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Isomorphisms between spaces of multilinear mappings or homogeneous polynomials

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Abstract. Let E and F be Banach spaces. Our objective in this work is to find conditions under which, whenever the topological dual spaces E' and F' are isomorphic, the spaces of multilinear mappings (resp. homogeneous polynomials) on E and F are isomorphic as well. We also examine the corresponding problem for the spaces of multilinear mappings (resp. homogeneous polynomials) of a certain type, for instance of finite, nuclear, compact or weakly compact type.

Keywords: Banach space , Polynomials , Isomorphisms.

MSC 2000 classification: primary 46G20, secondary 46G25

Notation

Throughout the whole paper D,E, F and G always denote Banach spaces over the same field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. $\mathbb{N}$ denotes the set of all positive integers. $L(^mE;G)$ denotes the vector space of all continuous m-linear mappings from E^m into G. $L(^mE;G)$ is a Banach space under its natural norm. If $G = \mathbb{K}$, we write $L(^mE;\mathbb{K}) = L(^mE)$. If $m = 1$, we write $L(^1E;G) = L(E;G)$. If $m = 1$ and $G = \mathbb{K}$, we write $L(E) = E'$, the topological dual of E. The mapping

$$I_m : L(^{m+n}E;G) \to L(^mE;L(^nE;G))$$

defined by $I_mA(x)(y) = A(x,y)$ for all $A \in L(^{m+n}E;G), x \in E^m, y \in E^n$, is an isometric isomorphism. Likewise the mapping

$$T^t : A \in L(^mE;L(^nF;G)) \to A^t \in L(^nF;L(^mE;G))$$

defined by $A^t(y)(x) = A(x)(y)$ for all $A \in L(^mE;L(^nF;G)), x \in E^m, y \in F^n$, is an isometric isomorphism. Let $L^s(^mE;G)$ denote the subspace of all $A \in L(^mE;G)$ which are symmetric. Let $\mathcal{P}(^mE;G)$ denote the vector space of all continuous $m-$ homogeneous polynomials from E into G. If $G = \mathbb{K}$, we write

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$\mathcal{P}(^mE;\mathbb{K}) = \mathcal{P}(^mE)$. For each $A \in L(^mE;G)$ let $\widehat{A} \in \mathcal{P}(^mE;G)$ be defined by $\widehat{A}(x) = Ax^m$ for every $x \in E$. The mapping $A \to \widehat{A}$ induces a topological isomorphism between $L^s(^mE;G)$ and $\mathcal{P}(^mE;G)$.

1 Definition. A mapping $A \in L(^mE;G)$ is said to be of finite type if there exist $c_1,\ldots,c_n \in G$, $\varphi_{1i},\ldots,\varphi_{mi} \in E'$, $1 \le i \le n$ such that A can be written in the form: $A(x_1,\ldots,x_m) = \sum_{i=1}^n \varphi_{1i}(x_1)\cdots\varphi_{mi}(x_m)c_i$ for all $(x_1,\ldots,x_m) \in E^m$.

2 Definition. A polynomial $P \in \mathcal{P}(^mE;G)$ is said to be of finite type, if there exist $c_1,\ldots,c_n \in G$, $\varphi_1,\ldots,\varphi_n \in E'$ such that P can be written of the form: $P(x) = \sum_{i=1}^n \varphi_i(x)^m c_i$ for all $x \in E$.

3 Definition. A mapping $A \in L(^mE;G)$ is said to be nuclear, if there exist sequence $(\varphi_{ji})_{i\in\mathbb{N}}$ in E', $1 \le j \le m$ and $(c_i)_{i\in\mathbb{N}}$ in G with $\sum_{i=1}^\infty \|\varphi_{1i}\|\cdots\|\varphi_{mi}\|\|c_i\| < \infty$ such that $A(x_1,\ldots,x_m) = \sum_{i=1}^\infty \varphi_{1i}(x_1)\cdots\varphi_{mi}(x_m)c_i$ for all $(x_1,\ldots,x_m) \in E^m$.

4 Definition. A polynomial $P \in \mathcal{P}(^mE;G)$ is said to be nuclear, if there exist sequences $(\varphi_i)_{i\in\mathbb{N}}$ in E', and $(c_i)_{i\in\mathbb{N}}$ in G with $\sum_{i=1}^\infty \|\varphi_i\|^m\|c_i\| < \infty$ such that $P(x) = \sum_{i=1}^\infty \varphi_i(x)^m c_i$ for all $x \in E$.

Let $L_f(^mE;G)$ denote the space of all $A \in L(^mE;G)$ which are of finite type. Let $\mathcal{P}_f(^mE;G)$ denote the space of all $P \in \mathcal{P}(^mE;G)$ which are of finite type. Let $L_N(^mE;G)$ denote the space of all $A \in L(^mE;G)$ which are nuclear endowed with the nuclear norm $\|A\|_N = \inf\sum_{i=1}^\infty \|\varphi_{1i}\|\cdots\|\varphi_{mi}\|\|c_i\|$, where the infimum is taken over all sequences $(\varphi_{ji})_{i\in\mathbb{N}}$ and $(c_i)_{i\in\mathbb{N}}$ which satisfy the definition. When $G = \mathbb{K}$, we write $L_\Theta(^mE,\mathbb{K}) = L_\Theta(^mE)$, where $\Theta = f$ or N. Let $\mathcal{P}_N(^mE;G)$ denote the space of all $P \in \mathcal{P}(^mE;G)$ which are nuclear, endowed with the nuclear norm $\|P\|_N = \inf\sum_{i=1}^\infty \|\varphi_i\|^m\|c_i\|$, where the infimum is taken over all sequences $(\varphi_i)_{i\in\mathbb{N}}$ and $(c_i)_{i\in\mathbb{N}}$ which satisfy the definition. When $G = \mathbb{K}$, we write $\mathcal{P}_\Theta(^mE,\mathbb{K}) = \mathcal{P}_\Theta(^mE)$, where $\Theta = f$ or N. Let us recall that $A \in L(^mE;G)$ is a compact (resp. weakly compact) mapping if $A(B_{E^m})$ is relatively compact in G (resp. for the weak topology), where B_{E^m} denotes the closed unit ball of E^m. Let $L_K(^mE;G)$ (resp. $L_{WK}(^mE;G)$) denote the space of all $A \in L(^mE;G)$ which are compact (resp. weakly compact). Recall that $P \in \mathcal{P}(^mE;G)$ is a compact (resp. weakly compact) polynomial if $P(B_E)$ is relatively compact in G (resp. for the weak topology), where B_E denotes the closed unit ball of E. Let $\mathcal{P}_K(^mE;G)$ (resp. $\mathcal{P}_{WK}(^mE;G)$) denote the space of all compact (resp. weakly compact) homogeneous polynomials from E into G. We observe that in the case where $G = \mathbb{K}$, all the continuous homogeneous polynomials are compact (resp. weakly compact). For background information on multilinear mappings and homogeneous polynomials we refer to the books [7] and [12].

1 Isomorphisms between spaces of multilinear mappings or homogeneous polynomials

In this work, we use the Nicodemi sequences defined in [8] to prove all the theorems. We recall the definition of the Nicodemi sequences.

5 Definition. [8] Given a continuous linear operator $R_1 : L(E;G) \longrightarrow L(F;G)$, let $R_m : L(^mE;G) \longrightarrow L(^mF;G)$ be inductively defined by $R_{m+1}A = I_m^{-1}[R_m \circ (R_1 \circ I_m(A))^t]^t$ for all $A \in L(^{m+1}E;G)$ and $m \in \mathbb{N}$.

6 Example. [8, Example 1.2] Let $T_1 : E' \hookrightarrow E'''$ be the natural embedding and let $T_m : L(^mE) \to L(^mE'')$ be the Nicodemi sequence of operators beginning with T_1. This sequence is precisely the sequence of operators constructed by Aron and Berner in [1, Proposition 2.1].

7 Definition. [8, Proposition 4.1] Given $R_1 \in L(E';F')$, let $\widetilde{R_1} \in L(L(E;G'),L(F;G'))$ be defined by $\widetilde{R_1}A(y)(z) = R_1(\delta_z \circ A)(y)$ for all $A \in L(E;G')$, $y \in F$ *and* $z \in G$, where $\delta_z : G' \to \mathbb{K}$ is defined by $\delta_z(z') = z'(z)$ for all $z' \in G'$. If (R_m) and $(\widetilde{R_m})$ are the corresponding Nicodemi sequences, then $\widetilde{R_m}A(y)(z) = R_m(\delta_z \circ A)(y)$ for all $A \in L(^mE;G')$, $y \in F^m$ *and* $z \in G$.

8 Example. Let $T_m : L(^mE) \longrightarrow L(^mE'')$ be the Nicodemi sequence of operators beginning with the natural embedding $T_1 : E' \hookrightarrow E'''$, and let $\widetilde{T_m} : L(^mE;G') \longrightarrow L(^mE'';G')$ be the corresponding sequence for vector-valued multilinear mappings (See [8, Proposition 4.1]). We observe that as G' is a $\mathcal{C}_1$−space, the sequence $\widetilde{T_m} : L(^mE;G') \longrightarrow L(^mE'';G') \subset L(^mE'';G''')$ coincides with the sequence of operators constructed by Aron and Berner in [1, Proposition 2.1].

The next theorem shows the relationship between an arbitrary Nicodemi sequence of scalar-valued mappings and the Nicodemi sequence beginning with the natural embedding $E' \hookrightarrow E'''$.

9 Theorem. *Let $R_m : L(^mE) \longrightarrow L(^mF)$ be a Nicodemi sequence, let $T_m : L(^mE) \longrightarrow L(^mE'')$ be the Nicodemi sequence beginning with the natural embedding $T_1 = J_{E'} : E' \hookrightarrow E'''$, and let $J_F : F \hookrightarrow F''$ be the natural embedding. Then*

$$R_mA(y_1,\ldots,y_m) = T_mA(R_1'(J_Fy_1),\ldots,R_1'(J_Fy_m))$$

for all $A \in L(^mE)$, and $y_1,\ldots,y_m \in F$, where R_1' is the transpose of R_1.

Proof. We will prove this theorem by induction on m. If $A \in L(E) = E'$, we have

$$R_1A(y) = \langle J_Fy, R_1A\rangle = \langle R_1'(J_Fy), A\rangle = \langle T_1A, R_1'(J_Fy)\rangle = T_1A(R_1'(J_Fy))$$

for all $y \in F$. Now let us assume that the identity in Theorem 9 is true for $m-$ linear forms. Let $A \in L(^{m+1}E)$. We will prove that

$$T_{m+1}A(z_1,\dots,z_m,R_1'(J_Fy)) = T_m[(R_1 \circ I_mA)^t(y)](z_1,\dots,z_m) \tag{1}$$

for all $z_1,\dots,z_m \in E''$ and $y \in F$. From the definition of Nicodemi sequences, it follows that

$$T_{m+1}A(z_1,\dots,z_m,R_1'(J_Fy)) = T_m[(T_1 \circ I_mA)^t(R_1'(J_Fy))](z_1,\dots,z_m). \tag{2}$$

Therefore, to get (1) comparing with (2), it is enough to prove that

$$(R_1 \circ I_mA)^t(y) = (T_1 \circ I_mA)^t(R_1'(J_Fy)).$$

In fact,

$$\begin{aligned}(T_1 \circ I_mA)^t(R_1'(J_Fy))(x_1,\dots,x_m) &= T_1[I_mA(x_1,\dots,x_m)](R_1'(J_Fy))\\ &= \langle R_1'(J_Fy), I_mA(x_1,\dots,x_m)\rangle\\ &= \langle J_Fy, R_1[I_mA(x_1,\dots,x_m)]\rangle\\ &= \langle R_1[I_mA(x_1,\dots,x_m)], y\rangle\\ &= R_1[I_mA(x_1,\dots,x_m)](y)\\ &= (R_1 \circ I_mA)^t(y)(x_1,\dots,x_m)\end{aligned}$$

for all $x_1,\dots,x_m \in E$. Thus by the induction hypothesis and (1) it follows that for all $y_1,\dots,y_{m+1} \in F$

$$\begin{aligned}R_{m+1}A(y_1,\dots,y_{m+1}) =& R_m[(R_1 \circ I_mA)^t(y_{m+1})](y_1,\dots,y_m)\\ =& T_m[(R_1 \circ I_mA)^t(y_{m+1})](R_1'(J_Fy_1),\dots,R_1'(J_Fy_m))\\ =& T_{m+1}A(R_1'(J_Fy_1),\dots,R_1'(J_Fy_m),R_1'(J_Fy_{m+1})).\end{aligned}$$

QED

We next extend Theorem 9 to the case of a Nicodemi sequence of vector-valued mappings. Let us recall that each Nicodemi sequence (R_m) for scalar-valued mappings yields a Nicodemi sequence $(\widetilde{R_m})$ for vector - valued mappings.

10 Theorem. *Let $R_m : L(^mE) \longrightarrow L(^mF)$ be a Nicodemi sequence, and let $T_m : L(^mE) \longrightarrow L(^mE'')$ be the Nicodemi sequence beginning with the natural embedding $T_1 = J_{E'} : E' \hookrightarrow E'''$. Let*

$$\widetilde{R_m} : L(^mE;G') \to L(^mF;G')$$

and

$$\widetilde{T_m} : L(^m E; G') \to L(^m E''; G')$$

be the corresponding Nicodemi sequences for vector-valued multilinear mappings. Then

$$\widetilde{R_m} A(y_1, \ldots, y_m) = \widetilde{T_m} A(R_1'(J_F y_1), \ldots, R_1'(J_F y_m))$$

for all $A \in L(^m E; G')$*, and* $y_1, \ldots, y_m \in F$*, where* R_1' *is the transpose of* R_1*.*

Proof. We will prove this theorem by induction on m. If $A \in L(E; G')$, it follows that

$$\begin{aligned}
\widetilde{R_1} A(y)(z) =& R_1(\delta_z \circ A)(y) \\
=& \langle J_F y, R_1(\delta_z \circ A) \rangle \\
=& \langle R_1'(J_F y), (\delta_z \circ A) \rangle \\
=& \langle T_1(\delta_z \circ A), R_1'(J_F y) \rangle \\
=& \widetilde{T_1} A(R_1'(J_F y))(z)
\end{aligned}$$

for all $y \in F$ and $z \in G$. Now let us assume that the identity is true for $m-$linear forms. Let $A \in L(^{m+1} E; G')$. We prove initially that

$$\widetilde{T_{m+1}} A(z_1, \ldots, z_m, R_1'(J_F y)) = \widetilde{T_m}[(\widetilde{R_1} \circ I_m A)^t(y)](z_1, \ldots, z_m) \tag{3}$$

for all $z_1, \ldots, z_m \in E''$ and $y \in F$. From the definition of Nicodemi sequences, it follows that

$$\widetilde{T_{m+1}} A(z_1, \ldots, z_m, R_1'(J_F y)) = \widetilde{T_m}[(\widetilde{T_1} \circ I_m A)^t(R_1'(J_F y))](z_1, \ldots, z_m). \tag{4}$$

Therefore, to get (3) comparing with (4), it is enough to prove that

$$(\widetilde{R_1} \circ I_m A)^t(y) = (\widetilde{T_1} \circ I_m A)^t(R_1'(J_F y)).$$

In fact,

$$\begin{aligned}
(\widetilde{T_1} \circ I_m A)^t(R_1'(J_F y))(x_1, \ldots, x_m)(z) =& \widetilde{T_1}[I_m A(x_1, \ldots, x_m)](R_1'(J_F y))(z) \\
=& T_1[\delta_z \circ I_m A(x_1, \ldots, x_m)](R_1'(J_F y)) \\
=& \langle R_1'(J_F y), \delta_z \circ I_m A(x_1, \ldots, x_m) \rangle \\
=& \langle J_F y, R_1[\delta_z \circ I_m A(x_1, \ldots, x_m)] \rangle \\
=& \langle R_1[\delta_z \circ I_m A(x_1, \ldots, x_m)], y \rangle \\
=& R_1[\delta_z \circ I_m A(x_1, \ldots, x_m)](y) \\
=& (\widetilde{R_1} \circ I_m A)^t(y)(x_1, \ldots, x_m)(z)
\end{aligned}$$

for all $x_1, \dots, x_m \in E$ and $z \in G$. Thus, from the induction hypothesis and (3), it follows that for all $y_1, \dots, y_{m+1} \in F$

$$\begin{aligned}\widetilde{R_{m+1}}A(y_1, \dots, y_{m+1}) =& \widetilde{R_m}[(\widetilde{R_1} \circ I_m A)^t(y_{m+1})](y_1, \dots, y_m)\\ =& \widetilde{T_m}[(\widetilde{R_1} \circ I_m A)^t(y_{m+1})](R_1'(J_F y_1), \dots, R_1'(J_F y_m))\\ =& \widetilde{T_{m+1}}A(R_1'(J_F y_1), \dots, R_1'(J_F y_m), R_1'(J_F y_{m+1})).\end{aligned}$$

QED

Recall that a Banach space E is said to be Arens - regular if all linear operator $E \longrightarrow E'$ are weakly compact, and symmetrically Arens - regular if this is so for all symmetric linear operators. An operator $T : E \to E'$ is said to be symmetric if $Tx(y) = Ty(x)$ for all $x, y \in E$(see [3] and [9]). Let us recall that if E is symmetrically Arens - regular, then $T_m A \in L^s(^m E'')$ for all $A \in L^s(^m E)$, where (T_m) is the Nicodemi sequence beginning with the natural embedding $T_1 = J_{E'} : E' \hookrightarrow E'''$ (see [2, Theorem 8.3]). Now, we are ready to study the theorems of preservation of symmetric multilinear mappings.

11 Theorem. *Let $R_m : L(^m E) \longrightarrow L(^m F)$ be a Nicodemi sequence and let $T_m : L(^m E) \longrightarrow L(^m E'')$ be the Nicodemi sequence beginning with the natural embedding $T_1 = J_{E'} : E' \hookrightarrow E'''$. If $T_m A$ is symmetric, then $R_m A$ is also symmetric. In particular, if E is symmetrically Arens - regular, then $R_m A \in L^s(^m F)$ for all $A \in L^s(^m E)$.*

Proof. By Theorem 9 we have that

$$R_m A(y_1, \dots, y_m) = T_m A(R_1'(J_F y_1), \dots, R_1'(J_F y_m)).$$

Thus, if $T_m A$ is symmetric, then $R_m A$ is also symmetric. Now if E is symmetrically Arens - regular, we have that the Aron - Berner extension $T_m A$ is symmetric for all $A \in L^s(^m E)$(See [2, Proposition 8.3]). We conclude that $R_m A \in L^s(^m F)$ for all $A \in L^s(^m E)$. QED

12 Theorem. *Let $R_m : L(^m E) \longrightarrow L(^m F)$ be a Nicodemi sequence and let $\widetilde{R_m} : L(^m E; G') \longrightarrow L(^m F; G')$ be the corresponding Nicodemi sequence for vector-valued multilinear mappings. If E is symmetrically Arens - regular, then $\widetilde{R_m} A \in L^s(^m F; G')$ for all $A \in L^s(^m E; G')$.*

Proof. By [8, Proposition 4.1], we have that

$$\widetilde{R_m}A(y)(z) = R_m(\delta_z \circ A)(y) \tag{5}$$

for all $A \in L(^m E, G')$, $y \in F^m$ *and* $z \in G$. The identity (5) and Theorem 11 imply that if A is symmetric, then $\widetilde{R_m}A$ is also symmetric. QED

In Theorem 13 we will denote $J_F(y)$ by y for all $y \in F$ and $J_E(x)$ by x for all $x \in E$.

13 Theorem. *Let $R_m : L(^mE) \longrightarrow L(^mF)$ be a Nicodemi sequence, let $T_m : L(^mE) \longrightarrow L(^mE'')$ be the Nicodemi sequence beginning with the natural embedding $T_1 = J_{E'} : E' \hookrightarrow E'''$, and let $Q_m : L(^mF) \longrightarrow L(^mF'')$ be the Nicodemi sequence beginning with the natural embedding $Q_1 = J_{F'} : F' \hookrightarrow F'''$. If T_mA is symmetric, then*

$$Q_m \circ R_mA(w_1, \dots, w_m) = T_mA(R_1'w_1, \dots, R_1'w_m)$$

for all $A \in L(^mE)$, $w_j \in F'', j = 1, \dots, m$, and $m \in \mathbb{N}$.

Proof. We will prove by induction on $k \in \mathbb{N}$ that

$$Q_m \circ R_mA(w_1, \dots, w_k, y_{k+1}, \dots, y_m) = \\ T_mA(R_1'(w_1), \dots, R_1'(w_k), R_1'(y_{k+1}), \dots, R_1'(y_m))$$

for all $y_j \in F$ and $w_j \in F''$. Recall that (i) Q_mB and T_mA are weak* continuous in its first variable for all $B \in L(^mF)$ and $A \in L(^mE)$ by [8, Proposition 5.1]; (ii) elements of F'' and of $J_F(F)$ commute in variables of Q_mB by [8, Lemma 3.4]; (iii) R_1' is $\sigma(F'', F') - \sigma(E'', E')$ continuous; (iv) by [8, Proposition 2.1] , we have that $T_mA(x_1, \dots, x_m) = A(x_1, \dots, x_m)$ for all $A \in L(^mE)$ and $x_1, \dots, x_m \in E$; and $Q_mB(y_1, \dots, y_m) = B(y_1, \dots, y_m)$ for all $B \in L(^mF)$ and $y_1, \dots, y_m \in F$. If $k = 1$, by Goldstine´s theorem, there is a net $(y_\alpha) \subset F$ such that $y_\alpha \longrightarrow w_1$ for the topology $\sigma(F'', F')$. Then by Theorem 9

$$\begin{aligned} Q_m \circ R_mA(w_1, y_2 \dots, y_m) &= Q_m(R_mA)(w_1, y_2 \dots, y_m) \\ &= \lim_\alpha Q_m(R_mA)(y_\alpha, y_2 \dots, y_m) \\ &= \lim_\alpha (R_mA)(y_\alpha, y_2 \dots, y_m) \\ &= \lim_\alpha T_mA(R_1'y_\alpha, R_1'y_2, \dots, R_1'y_m) \\ &= T_mA(R_1'w_1, R_1'y_2, \dots, R_1'y_m). \end{aligned}$$

Now assuming that the identity holds for k, we will prove that the identity holds for $k+1$. By the induction hypothesis and T_mA being symmetric, it follows that

$$\begin{aligned} &Q_m \circ R_mA(w_1, \dots, w_{k+1}, y_{k+2}, \dots, y_m) \\ &\quad = Q_m(R_mA)(w_1, \dots, w_{k+1}, y_{k+2}, \dots, y_m) \\ &\quad = \lim_\alpha Q_m(R_mA)(y_\alpha, w_2, \dots, w_{k+1}, y_{k+2}, \dots, y_m) \\ &\quad = \lim_\alpha Q_m(R_mA)(w_2, \dots, w_{k+1}, y_\alpha, y_{k+2}, \dots, y_m) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\alpha} T_m A(R_1' w_2, \ldots, R_1' w_{k+1}, R_1' y_\alpha, R_1' y_{k+2}, \ldots, R_1' y_m)\\
&= \lim_{\alpha} T_m A(R_1' y_\alpha, R_1' w_2, \ldots, R_1' w_{k+1}, R_1' y_{k+2}, \ldots, R_1' y_m)\\
&= T_m A(R_1' w_1, \ldots, R_1' w_{k+1}, R_1' y_{k+2}, \ldots, R_1' y_m).
\end{aligned}
$$

QED

Next we will prove that each isomorphism between E' and F' induces an isomorphism between $L(^mE;G')$ and $L(^mF;G')$ for all $m \in \mathbb{N}$. If E and F are symmetrically Arens - regular, then each isomorphism between E' and F' induces an isomorphism between $\mathcal{P}(^mE;G')$ and $\mathcal{P}(^mF;G')$ for all $m \in \mathbb{N}$.

Given a continuous linear operator

$$R_m : L(^mE;G) \longrightarrow L(^mF;G),$$

we define

$$U_m : A \in L(^nD; L(^mE;G)) \longrightarrow R_m \circ A \in L(^nD; L(^mF;G)).$$

We observe that if R_m is an isomorphism then U_m is also an isomorphism, whose inverse

$$U_m^{-1} : L(^nD; L(^mF;G)) \longrightarrow L(^nD; L(^mE;G))$$

is defined by $U_m^{-1}(B) = R_m^{-1} \circ B$ for all $B \in L(^nD; L(^mF;G))$ where R_m^{-1} is the inverse of R_m. Thus, with the previous notations, it is possible to rewrite the definition of the Nicodemi operators in the following way:

14 Lemma. *Given an operator*

$$R_m : L(^mE;G) \longrightarrow L(^mF;G),$$

the operator

$$R_{m+1} : L(^{m+1}E;G) \longrightarrow L(^{m+1}F;G)$$

is given by

$$R_{m+1}(A) = I_m^{-1}[R_m \circ (R_1 \circ I_m(A))^t]^t = I_m^{-1} \circ T^t \circ U_m \circ T^t \circ U_1 \circ I_m(A)$$

for all $A \in L(^{m+1}E;G)$.

The following theorem was obtained in [4]. But we will need our proof in the proof of the subsequent theorem.

15 Theorem. *If E' and F' are isomorphic, then $L(^mE)$ and $L(^mF)$ are isomorphic for all $m \in \mathbb{N}$.*

Proof. Since E' and F' are isomorphic, there exists an isomorphism $R_1 : E' \longrightarrow F'$. Let $R_m : L(^mE) \longrightarrow L(^mF)$ be the Nicodemi sequence beginning with R_1. We will prove by induction on $m \in \mathbb{N}$ that R_m is an isomorphism between $L(^mE)$ and $L(^mF)$ for all $m \in \mathbb{N}$. By hypothesis $R_1 : E' \longrightarrow F'$ is an isomorphism. Assuming that R_1 and R_m are isomorphisms, we show that R_{m+1} is also an isomorphism. In fact, by Lemma 1 it is possible to rewrite

$$R_{m+1} = I_m^{-1} \circ T^t \circ U_m \circ T^t \circ U_1 \circ I_m.$$

Since R_1 and R_m are isomorphisms, we have that U_1 and U_m are also isomorphisms. Thus R_{m+1} is an isomorphism between $L(^{m+1}E)$ and $L(^{m+1}F)$ being a composite of isomorphisms. Therefore $L(^mE)$ and $L(^mF)$ are isomorphic for all $m \in \mathbb{N}$. QED

16 Theorem. *If E' and F' are isomorphic, then $L(^mE; G')$ and $L(^mF; G')$ are isomorphic for all $m \in \mathbb{N}$.*

Proof. Since E' and F' are isomorphic, there exists an isomorphism $R_1 : E' \longrightarrow F'$. Let $R_m : L(^mE) \longrightarrow L(^mF)$ be the Nicodemi sequence beginning with R_1. Let

$$\widetilde{R_m} : L(^mE, G') \longrightarrow L(^mF, G')$$

be the corresponding Nicodemi sequence for vector-valued multilinear mappings. We observe that

$$(\delta_z \circ \widetilde{R_m}A) = R_m(\delta_z \circ A) \tag{6}$$

for all $z \in G$ and $A \in L(^mE; G')$. In fact, by [8, Proposition 4.1],

$$(\delta_z \circ \widetilde{R_m}A)(y) = \widetilde{R_m}A(y)(z) = R_m(\delta_z \circ A)(y)$$

for all $y \in F^m$. It follows from the proof of Theorem 15 that R_m is an isomorphism for all $m \in \mathbb{N}$. Let $S_m : L(^mF) \longrightarrow L(^mE)$ denote the inverse of R_m for all $m \in \mathbb{N}$. We define

$$\widetilde{S_m} : L(^mF; G') \longrightarrow L(^mE; G')$$

by $\widetilde{S_m}B(x)(z) = S_m(\delta_z \circ B)(x)$ for all $B \in L(^mF; G')$, $x \in E^m$, $z \in G$ and $m \in \mathbb{N}$. Thus $\widetilde{S_m}$ is linear and continuous. We will show that $\widetilde{R_m}$ is an isomorphism between $L({}^mE;G')$ and $L({}^mF;G')$ for all $m \in \mathbb{N}$. In fact, we have that by (6)

$$\begin{aligned}\widetilde{S_m} \circ \widetilde{R_m}A(x)(z) =& \widetilde{S_m}(\widetilde{R_m}A)(x)(z)\\ =& S_m(\delta_z \circ \widetilde{R_m}A)(x)\\ =& S_m(R_m(\delta_z \circ A))(x)\\ =& [S_m \circ R_m(\delta_z \circ A)](x)\\ =& (\delta_z \circ A)(x)\\ =& A(x)(z)\end{aligned}$$

for all $A \in L({}^mE, G'), x \in E^m$ and $z \in G$. In a similar way, we can get that $(\widetilde{R_m} \circ \widetilde{S_m})B = B$ for all $B \in L({}^mF; G')$. QED

17 Theorem. *If E and F are symmetrically Arens - regular, and E' and F' are isomorphic, then $L^s({}^mE)$ and $L^s({}^mF)$ are isomorphic for all $m \in \mathbb{N}$.*

Proof. We write $J_E(x) = x$ for all $x \in E$. Recall that by [8, Proposition 2.1] $T_mA(x_1, \ldots, x_m) = A(x_1, \ldots, x_m)$ for all $A \in L({}^mE)$ and $x_1, \ldots, x_m \in E$. Since E' and F' are isomorphic, there exists an isomorphism $R_1 : E' \longrightarrow F'$. Let $R_m : L({}^mE) \longrightarrow L({}^mF)$ be the Nicodemi sequence beginning with R_1. Let $S_1 = R_1^{-1} : F' \longrightarrow E'$ be the inverse of R_1 and let $S_m : L({}^mF) \longrightarrow L({}^mE)$ be the Nicodemi sequence beginning with S_1. Since F is symmetrically Arens - regular, we have by Theorem 11 that $S_m(L^s({}^mF)) \subset L^s({}^mE)$. By Theorem 9 we have that $S_mB(x_1, \ldots, x_m) = Q_mB(S_1'x_1, \ldots, S_1'x_m)$ for all $B \in L({}^mF)$, and $x_1, \ldots, x_m \in E$, where S_1' is the transpose of S_1. In particular, we have that

$$S_m(R_mA)(x_1, \ldots, x_m) = Q_m(R_mA)(S_1'x_1, \ldots, S_1'x_m) \tag{7}$$

for all $A \in L({}^mE)$. On the other hand, since E is symmetrically Arens - regular, it follows from Theorem 11 that $R_m(L^s({}^mE)) \subset L^s({}^mF)$. Moreover, by Theorem 13 we have that

$$Q_m(R_mA)(S_1'x_1, \ldots, S_1'x_m) = T_mA(R_1'(S_1'x_1), \ldots, R_1'(S_1'x_m)), \tag{8}$$

for all $A \in L^s({}^mE)$. Therefore we conclude by (7) and (8) that

$$\begin{aligned}S_m(R_mA)(x_1, \ldots, x_m) =& Q_m(R_mA)(S_1'x_1, \ldots, S_1'x_m)\\ =& T_mA(R_1'(S_1'x_1), \ldots, R_1'(S_1'x_m))\\ =& T_mA(x_1, \ldots, x_m)\\ =& A(x_1, \ldots, x_m),\end{aligned}$$

for all $A \in L^s({}^mE)$, that is

$$(S_m \circ R_m)A = A \tag{9}$$

for all $A \in L^s(^mE)$. In an analogous way, we can prove that $(R_m \circ S_m)B = B$ for all $B \in L^s(^mF)$. QED

The following Corollary 18 was proven by Lassalle - Zalduendo in [11] and by F. Cabello Sánchez, J. Castillo and R. García in [4] by a different method.

18 Corollary. *If E and F are symmetrically Arens - regular, and E' and F' are isomorphic, then $\mathcal{P}(^mE)$ and $\mathcal{P}(^mF)$ are isomorphic for all $m \in \mathbb{N}$.*

19 Theorem. *If E and F are symmetrically Arens - regular, and E' and F' are isomorphic, then $L^s(^mE;G')$ and $L^s(^mF;G')$ are isomorphic for all $m \in \mathbb{N}$.*

Proof. Since E' and F' are isomorphic, there exists an isomorphism $R_1 : E' \longrightarrow F'$. Let $R_m : L(^mE) \longrightarrow L(^mF)$ be the Nicodemi sequence beginning with R_1. Let $S_1 = R_1^{-1} : F' \longrightarrow E'$ be the inverse of R_1, and let $S_m : L(^mF) \longrightarrow L(^mE)$ be the Nicodemi sequence beginning with S_1. Let $\widetilde{R_m} : L(^mE;G') \longrightarrow L(^mF;G')$ and $\widetilde{S_m} : L(^mF;G') \longrightarrow L(^mE;G')$ be the corresponding Nicodemi sequence for vector-valued multilinear mappings. By Theorem 12 we have that

$$\widetilde{R_m}(L^s(^mE;G')) \subset L^s(^mF;G'). \tag{10}$$

and

$$\widetilde{S_m}(L^s(^mF;G')) \subset L^s(^mE;G'). \tag{11}$$

It follows from the proof of Theorem 16 that $\widetilde{R_m}$ is an isomorphism between $L(^mE;G')$ and $L(^mF;G')$ such that $\widetilde{R_m}^{-1} = \widetilde{S_m}$. By (10) and (11), we have that $\widetilde{R_m}|_{L^s(^mE;G')}$ is an isomorphism between $L^s(^mE;G')$ and $L^s(^mF;G')$. QED

The following Corollary 20 was proven for Carando - Lassalle in [5] by a different method.

20 Corollary. *If E and F are symmetrically Arens - regular, and E' and F' are isomorphic, then $\mathcal{P}(^mE;G')$ and $\mathcal{P}(^mF;G')$ are isomorphic for all $m \in \mathbb{N}$.*

2 Isomorphisms between spaces of nuclear multilinear mappings or homogeneous polynomials.

We will use the following notation:

$$(\varphi_1 \otimes \varphi_2 \otimes c)(x_1, x_2) = \varphi_1(x_1)\varphi_2(x_2)c$$

for all $\varphi_1, \varphi_2 \in E', c \in F$ and $x_1, x_2 \in E$. The next lemma comes essentially from Aron and Berner [1].

21 Lemma. *[1, Proposition 2.2] Let* $T_m : L(^mE) \longrightarrow L(^mE'')$ *be the Nicodemi sequence beginning with the natural embedding* $T_1 = J_{E'} : E' \hookrightarrow E'''$. *Let* $A \in L_f(^mE)$ *and let* $c_1, \ldots, c_n \in \mathbb{K}$, $\varphi_{1i}, \ldots, \varphi_{mi} \in E', 1 \leq i \leq n$ *such that*

$$A(x_1, \ldots, x_m) = \sum_{i=1}^{n} \varphi_{1i}(x_1) \cdots \varphi_{mi}(x_m) c_i$$

for all $(x_1, \ldots, x_m) \in E^m$, *then*

$$T_m(A)(x_1'', \ldots, x_m'') = \sum_{i=1}^{n} (T_1\varphi_{1i})(x_1'') \cdots (T_1\varphi_{mi})(x_m'') c_i$$

for all $(x_1'', \ldots, x_m'') \in (E'')^m$. *In particular* $T_m(A) \in L_f(^mE'')$.

22 Lemma. *Let* $T_m : L(^mE) \longrightarrow L(^mE'')$ *be the Nicodemi sequence beginning with the natural embedding* $T_1 = J_{E'} : E' \hookrightarrow E'''$. *Let* $A \in L_N(^mE)$ *and let* $(\varphi_{ji})_{i\in\mathbb{N}}$ *in* E', $1 \leq j \leq m$ *and* $(c_i)_{i\in\mathbb{N}}$ *in* $\mathbb{K}$ *with*

$$\sum_{i=1}^{\infty} \|\varphi_{1i}\| \cdots \|\varphi_{mi}\| \|c_i\| < \infty$$

such that

$$A(x_1, \ldots, x_m) = \sum_{i=1}^{\infty} \varphi_{1i}(x_1) \cdots \varphi_{mi}(x_m) c_i$$

for all $(x_1, \ldots, x_m) \in E^m$. *Then* $T_m(A)$ *has the following form:*

$$T_m(A)(x_1'', \ldots, x_m'') = \sum_{i=1}^{\infty} (T_1\varphi_{1i})(x_1'') \cdots (T_1\varphi_{mi})(x_m'') c_i$$

for all $(x_1'', \ldots, x_m'') \in (E'')^m$. *In particular* $T_m(A) \in L_N(^mE'')$.

Proof. Let $A_n = \sum_{i=1}^{n} \varphi_{1i} \otimes \cdots \otimes \varphi_{mi} \otimes c_i$ for all $n \in \mathbb{N}$. Then $A_n \in L_f(^mE)$ and $A_n \to A$ for the nuclear norm, that is

$$\|A_n - A\|_N \longrightarrow 0. \tag{12}$$

By Lemma 21 $T_m A_n$ has the following form: $T_m A_n = \sum_{i=1}^{n} T_1\varphi_{1i} \otimes \cdots \otimes T_1\varphi_{mi} \otimes c_i$ for all $n \in \mathbb{N}$. It follows that $T_m A_n \longrightarrow \sum_{i=1}^{\infty} T_1\varphi_{1i} \otimes \cdots \otimes T_1\varphi_{mi} \otimes c_i$ for the nuclear norm. In particular

$$T_m A_n \longrightarrow \sum_{i=1}^{\infty} T_1\varphi_{1i} \otimes \cdots \otimes T_1\varphi_{mi} \otimes c_i \tag{13}$$

pointwise. On the other hand, we have that

$$T_m A_n \to T_m A \tag{14}$$

pointwise. Then by (12)

$$\begin{aligned}\|T_m A(x_1'', \dots, x_m'') - T_m A_n(x_1'', \dots, x_m'')\| \\ &= \|(T_m A - T_m A_n)(x_1'', \dots, x_m'')\| \\ &= \|T_m(A - A_n)(x_1'', \dots, x_m'')\| \\ &\leq \|T_m\| \|A - A_n\| \|(x_1'', \dots, x_m'')\| \\ &\leq \|T_m\| \|A - A_n\|_N \|(x_1'', \dots, x_m'')\| \longrightarrow 0\end{aligned}$$

for all $x_1'', \dots, x_m'' \in E''$. By (13) and (14), we have that

$$T_m(A)(x_1'', \dots, x_m'') = \sum_{i=1}^{\infty} (T_1\varphi_{1i})(x_1'') \cdots (T_1\varphi_{mi})(x_m'')c_i$$

for all $x_1'', \dots, x_m'' \in E''$ and therefore $T_m(A) \in L_N(^m E'')$. QED

Next we will see that the operators from the Nicodemi sequence preserve multilinear mappings of finite type and nuclear multilinear mappings. We consider first the case of scalar-valued multilinear mappings.

23 Theorem. *Let $R_m : L(^m E) \longrightarrow L(^m F)$ be a Nicodemi sequence. Then $R_m A \in L_\Theta(^m F)$ for all $A \in L_\Theta(^m E)$, where $\Theta = f$ or N.*

Proof. We will write the proof in detail in the case $\Theta = N$. In the case $\Theta = f$, the proof is similar. Given $A \in L_N(^m E)$, there exist sequences $(\varphi_{ji})_{i \in \mathbb{N}}$ in E', $1 \leq j \leq m$, and $(c_i)_{i \in \mathbb{N}}$ in $\mathbb{K}$, with $\sum_{i=1}^{\infty} \|\varphi_{1i}\| \cdots \|\varphi_{mi}\| |c_i| < \infty$ such that $A(x_1, \dots, x_m) = \sum_{i=1}^{\infty} \varphi_{1i}(x_1) \cdots \varphi_{mi}(x_m) c_i$ for all $(x_1, \dots, x_m) \in E^m$. Let $T_m : L(^m E) \longrightarrow L(^m E'')$ be the Nicodemi sequence beginning with the natural embedding $T_1 = J_{E'} : E' \hookrightarrow E'''$. By Theorem 9 we have that

$$R_m A(y_1, \dots, y_m) = T_m A(R_1'(J_F y_1), \dots, R_1'(J_F y_m))$$

for all $A \in L(^m E)$, and $y_1, \dots, y_m \in F$, where R_1' is the transpose of R_1 and by Lemma 22 we have that

$$T_m(A)(x_1'', \dots, x_m'') = \sum_{i=1}^{\infty} (T_1\varphi_{1i})(x_1'') \cdots (T_1\varphi_{mi})(x_m'')c_i$$

for all $(x_1'', \dots, x_m'') \in (E'')^m$. Therefore

$$\begin{aligned} R_mA(y_1,\dots,y_m) =& T_mA(R'_1(J_Fy_1),\dots,R'_1(J_Fy_m)) \\ =& \sum_{i=1}^{\infty}(T_1\varphi_{1i})(R'_1J_Fy_1)\cdots(T_1\varphi_{mi})(R'_1J_Fy_m)c_i \\ =& \sum_{i=1}^{\infty}R_1\varphi_{1i}(y_1)\cdots R_1\varphi_{mi}(y_m)c_i \end{aligned}$$

for all $y_1,\dots,y_m \in F$ because $< T_1\varphi, R'_1(J_Fy) >=< J_{E'}\varphi, R'_1(J_Fy) >=$ $< R'_1(J_Fy),\varphi >=< J_Fy, R_1\varphi >=< R_1\varphi, y >$, for all $\varphi \in E'$ and all $y \in F$. Therefore

$$R_mA = \sum_{i=1}^{\infty} R_1\varphi_{1i}\otimes\cdots\otimes R_1\varphi_{mi}\otimes c_i \tag{15}$$

and then $R_mA \in L_N(^mF)$. QED

The next theorem considers the case of vector-valued multilinear mappings.

24 Theorem. *Let $R_m : L(^mE) \longrightarrow L(^mF)$ be a Nicodemi sequence and let $\widetilde{R_m} : L(^mE;G') \longrightarrow L(^mF;G')$ be the corresponding sequence for vector-valued multilinear mappings. Then $\widetilde{R_m}A \in L_\Theta(^mF;G')$ for all $A \in L_\Theta(^mE;G')$ where $\Theta = f$ or N.*

Proof. We will only write the proof in the case $\Theta = N$. In the case $\Theta = f$, the proof is similar. Given $A \in L_N(^mE;G')$ there exist sequences $(\varphi_{ji})_{i\in\mathbb{N}}$ in E', $1 \le j \le m$, and $(c_i)_{i\in\mathbb{N}}$ in G', with $\sum_{i=1}^{\infty}\|\varphi_{1i}\|\cdots\|\varphi_{mi}\|\|c_i\| < \infty$ such that $A = \sum_{i=1}^{\infty}\varphi_{1i}\otimes\cdots\otimes\varphi_{mi}\otimes c_i$. Then $\delta_z\circ A = \sum_{i=1}^{\infty}\varphi_{1i}\otimes\cdots\otimes\varphi_{mi}\otimes c_i(z)$ for all $z \in G$, and clearly $\delta_z\circ A \in L_N(^mE)$. We have by (15) that

$$R_m(\delta_z\circ A) = \sum_{i=1}^{\infty}R_1\varphi_{1i}\otimes\cdots\otimes R_1\varphi_{mi}\otimes c_i(z)$$

for all $z \in G$. Then

$$\begin{aligned} \widetilde{R_m}A(y_1,\dots,y_m)(z) =& R_m(\delta_z\circ A)(y_1,\dots,y_m) \\ =& \sum_{i=1}^{\infty}R_1\varphi_{1i}(y_1)\cdots R_1\varphi_{mi}(y_m)c_i(z) \end{aligned}$$

for all $z \in G$ and $y_1,\dots,y_m \in F$. We conclude that

$$\widetilde{R_m}A = \sum_{i=1}^{\infty}R_1\varphi_{1i}\otimes\cdots\otimes R_1\varphi_{mi}\otimes c_i \tag{16}$$

and then $\widetilde{R_m}A \in L_N(^mF;G')$. QED

25 Definition. Given a Nicodemi sequence $R_m : L(^mE;G) \longrightarrow L(^mF;G)$, we define

$$\widehat{R}_m : \mathcal{P}(^mE;G) \longrightarrow \mathcal{P}(^mF;G)$$

by $\widehat{R}_m\widehat{A} = \widehat{R_mA}$ por every symmetric $A \in L(^mE;G)$.

26 Lemma. *Let $T_m : L(^mE) \longrightarrow L(^mE'')$ be the Nicodemi sequence beginning with the natural embedding $T_1 = J_{E'} : E' \hookrightarrow E'''$. Let $P \in \mathcal{P}_N(^mE)$ and let $c_1, \dots, c_n \in \mathbb{K}$, $\varphi_1, \dots, \varphi_m \in E'$ such that*

$$P = \sum_{i=1}^{\infty} \varphi_i^m \otimes c_i.$$

Then

$$\widehat{T}_m(P) = \sum_{i=1}^{\infty} (T_1\varphi_i)^m \otimes c_i.$$

In particular $\widehat{T}_m(P) \in \mathcal{P}_N(^mE'')$.

The proof of Lemma 26 is similar to the proof of Lemma 22 and is omitted. Next we will see that the operators from the Nicodemi sequence preserve homogeneous polynomials of finite type and nuclear homogeneous polynomials. We consider first the case of scalar-valued homogeneous polynomials.

27 Theorem. *Let $R_m : L(^mE) \longrightarrow L(^mF)$ be a Nicodemi sequence. Then $\widehat{R}_mP \in \mathcal{P}_\Theta(^mF)$ for all $P \in \mathcal{P}_\Theta(^mE)$ where $\Theta = f$ or N.*

Proof. We will only write the proof in the case $\Theta = N$. In the case $\Theta = f$, the proof is similar. Given $P \in \mathcal{P}_N(^mE)$, there exist $(c_i)_{i\in\mathbb{N}} \in \mathbb{K}$, $(\varphi_i)_{i\in\mathbb{N}} \in E'$ such that P can be written in the form: $P(x) = \sum_{i=1}^{\infty} \varphi_i(x)^m c_i$ for all $x \in E$. Let $T_m : L(^mE) \longrightarrow L(^mE'')$ be the Nicodemi sequence beginning with the natural embedding $T_1 = J_{E'} : E' \hookrightarrow E'''$. By Theorem 9 we have that

$$R_mA(y_1, \dots, y_m) = T_mA(R_1'(J_F y_1), \dots, R_1'(J_F y_m))$$

for all $A \in L(^mE)$, and $y_1, \dots, y_m \in F$, where R_1' is the transpose of R_1 and by Lemma 26 we have that $\widehat{T}_m(P)$ has the following form:

$$\widehat{T}_m(P)(x'') = \sum_{i=1}^{\infty} (T_1\varphi_i)(x'')^m c_i$$

for all $x'' \in E''$. Therefore, A being the m-linear mapping associated with P, it follows that

$$\begin{aligned}
\widehat{R}_m P(y) =& R_m A(\underbrace{y, \ldots, y}_{m-times}) \\
=& T_m A(\underbrace{R_1'(J_F y), \ldots, R_1'(J_F y)}_{m-times}) \\
=& \widehat{T}_m P(R_1'(J_F y)) \\
=& \sum_{i=1}^{\infty} (T_1 \varphi_i)(R_1'(J_F y))^m c_i \\
=& \sum_{i=1}^{\infty} (R_1 \varphi_i y))^m c_i
\end{aligned}$$

for all $y \in F$. Therefore,

$$\widehat{R}_m P = \sum_{i=1}^{\infty} (R_1 \varphi_i)^m \otimes c_i \tag{17}$$

and then $\widehat{R}_m P \in \mathcal{P}_N({}^m F)$. QED

The next theorem consider the case of vector-valued homogeneous polynomials.

28 Theorem. *Let $R_m : L({}^m E) \longrightarrow L({}^m F)$ be a Nicodemi sequence, let $\widetilde{R_m} : L({}^m E; G') \longrightarrow L({}^m F; G')$ be the corresponding Nicodemi sequence for vector-valued multilinear mappings. Then $\widehat{\widetilde{R}}_m P \in \mathcal{P}_\Theta({}^m F; G')$ for all $P \in \mathcal{P}_\Theta({}^m E; G')$ where $\Theta = f$ or N.*

Proof. We will only write the proof in the case $\Theta = N$. In the case $\Theta = f$, the proof is similar. Given $\widehat{A} \in \mathcal{P}_N({}^m E; G')$, there exist sequences $(\varphi_i)_{i\in\mathbb{N}}$ in E', and $(c_i)_{i\in\mathbb{N}}$ in G' with $\sum_{i=1}^{\infty} \|\varphi_i\|^m \|c_i\| < \infty$ such that $\widehat{A}(x) = \sum_{i=1}^{\infty} \varphi_i(x)^m c_i$ for all $x \in E$, where $A \in L^s({}^m E; G')$. We observe that if $B = \sum_{i=1}^{\infty} \underbrace{\varphi_i \otimes \cdots \otimes \varphi_i}_{m-times} \otimes c_i$, then $B \in L_N({}^m E; G') \cap L^s({}^m E; G')$ and $\widehat{B} = \widehat{A}$. Then $A = B$ from the injectivity of the canonical isomorphism $L^s({}^m E; G') \longrightarrow \mathcal{P}({}^m E; G')$. Thus, we get that $A \in L_N({}^m E; G')$ and by (16)

$$\widetilde{R_m} A = \sum_{i=1}^{\infty} \underbrace{R_1 \varphi_i \otimes \cdots \otimes R_1 \varphi_i}_{m-times} \otimes c_i.$$

Since $\widehat{\widetilde{R}}_m \widehat{A} = \widehat{\widetilde{R_m} A}$, it follows that

$$\widehat{\widetilde{R}}_m P = \sum_{i=1}^{n} (R_1 \varphi_i)^m \otimes c_i \tag{18}$$

and then $\widehat{\widehat{R}}_m P \in \mathcal{P}_N({}^m F; G')$. QED

We remark that if E is a closed subspace of F, then Aron and Berner [1, Theorem 2.1] proved that the restriction mapping

$$\mathcal{P}_N({}^m F; G) \longrightarrow \mathcal{P}_N({}^m E; G)$$

is surjective for each $m \in \mathbb{N}$, but even in the case of the natural embedding $J_E : E \hookrightarrow E''$, they did not study the problem of existence of a linear extension operator

$$T_m : \mathcal{P}_N({}^m E) \longrightarrow \mathcal{P}_N({}^m E'')$$

for each $m \in \mathbb{N}$. Next we will see that each isomorphism between E' and F' induces an isomorphism between $L_\Theta({}^m E; G')$ and $L_\Theta({}^m F; G')$ for each $m \in \mathbb{N}$ and induces also an isomorphism between $\mathcal{P}_\Theta({}^m E; G')$ and $\mathcal{P}_\Theta({}^m F; G')$ for each $m \in \mathbb{N}$, where $\Theta = f$ or N.

29 Theorem. *If E' and F' are isomorphic, then $L_\Theta({}^m E)$ and $L_\Theta({}^m F)$ are isomorphic, for all $m \in \mathbb{N}$, where $\Theta = f$ or N.*

Proof. We will only write the proof in the case $\Theta = N$. In the case $\Theta = f$, the proof is similar. We use the notations from the proof of Theorem 17 By Theorem 23 we have that $R_m(L_N({}^m E)) \subset L_N({}^m F)$. As in the proof of Theorem 23 we can prove that $S_m(R_m A) = \sum_{i=1}^\infty S_1(R_1\varphi_{1i}) \otimes \cdots \otimes S_1(R_1\varphi_{mi}) \otimes c_i$. Since $S_1 \circ R_1$ is the identity, we have that $S_m(R_m A) = \sum_{i=1}^\infty \varphi_{1i} \otimes \cdots \otimes \varphi_{mi} \otimes c_i = A$ for all $A \in L_N({}^m E)$. On the other hand, in a similar way, we can get that $S_m(L_N({}^m F)) \subset L_N({}^m E)$ and $R_m(S_m B) = B$ for all $B \in L_N({}^m F)$. We conclude that $L_N({}^m E)$ and $L_N({}^m F)$ are isomorphic for all $m \in \mathbb{N}$. QED

30 Theorem. *If E' and F' are isomorphic, then $L_\Theta({}^m E; G')$ and $L_\Theta({}^m F; G')$ are isomorphic for all $m \in \mathbb{N}$, where $\Theta = f$ or N.*

Proof. We will only write the proof in the case $\Theta = N$. In the case $\Theta = f$, the proof is similar. We use the notations from the proof of Theorem 19. By Theorem 24 we have that $\widetilde{R_m}(L_N({}^m E; G')) \subset L_N({}^m F; G')$ and $\widetilde{S_m}(L_N({}^m F; G')) \subset L_N({}^m E; G')$. As in the proof of Theorem 24 we can prove that $\widetilde{S_m}(\widetilde{R_m} A) = \sum_{i=1}^\infty S_1(R_1\varphi_{1i}) \otimes \cdots \otimes S_1(R_1\varphi_{mi}) \otimes c_i$. Since $S_1 \circ R_1$ is the identity, we have that $\widetilde{S_m}(\widetilde{R_m} A) = \sum_{i=1}^\infty \varphi_{1i} \otimes \cdots \otimes \varphi_{mi} \otimes c_i = A$ for all $A \in L_N({}^m E; G')$. On the other hand, in a similar way, we can get that $\widetilde{R_m}(\widetilde{S_m} B) = B$ for all $B \in L_N({}^m F; G')$. We conclude that $L_N({}^m E; G')$ and $L_N({}^m F; G')$ are isomorphic for all $m \in \mathbb{N}$. QED

31 Theorem. *If E' and F' are isomorphic, then $\mathcal{P}_\Theta({}^m E)$ and $\mathcal{P}_\Theta({}^m F)$ are isomorphic for all $m \in \mathbb{N}$, where $\Theta = f$ or N.*

Proof. We will only write the proof in the case $\Theta = N$. In the case $\Theta = f$, the proof is similar. We use the notations from the proof of Theorem 17. By Theorem 27 we have that $\widehat{R}_m(\mathcal{P}_N(^mE)) \subset \mathcal{P}_N(^mF)$ and $\widehat{S}_m(\mathcal{P}_N(^mF)) \subset \mathcal{P}_N(^mE)$. As in the proof of Theorem 27 we can prove that $\widehat{S}_m(\widehat{R}_m P) = \sum_{i=1}^{\infty}(S_1(R_1\varphi_i))^m \otimes c_i$. Since $S_1 \circ R_1$ is the identity, we have that $\widehat{S}_m(\widehat{R}_m P) = \sum_{i=1}^{\infty}(\varphi_i)^m \otimes c_i = P$ for all $P \in \mathcal{P}_N(^mE)$. On the other hand, in a similar way, we can get that $\widehat{R}_m(\widehat{S}_m Q) = Q$ for all $Q \in \mathcal{P}_N(^mF)$. We conclude that $\mathcal{P}_N(^mE)$ and $\mathcal{P}_N(^mF)$ are isomorphic for all $m \in \mathbb{N}$. QED

32 Theorem. *If E' and F' are isomorphic, then $\mathcal{P}_\Theta(^mE; G')$ and $\mathcal{P}_\Theta(^mF; G')$ are isomorphic for all $m \in \mathbb{N}$, where $\Theta = f$ or N.*

Proof. We will only write the proof in the case of $\Theta = N$. In the case of $\Theta = f$ the proof is similar. We use the notations from the proof of Theorem 19. By Theorem 28 we have that $\widehat{\widehat{R}}_m(\mathcal{P}_N(^mE; G')) \subset \mathcal{P}_N(^mF; G')$ and $\widehat{\widehat{S}}_m(\mathcal{P}_N(^mF; G')) \subset \mathcal{P}_N(^mE; G')$. As in the proof of Theorem 28 we can prove that $\widehat{\widehat{S}}_m(\widehat{\widehat{R}}_m P) = \sum_{i=1}^{\infty}(S_1(R_1\varphi_i))^m \otimes c_i$. Since $S_1 \circ R_1$ is the identity, we have that $\widehat{\widehat{S}}_m(\widehat{\widehat{R}}_m P) = \sum_{i=1}^{\infty}(\varphi_i)^m \otimes c_i = P$ for all $P \in \mathcal{P}_N(^mE; G')$. On the other hand, in a similar way, we can get that $\widehat{\widehat{R}}_m(\widehat{\widehat{S}}_m Q) = Q$ for all $Q \in \mathcal{P}_N(^mF; G')$. We conclude that $\mathcal{P}_N(^mE; G')$ and $\mathcal{P}_N(^mF; G')$ are isomorphic for all $m \in \mathbb{N}$. QED

Let us notice that in Theorems 29, 30, 31 and 32 we do not need the Arens regularity hypothesis.

3 Isomorphisms between spaces of compact or weakly compact multilinear mappings or homogeneous polynomials.

We first show that the operators $\widetilde{R_m}$ from the Nicodemi sequence preserve compact and weakly compact multilinear mappings.

33 Theorem. *Let $R_m : L(^mE) \longrightarrow L(^mF)$ be a Nicodemi sequence and let $\widetilde{R_m} : L(^mE; G') \longrightarrow L(^mF; G')$ be the corresponding Nicodemi sequence for vector-valued multilinear mappings. Then $\widetilde{R_m}A \in L_\Theta(^mF; G')$ for each $A \in L_\Theta(^mE; G')$, where $\Theta = K$ or WK.*

Proof. Let $T_m : L(^mE) \longrightarrow L(^mE'')$ be the Nicodemi sequence beginning with the natural embedding $T_1 : E' \hookrightarrow E'''$ and $\widetilde{T_m} : L(^mE; G') \longrightarrow L(^mE''; G')$ be the corresponding Nicodemi sequence for vector-valued multilinear mappings.

This sequence coincides with the sequence of operators constructed by Aron-Berner in [1, Proposition 2.1]. Therefore, by [1, Proposition 2.1], we get that $\widetilde{T_m}A \in L_\Theta(^mE''; G')$ for each $A \in L_\Theta(^mE; G')$, where $\Theta = K$ or WK. By Theorem 10 we have that

$$\widetilde{R_m}A(y_1, \ldots, y_m) = \widetilde{T_m}A(R_1'(J_F y_1), \ldots, R_1'(J_F y_m))$$

for all $A \in L(^mE; G')$ and $y_1, \ldots, y_m \in F$, where R_1' is the transpose of R_1. If $\widetilde{T_m}A \in L_\Theta(^mE''; G')$, it follows that $\widetilde{R_m}A \in L_\Theta(^mF; G')$, where $\Theta = K$ or WK. Therefore we conclude that $\widetilde{R_m}A \in L_\Theta(^mF; G')$ for each $A \in L_\Theta(^mE; G')$, where $\Theta = K$ or WK. QED

We next show that the operators $\widehat{\widetilde{R}}_m$ from the the Nicodemi sequence preserve compact and weakly compact homogeneous polynomials.

34 Theorem. *Let $R_m : L(^mE) \longrightarrow L(^mF)$ be a Nicodemi sequence, let $\widetilde{R_m} : L(^mE; G') \longrightarrow L(^mF; G')$ be the corresponding Nicodemi sequence for vector-valued multilinear mappings. Then $\widehat{\widetilde{R}}_m P \in \mathcal{P}_\Theta(^mF; G')$ for each $P \in \mathcal{P}_\Theta(^mE; G')$, where $\Theta = K$ or WK.*

Proof. Let (T_m) and $(\widetilde{T_m})$ be the sequences from the proof of Theorem 33. The sequence $(\widetilde{T_m})$ coincides with the sequence of operators constructed by Aron-Berner in [1, Proposition 2.1]. Then the sequence $\widehat{\widetilde{T}}_m : \mathcal{P}(^mE; G') \longrightarrow \mathcal{P}(^mE''; G')$ defined by $\widehat{\widetilde{T}}_m \widehat{A} = \widehat{\widetilde{T_m}A}$ for each $A \in L^s(^mE; G')$ coincides with the sequence of operators constructed by Aron-Berner in [1, Corollary 2.2]. Therefore $\widehat{\widetilde{T}}_m P \in \mathcal{P}_\Theta(^mE''; G')$ for each $P \in \mathcal{P}_\Theta(^mE; G')$, where $\Theta = K$ or WK. By Theorem 10 we have that

$$\widehat{\widetilde{R}}_m P(y) = \widehat{\widetilde{T}}_m P(R_1'(J_F y))$$

for all $P \in \mathcal{P}(^mE; G')$, and $y \in F$, where R_1' is the transpose of R_1. It follows that $\widehat{\widetilde{R}}_m P \in \mathcal{P}_\Theta(^mF; G')$ for each $\widehat{\widetilde{T}}_m P \in \mathcal{P}_\Theta(^mE''; G')$, where $\Theta = K$ or WK. We conclude that $\widehat{\widetilde{R}}_m P \in \mathcal{P}_\Theta(^mF; G')$ for each $P \in \mathcal{P}_\Theta(^mE; G')$, where $\Theta = K$ or WK. QED

Let $L^s_\Theta(^mE; G) = L^s(^mE; G) \cap L_\Theta(^mE; G)$, where $\Theta = K$ or WK. We next show that if E and F are symmetrically Arens - regular, then each isomorphism between E' and F' induces an isomorphism between $\mathcal{P}_\Theta(^mE; G')$ and $\mathcal{P}_\Theta(^mF; G')$ for each $m \in \mathbb{N}$, where $\Theta = K$ or WK.

35 Theorem. *If E and F are symmetrically Arens - regular, and E' and F' are isomorphic, then $L^s_\Theta(^mE; G')$ and $L^s_\Theta(^mF; G')$ are isomorphic for all $m \in \mathbb{N}$, where $\Theta = K$ or WK.*

Proof. We use the notations from the proof of Theorem 19. By Theorems 12 and 33 we have that, for $\Theta = K$ or WK

$$\widetilde{R_m}(L^s_\Theta({}^mE; G')) \subset L^s_\Theta({}^mF; G').$$

and

$$\widetilde{S_m}(L^s_\Theta({}^mF; G')) \subset L^s_\Theta({}^mE; G').$$

By (9) we have that $S_m \circ R_m|_{L^s({}^mE)}$ is the identity mapping. Using (6), we have that

$$\begin{aligned}\widetilde{S_m} \circ \widetilde{R_m}A(x)(z) =& \widetilde{S_m}(\widetilde{R_m}A)(x)(z)\\ =& S_m(\delta_z \circ \widetilde{R_m}A)(x)\\ =& S_m(R_m(\delta_z \circ A))(x)\\ =& [S_m \circ R_m(\delta_z \circ A)](x)\\ =& (\delta_z \circ A)(x)\\ =& A(x)(z)\end{aligned}$$

for all $A \in L^s_K({}^mE; G'), x \in E^m$ and $z \in G$, that is

$$(\widetilde{S_m} \circ \widetilde{R_m})A = A$$

for all $A \in L^s_\Theta({}^mE; G')$. In a similar way, we can prove that

$$(\widetilde{R_m} \circ \widetilde{S_m})B = B$$

for all $B \in L^s_\Theta({}^mF; G')$. Therefore, we get that $L^s_\Theta({}^mE; G')$ and $L^s_\Theta({}^mF; G')$ are isomorphic, where $\Theta = K$ or WK. QED

36 Theorem. *If E and F are symmetrically Arens - regular, and E' and F' are isomorphic, then $\mathcal{P}_\Theta({}^mE; G')$ and $\mathcal{P}_\Theta({}^mF; G')$ are isomorphic for all $m \in \mathbb{N}$, where $\Theta = K$ or WK.*

Proof. We use the notations from the proof of Theorem 32. By Theorem 34 we have that, for $\Theta = K$ or WK,

$$\widehat{\widehat{R}}_m(\mathcal{P}_\Theta({}^mE; G')) \subset \mathcal{P}_\Theta({}^mF; G').$$

and

$$\widehat{\widehat{S}}_m(\mathcal{P}_\Theta({}^mF; G')) \subset \mathcal{P}_\Theta({}^mE; G').$$

By (10) and (11) we have that for each $\widehat{A} \in \mathcal{P}_\Theta(^m E; G')$

$$\begin{aligned}(\widehat{\widetilde{S}}_m \circ \widehat{\widetilde{R}}_m)(\widehat{A}) =& \widehat{\widetilde{S}}_m(\widehat{\widetilde{R}}_m \widehat{A}) \\ =& \widehat{\widetilde{S}}_m(\widehat{\widetilde{R}_m A}) \\ =& \widehat{\widetilde{S}_m(\widetilde{R}_m A)} \\ =& \widehat{A}\end{aligned}$$

That is,

$$(\widehat{\widetilde{S}}_m \circ \widehat{\widetilde{R}}_m)\widehat{A} = \widehat{A}$$

for all $\widehat{A} \in \mathcal{P}_\Theta(^m E; G')$. Similarly we can prove that

$$(\widehat{\widetilde{R}}_m \circ \widehat{\widetilde{S}}_m)\widehat{B} = \widehat{B}$$

for all $\widehat{B} \in \mathcal{P}_\Theta(^m F; G')$. Thus we conclude that $\mathcal{P}_\Theta(^m E; G')$ and $\mathcal{P}_\Theta(^m F; G')$ are isomorphic for all $m \in \mathbb{N}$, where $\Theta = K$ or WK. QED

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