists exactly one $\widetilde{n} \in \Psi$ such that $\widetilde{n}(H) = H'$.
Vice versa if $G = N \rtimes H$ is a split extension, then $\forall\, n \in N^*$, $nHn^{-1} \neq H$ and

$$\mathcal{F} = \{N\} \cup \{\widetilde{n}(H) \mid n \in N\} = \{N\} \cup \widetilde{N}(H).$$ QED

34 Remark. Note that in Corollary 32 we have shown that if the set $\mathcal{F}^*$ is made up of only one conjugacy class, then a family Ψ which fulfills property $(*)$ always exists and can be chosen equal to $\widetilde{N}$, and so it is a family of collineations.

6 Examples

In this section we collect some examples. The first one is an example of a finite apt-structure with a principal line, and thus of a translation transversal design, with a family Ψ fulfilling property $(*)$ and made up of collineations. Note that, so far, we do not know any example of apt-structures with a principal line which do not fulfill property $(*)$.

35 Example. Let $N = \mathbb{Z}_7 = \{0, 1, \ldots, 6\}$ be the cyclic group of seven elements and $H = \{1, 2, 4\}$ be the group of squares of the Galois field $\mathbb{Z}_7$ different from 0, and consider the action of H on N given by left multiplication. It is easy to show that $G := N \rtimes H$ is a Frobenius group. Let us consider the automorphism $\varphi \in \mathrm{Aut}(N, \cdot)$ given, for all $x \in N$, by $\varphi(x) = 2x$. It is easy to show by direct computations that the family of automorphisms Ψ obtained by extending in a natural way the projections between the line H and the other lines of $\mathcal{F}^*$ by φ is a family of fixed-point-free collineations of the geometric structure.

The next two examples show that in the case of Hughes-Thompson groups and Frobenius groups (finite or infinite), whenever a family Ψ exists, it is made up of collineations.

36 Example. Let $G = H_p \rtimes H$ be a (finite) Hughes-Thompson group and $\mathcal{F}$ the usual partition made up of the group H_p and all the groups of order p which are external to H_p. It is known that this partition is *characteristic*, in the sense that it is fixed by all the automorphisms of the group G, and this shows that any automorphism of the group is, in fact, a collineation.

If we assume that a family Ψ fulfilling property $(*)$ exists, then we can check this directly. For, if F is a group in $\mathcal{F}^*$, $\psi \in \Psi$ and $x \in \psi(F) \cap H_p$, then

$$\psi^{-1}(x) \in F \cap \psi^{-1}(H_p) = F \cap H_p = \{1\},$$

which shows that $x = 1$. $\psi(F)$ is, therefore, a group of order p which is external to H_p, ad so belongs to $\mathcal{F}$. Note also that, in general, in a Hughes-Thompson group not all the subgroups of $\mathcal{F}^*$ are conjugate to each other, thus in general the family Ψ is different from the group of innner automorphisms

37 Example. Let $G = N \rtimes H$ be a Frobenius group (finite or infinite) and $\mathcal{F}$ be the principal Frobenius partition. This partition is characteristic and so, as in the previous example, any automorphism of the group is a collineation.

Again we can easily see this also by direct computations. Let $F \in \mathcal{F}^*$ and $\psi \in \Psi$, $H' \in \mathcal{F}^*$ be such that $H' = \psi(H)$. If $f \in F \setminus \{1\}$, then $\psi(f) \in G \setminus N$, and so there exists $n \in N$ such that $\psi(f) \in \widetilde{n}(H')$, therefore

$$f \in \psi^{-1}\left(\widetilde{n}(H')\right) = \widetilde{\psi^{-1}(n)}(H),$$

and so, from $f \in \widetilde{\psi^{-1}(n)}(H) \cap F$, we can conclude that $F = \widetilde{\psi^{-1}(n)}(H)$. From this it follows easily that

$$\psi(F) = \psi\left(\widetilde{\psi^{-1}(n)}(H)\right) = \widetilde{n}\left(\psi(H)\right) = \widetilde{n}(H'),$$

which shows that $\psi(F)$ is conjugate to H', and so to H, as to say that $\psi(F) \in \mathcal{F}$.

In this case, in particular, the inner automorphisms of G form a family Ψ which fulfill property $(*)$.

Example 38 below is an example of a kinematic space with a principal line in which the lines of the normal partition $\mathcal{F}$ different from the principal line are parted into different conjugacy classes. A family Ψ is showed, which is different from the family of inner automorphisms of the group.

38 Example (see Franchi, [5, Ex. 1])**.** Let p be an odd prime, m, n positive integers such that $n < p$ and write q for p^m. Denote by $\mathbb{F}_q$ the Galois field of order q, by $\mathbb{F}_p$ its prime subfield and by V the vector space $\mathbb{F}_q^n$. Let $a_1, a_2, \ldots, a_m$ be $\mathbb{F}_p$-independent elements of $\mathbb{F}_q$ and define the matrices A_i to be the matrices of $GL(n, q)$ which have ones on the principal diagonal, a_i on the first sub-diagonal and zeroes elsewhere. Denote by H the subgroup $H = \langle A_1, \ldots, A_m \rangle \leqslant GL(n, q)$. In [5] it is proved that each A_i has order p and H is an elementary abelian p-group of order $p^m = q$. Consider now the action of H on V by right multiplication of row vectors and define G to be the semidirect product $V \rtimes H$.

Let now $\underline{w} = (0, 1, \ldots, 1) \in V$ and, for each $x \in \mathbb{F}_q$, define $\underline{w}_x^i := a_1^{-1} a_i x \underline{w}$ and $H_x := \langle \underline{w}_x^1 A_1, \ldots, \underline{w}_x^m A_m \rangle \leqslant G$. In [5] it is proved that, for any $x \in \mathbb{F}_q$, $H_x \cap V = 1$, $H_x \cap H_y^g \neq 1$ implies that $x = y$ and $g \in N_G(H_x)$, and $\mathcal{F} := \{V\} \cup \{H_x^g \mid x \in \mathbb{F}_q,\ g \in G\}$ is a normal partition of G such that its components split into $q + 1$ conjugacy classes.

It is now easy to check that, for any $r \in \mathbb{F}_p$, the map $\psi_r : G \longrightarrow G$ defined by $\psi_r(\underline{v}, A) = (r\underline{v}, A)$ is an automorphism of the group G which maps H to another line of $\mathcal{F}^*$, and so the family $\Psi := \{\psi_r \mid r \in \mathbb{F}_p\}$ fulfills property $(*)$.

Finally, Example 39 is an example of a kinematic space with a principal line built starting from a natural generalization of the 4-dimensional real algebra of the Study quaternions, which in the classical case arises considering the motions of the real euclidean plane. The group associated to this kinematic space is a Frobenius group (finite or infinite) with an abelian kernel and the family Ψ fulfilling property $(*)$ is made up of inner automorphisms.

39 Example. Let $\mathbb{F}$ be a field of characteristic different from 2 and fix $s \in \mathbb{F}$ which is not a square of $\mathbb{F}$. Write $\mathbb{H}_S$ for the four-dimensional vector space over $\mathbb{F}$ with basis $\mathscr{B} = (1, i, \varepsilon, i\varepsilon)$, where i and ε are such that $i^2 = s$, $\varepsilon^2 = 0$ and $i\varepsilon = -\varepsilon i$. Then $\mathbb{H}_S$ is in a natural way an algebra over $\mathbb{F}$ that, in analogy with the case $\mathbb{F} = \mathbb{R}$, we shall call the *Study quaternion algebra* (see [11]). Moreover $M := \mathbb{F}\varepsilon + \mathbb{F}i\varepsilon$ is the only maximal ideal of $\mathbb{H}_S$, which is then a local algebra. It is a straightforward computation to check that $\mathbb{H}_S$ is a *kinematic algebra* or *quadratic algebra*, that is for any $h \in \mathbb{H}_S$, $h^2 \in \mathbb{F}h + \mathbb{F}$, thus, the set $(\mathbb{H}_S \setminus M)/\mathbb{F}^*$ can be made into a kinematic space (see [10]) and, in particular, it is a Frobenius group with an abelian kernel. In this space, that can be identified with the 3-dimensional projective space $\mathbb{H}_S/\mathbb{F}^*$, deprived of the projective line $M/\mathbb{F}^*$, we can define a new incidence structure in the following way: we call line any (affine) plane of the pencil defined by $M/\mathbb{F}^*$ and any (projective) line which does not intersect M/F^*. We can also introduce a new parallelism: any two lines of the pencil defined by $M/\mathbb{F}^*$ are parallel and two projective lines which do not intersect M/F^* are parallel if and only if they differ for the (left) multiplication by an element of $(\mathbb{H}_S \setminus M)/\mathbb{F}^*$. It turns out that the linear space defined in this way is a kinematic space in which the affine planes considered above are principal lines.

A family Ψ of automorphisms fulfilling property $(*)$ is, for example, the group of inner automorphisms of $((\mathbb{H}_S \setminus M)/\mathcal{F}^*, \cdot)$, since any two projective lines through 1 are conjugate.

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