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On the extrinsic principal directions of Riemannian submanifolds

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Abstract. The Casorati curvature of a submanifold M^n of a Riemannian manifold $\widetilde{M}^{n+m}$ is known to be the normalized square of the length of the second fundamental form, $C = \frac{1}{n}\|h\|^2$, i.e., in particular, for hypersurfaces, $C = \frac{1}{n}(k_1^2 + \cdots + k_n^2)$, whereby $k_1, \ldots, k_n$ are the principal normal curvatures of these hypersurfaces. In this paper we in addition define the Casorati curvature of a submanifold M^n in a Riemannian manifold $\widetilde{M}^{n+m}$ at any point p of M^n in any tangent direction u of M^n. The principal extrinsic (Casorati) directions of a submanifold at a point are defined as an extension of the principal directions of a hypersurface M^n at a point in $\widetilde{M}^{n+1}$. A geometrical interpretation of the Casorati curvature of M^n in $\widetilde{M}^{n+m}$ at p in the direction u is given. A characterization of normally flat submanifolds in Euclidean spaces is given in terms of a relation between the Casorati curvatures and the normal curvatures of these submanifolds.

Keywords: Casorati curvature, principal direction, normal curvature, squared length of the second fundamental form.

MSC 2000 classification: 53A07, 53B20, 53C42.

1 Introduction

In Section 2 we define *the Casorati curvature* $c(u)$ of a submanifold M^n in a Riemannian manifold $\widetilde{M}^{n+m}$ *at any point* p of M^n *for any unit tangent vector* u, in terms of the *Casorati operator* A^C of M^n in $\widetilde{M}^{n+m}$, and we define *the Casorati curvature* $C : M^n \to \mathbb{R} : p \mapsto C(p)$ of a submanifold M^n in $\widetilde{M}^{n+m}$ "*as such*", $C(p)$ then being the mean value of all Casorati curvatures $c(u)$ at p or, equivalently, the trace of A^C divided by n. In Section 3, we explain the *geometrical meaning* of the Casorati curvatures. First, we will give a geometrical meaning for *hypersurfaces*, extending the original idea of Casorati concerning surfaces in $\mathbb{E}^3$. Next, we will give a geometrical interpretation of the Casorati curvatures for *general submanifolds* M^n in $\widetilde{M}^{n+m}$, making use of the definition of *the angle between subspaces of inner product spaces*: this is essentially C. Jordan's extension [7] of Euler's approach to the geometry of surfaces in $\mathbb{E}^3$ and also the first step of Trenčevski's extension [11] of Euler's approach to the geometry of surfaces in $\mathbb{E}^3$ as well as of C. Jordan's "Frenet"-approach to the geometry of curves in $\mathbb{E}^{1+m}$, which actually both originate in Euler's view on the curvature of the Euclidean planar curves. In Section 4 we study the Casorati curvatures of *submanifolds* M^n *in Euclidean spaces* $\mathbb{E}^{n+m}$ in relation with *the curvatures of the normal sections of these submanifolds.* In particular, this study yields a new characterization of submanifolds M^n with *flat normal connection* in $\mathbb{E}^{n+m}$.

2 Basic formulae and definitions

Let (M^n, g) be an n-dimensional Riemannian manifold and m-codimensional submanifold of a Riemannian manifold $(\widetilde{M}^{n+m}, \widetilde{g})$, and let ∇ and $\widetilde{\nabla}$ be the *Levi-Civita connections* of g and $\widetilde{g}$, respectively. The *tangent* vector fields on M^n will be denoted by $X, Y, \ldots$ and the *normal* vector fields on M^n in $\widetilde{M}^{n+m}$ will be denoted $\xi, \eta, \ldots$, and so *the formulae of Gauss and Weingarten* which concern the decompositions of the vector fields $\widetilde{\nabla}_X Y$ and $\widetilde{\nabla}_X \xi$, respectively, into their tangential and normal components along the submanifold M^n in the ambient space $\widetilde{M}^{n+m}$ are given by

$$\begin{aligned}\widetilde{\nabla}_X Y &= \nabla_X Y + h(X, Y),\\ \widetilde{\nabla}_X \xi &= -A_\xi(X) + \nabla^\perp_X \xi,\end{aligned}$$

respectively, whereby h, A_ξ and $\nabla^\perp$ denote the *second fundamental form*, the *shape operator* or *Weingarten map* associated to ξ and the *normal connection* of the submanifold M^n in the ambient manifold $\widetilde{M}^{n+m}$, the shape operators A_ξ being related to the second fundamental form h by

$$\widetilde{g}(h(X, Y), \xi) = g(A_\xi(X), Y).$$

The *mean curvature vector field* $\vec{H}$ of M^n in $\widetilde{M}^{n+m}$ is defined by

$$\vec{H} = \frac{1}{n}\mathrm{tr}h = \frac{1}{n}\sum_i h(E_i, E_i),$$

whereby $\{E_1, \ldots, E_n\}$,$(i, j, \ldots \in \{1, 2, \ldots, n\})$, is any local orthonormal tangent frame field, such that,

$$\vec{H} = \frac{1}{n}\sum_\alpha (\mathrm{tr}A_\alpha)\xi_\alpha,$$

whereby $A_\alpha \equiv A_{\xi_\alpha}$ and $\{\xi_1, \ldots, \xi_m\}$, $(\alpha, \beta, \ldots \in \{1, 2, \ldots, m\})$, is any local orthonormal normal frame field on M^n in $\widetilde{M}^{n+m}$. Let R and $\widetilde{R}$ denote the $(0, 4)$ *Riemann-Christoffel curvature tensors* of M^n and $\widetilde{M}^{n+m}$, respectively. Then, the *equation of Gauss* is given by

$$\widetilde{R}(X, Y, Z, W) = R(X, Y, Z, W) + \widetilde{g}(h(X, Z), h(Y, W)) - \widetilde{g}(h(X, W), h(Y, Z)),$$

which, in particular, for submanifolds M^n in *real space forms* $\widetilde{M}^{n+m}(c)$, becomes

$$R(X, Y, Z, W) = c\, g((X \wedge Y)Z, W) + \widetilde{g}(h(X, W), h(Y, Z)) - \widetilde{g}(h(X, Z), h(Y, W)),$$

whereby $(X \wedge Y)\, Z = g(Y, Z)X - g(X, Z)Y$ (see e.g. [2]). In the latter situation, i.e. for submanifolds M^n in $\widetilde{M}^{n+m}(c)$, by contraction, it follows that

$$S(Y, Z) = (n - 1)\, c\, g(Y, Z) + g(A_{n\vec{H}}(Y), Z) - g(A^C(Y), Z),$$

whereby S denotes the $(0, 2)$ *Ricci tensor* of the Riemannian manifold (M^n, g) and whereby we have put $A^C = \sum_\alpha A_\alpha^2$. And, still for submanifolds M^n of real space forms $\widetilde{M}^{n+m}(c)$, it follows from the *equation of Ricci*,

$$R^\perp(X, Y, \xi, \eta) = g([A_\xi, A_\eta](X), Y),$$

whereby $[\cdot, \cdot]$ denotes the commutator of $(1, 1)$ tensor fields, that *the normal connection is flat or trivial*, i.e. $R^\perp = 0$, if and only if *all shape operators can be diagonalized simultaneously*.

Also for general submanifolds M^n in arbitrary Riemannian spaces $\widetilde{M}^{n+m}$, we will further consider the $(1, 1)$ tensor field $A^C = \sum_\alpha A_\alpha^2$, which clearly is independent of the choice of local orthonormal normal frame field $\{\xi_1, \ldots, \xi_m\}$, and we propose to call A^C *the Casorati operator* of M^n in $\widetilde{M}^{n+m}$; (for motivation of this nomenclature, see section 3). In [4] and [5], the *Casorati curvature* $C : M^n \to \mathbb{R}$ of a submanifold M in a Riemannian manifold $\widetilde{M}^{n+m}$ is defined as the scalar valued extrinsic invariant $C = \frac{1}{n}\|h\|^2 = \frac{1}{n}\sum_{\alpha,i,j}(h_{ij}^\alpha)^2 = \frac{1}{n}\mathrm{tr}A^C$, whereby h_{ij}^α denote the components of the second fundamental form h with respect to any orthonormal frame field $\{E_1, \ldots, E_n, \xi_1, \ldots, \xi_m\}$ on M^n in $\widetilde{M}^{n+m}$; $h(E_i, E_j) = \sum_\alpha h_{ij}^\alpha \xi_\alpha$, $h_{ij}^\alpha = \widetilde{g}(h(E_i, E_j), \xi_\alpha) = g(A_\alpha(E_i), E_j)$. Clearly, a submanifold M^n in $\widetilde{M}^{n+m}$ is *totally geodesic* $(h = 0)$ if and only if its Casorati curvature function vanishes at all points of M^n $(C \equiv 0)$.

Since for each normal vector field ξ on M^n in $\widetilde{M}^{n+m}$ the corresponding shape operator A_ξ is a *symmetric* $(1, 1)$ tensor field on M^n, by *the principal axis theorem*, at every point p of M^n, all eigenvalues $\lambda_1^\xi \geq \cdots \geq \lambda_n^\xi$ of $A_{\xi(p)}$ are real and there do exist n orthonormal eigenvectors $e_1, \ldots, e_n$ of $A_{\xi(p)}$, $A_{\xi(p)}(e_i) =$

$\lambda_i^\xi e_i$; the tangent directions on M^n determined by e_i are called *the principal directions* of M^n in $\widetilde{M}^{n+m}$ at p *with respect to* ξ, and the numbers λ_i^ξ are called *the principal normal curvatures* of M^n in $\widetilde{M}^{n+m}$ at p *with respect to* ξ. Similarly, the Casorati operator A^C of M^n in $\widetilde{M}^{n+m}$ being a symmetric $(1,1)$ tensor field on M^n, at every point p of M^n all eigenvalues $c_1 \geq \cdots \geq c_n \geq 0$ are real and there do exist n orthonormal eigenvectors $f_1, \ldots, f_n$ of A_p^C, $A_p^C(f_j) = c_j f_j$; the tangent directions on M^n determined by f_j are called *the extrinsic principal directions* or *the Casorati principal directions* of M^n in $\widetilde{M}^{n+m}$ at p, and the numbers c_j are called *the principal Casorati curvatures* of M^n in $\widetilde{M}^{n+m}$. Clearly, $C(p) = \frac{1}{n}\sum_j c_j$, i.e. at every point p of a submanifold M^n in $\widetilde{M}^{n+m}$ the scalar valued Casorati curvature of M^n in $\widetilde{M}^{n+m}$ is the arithmetic average of the principal Casorati curvatures $c_1, \ldots, c_n$ of M^n in $\widetilde{M}^{n+m}$ at p. From the contracted Gauss equation it can be trivially observed that, in particular, for the minimal and the pseudo-umbilical submanifolds and for the submanifolds with flat normal connection in real space forms, *the intrinsic principal directions* of the Riemannian manifold (M^n, g), i.e. the eigendirections of its symmetric $(0,2)$ Ricci tensor S, or, equivalently, of its symmetric $(1,1)$ Ricci operator $\widetilde{S}$, $S(Y,Z) = g(\widetilde{S}(Y), Z) = g(Y, \widetilde{S}(Z))$, *and the extrinsic principal directions*, as determined by its $(1,1)$ Casorati operator A^C, actually *do coincide*.

For the special situation of hypersurfaces M^n in ambient spaces $\widetilde{M}^{n+1}$, the formulae of Gauss and Weingarten are usually written as $\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\xi$ and $\widetilde{\nabla}_X \xi = -A(X)$, whereby h now denotes the *scalar valued second fundamental form* corresponding to a local unit normal vector field ξ on M^n in $\widetilde{M}^{n+1}$ and A denotes the shape operator or Weingarten map of M^n associated with ξ, such that $h(X,Y) = g(A(X), Y)$. As customary, the *principal curvatures* at some point p of M^n in $\widetilde{M}^{n+1}$ corresponding to $\xi(p)$ are denoted by $k_1 \geq \cdots \geq k_n$, i.e. $k_1, \ldots, k_n$ are the eigenvalues of A, say, corresponding to orthonormal vectors $e_1, \ldots, e_n$ at p, $A(e_i) = k_i e_i$, or, still: $k_1, \ldots, k_n$ are *the critical values of the normal curvature function* $k(u) = g(A(u), u)$ of M^n in $\widetilde{M}^{n+1}$ at p, $k : S_p^{n-1}(1) \subset T_p M^n \to \mathbb{R} : u \mapsto k(u)$; and, putting $u = \sum_i e_i \cdot \cos\alpha_i$, $\cos\alpha_i = g(u, e_i)$, *the formula of Euler*, in some sense, the 1760 starting point of the systematic study of the differential geometry of surfaces M^2 in $\mathbb{E}^3$, then readily follows as $k(u) = \sum_i k_i \cdot \cos^2\alpha_i$. For hypersurfaces M^n in $\widetilde{M}^{n+1}$, of course, having $A^C = A^2$, it follows that *the Euler principal directions and the Casorati principal directions are the same* and concerning the corresponding curvatures it follows, up to ordering (which may be influenced by the signs of the Euler curvatures k_i), that $c_j = k_j^2$, so that for hypersurfaces M^n in $\widetilde{M}^{n+1}$,

$$C(p) = \frac{1}{n}\sum_j k_j^2.$$

Returning to the situation of general submanifolds M^n in arbitrary Riemannian spaces $\widetilde{M}^{n+m}$, we insist on the fact that, when defining *the Casorati curvature* $c(u)$ of M^n in $\widetilde{M}^{n+m}$ at p in an arbitrary tangent direction u by $c(u) = g(A_p^C(u), u), c : S_p^{n-1}(1) \subset T_pM^n \to \mathbb{R} : u \mapsto c(u)$, then it can be readily observed that *the principal Casorati curvatures* $c_1, \dots, c_n$ of M^n in $\widetilde{M}^{n+m}$ at p are *the critical values* of this function $c : S_p^{n-1}(1) \subset T_pM^n \to \mathbb{R} : u \mapsto c(u)$, and these critical values are attained in *the principal Casorati directions* $f_1, \dots, f_n$ of T_pM^n, and, in complete analogy with the formula of Euler for the normal curvatures $k(u)$ in tangent directions u, from $A_p^C(f_j) = c_j f_j$ and $c(u) = g(A_p^C(u), u)$, putting $u = \sum_j f_j \cdot \cos\beta_j$, $\cos\beta_j = g(u, f_j)$, follows the similar formula $c(u) = \sum_j c_j \cdot \cos^2\beta_j$ for the Casorati curvatures $c(u)$ in tangent directions u. And, although, apart from the presentation and the terminology and a triviality, the above was essentially seen in 1874 by C. Jordan [7], and was taken up again since halfway the 19nineties by K. Trenčevski [10,12,13] in his fine new approach to submanifold theory, we will formulate some of the above, for eventual easier later references, in the following.

1 Theorem. *For every submanifold M^n in any ambient Riemannian space $\widetilde{M}^{n+m}$ of arbitrary dimension $n \geq 2$ and co-dimension $m \geq 1$, its Casorati operator A^C is a canonically determined extrinsic $(1,1)$ tensor on M^n whose eigenvalues $c_1 \geq \cdots \geq c_n \geq 0$ at any point p in M^n are the critical values of the Casorati curvatures $c(u) = g(A_p^C(u), u)$ taken in all directions u to M^n at p, and which values are attained in the extrinsic Casorati principal directions of M^n in $\widetilde{M}^{n+m}$ as determined by n orthonormal eigenvectors $f_1, \dots, f_n$ of A^C at p, and the Casorati curvature $c(u)$ at p in the direction $u = \sum_j f_j \cdot \cos\beta_j$ is given by $c(u) = \sum_j c_j \cdot \cos^2\beta_j$, while the Casorati curvature as such of M^n in $\widetilde{M}^{n+m}$ at p is given by $C(p) = \frac{1}{n}\|h\|^2(p) = \frac{1}{n}\sum_j c_j$.*

And, next, we similarly formulate the above made observations related to the contracted Gauss equation in the following.

2 Theorem. *For submanifolds M^n in real space forms $\widetilde{M}^{n+m}(c)$, under each of the following conditions the (extrinsic) Casorati principal directions and (intrinsic) Ricci principal directions do coincide: (1) M^n is minimal in $\widetilde{M}^{n+m}(c)$, (2) M^n is pseudo-umbilical in $\widetilde{M}^{n+m}(c)$ and (3) the normal bundle of M^n in $\widetilde{M}^{n+m}(c)$ is trivial.*

Focussing, in particular, on submanifolds M^n in Euclidean ambient spaces $\mathbb{E}^{n+m}$, with respect to (1) and (2) in this proposition, we would like to add the following comments. The *minimal* submanifolds M^n in $\mathbb{E}^{n+m}$, i.e. the submanifolds for which $\vec{H} = \vec{0}$, are the submanifolds M^n in $\mathbb{E}^{n+m}$ which assume a shape for which the resulting "*surface tension*" $H^2 = \widetilde{g}(\vec{H}, \vec{H})$ vanishes; (according to *the formula of Beltrami*, $\triangle\vec{x} = -n\vec{H}$, whereby $\vec{x}$ is *the position vector* *field* of M^n in $\mathbb{E}^{n+m}$ and $\triangle$ is *the Riemannian Laplace operator* of (M^n, g)). In general, of course, a submanifold M^n in $\mathbb{E}^{n+m}$ does experience a (non-zero) "surface tension", and the *pseudo-umbilical* submanifolds can be interpreted as the submanifolds M^n in $\mathbb{E}^{n+m}$ for which *this tension is distributed evenly over all tangent directions* on M^n, since, by definition, pseudo-umbilicity means that the shape operator $A_{\vec{H}}$ is proportional to the identity transformation, or, still, that $\vec{H}$ determines *an umbilical normal direction* on M^n in $\mathbb{E}^{n+m}$; (the *totally umbilical submanifolds* thus can be seen as the *most special pseudo-umbilical submanifolds* as, likewise, the *totally geodesic submanifolds* can be regarded as the *most special minimal submanifolds*). The point that we would like to make is the following: the minimal and pseudo-umbilical submanifolds M^n assume such shapes in the ambient spaces $\mathbb{E}^{n+m}$ that they "succeed" in completely avoiding any surface tension at all, or, if such tension is really unavoidable, then they "succeed" in distributing this tension uniformly in all tangent directions at all of their points. In general however, leaving in the present sense "trivial", i.e. totally geodesic submanifolds (= n-dimensional affine subspaces of $\mathbb{E}^{n+m}$) and totally umbilical submanifolds (= round n-spheres $\mathbb{S}^n$ in subspaces $\mathbb{E}^{n+1}$ of $\mathbb{E}^{n+m}$) out of discussion, minimal and pseudo-umbilical submanifolds are not at all tangentially isotropic: at each point they have n mutually orthogonal tangent directions which are very important from the extrinsic geometric point of view, since their Casorati curvatures, which are the most natural extrinsic scalar valued curvatures which can be associated with tangent directions, attain their critical values in these directions. What the proposition asserts is that a.o. minimal and pseudo-umbilical submanifolds M^n do assume shapes in $\mathbb{E}^{n+m}$ such that, at each point, their Casorati principal directions exactly lie in the n mutually orthogonal tangent directions of the Riemannian manifolds (M^n, g) in which the Ricci curvatures, which are their most natural intrinsic scalar valued curvatures which can be associated with tangent directions, attain their critical values.

3 On the geometrical meaning of the Casorati curvatures

Having in mind that for hypersurfaces M^n in general Riemannian spaces $\widetilde{M}^{n+1}$, of course, comparisons between directions of normals $N(p) \in T_p\widetilde{M}^{n+1}$ and $N(q) \in T_q\widetilde{M}^{n+1}$ on M^n in $\widetilde{M}^{n+1}$ at nearby points p and q of M^n, can only be made, in a sensible way, via the $\widetilde{\nabla}$-*parallel transport* between $T_p\widetilde{M}^{n+1}$ and $T_q\widetilde{M}^{n+1}$, we will now carry over in detail the essence of Casorati's approach towards the extrinsic curvatures for surfaces M^2 in $\mathbb{E}^3$ [1], to *hypersurfaces* M^n

in general Riemannian spaces $\widetilde{M}^{n+1}$. Let ξ be a local unit normal vector field on M^n in $\widetilde{M}^{n+1}$ around some point p of M^n. Let u be any unit tangent vector to M^n at p and consider on M^n any *arclength parameterized curve* δ passing through $p = \delta(0)$ with velocity $\delta'(0) = u$. Then, consider the $\widetilde{\nabla}$*-parallel transport of the normal vector* $\xi(p) \in T_p\widetilde{M}^{n+1}$ to M^n in $\widetilde{M}^{n+1}$ at p *along the curve* δ on M^n, and denote by $\xi^*(s)$ the vector thus obtained at a nearby point $q = \delta(s)$. The *angle* $\theta_u(s)$ between the unit normal vector $\xi(\delta(s)) = \xi(q) \in T_q\widetilde{M}^{n+1}$ to M^n in $\widetilde{M}^{n+1}$ at q and the unit vector $\xi^*(s) = \xi^*(q) \in T_q\widetilde{M}^{n+1}$ at $q = \delta(s)$ is given by

$$\cos\theta_u(s) = \widetilde{g}\Big(\xi(\delta(s)), \xi^*(s)\Big). \tag{1}$$

The classical Maclaurin expansion of $\cos\theta_u(s)$ readily gives

$$\cos\theta_u(s) = 1 - \frac{1}{2}\left(\frac{\mathrm{d}\theta_u}{\mathrm{d}s}(0)\right)^2 s^2 + O^{>2}(s), \tag{2}$$

and the Maclaurin expansion of $\widetilde{g}\Big(\xi(\delta(s)), \xi^*(s)\Big)$ is found to be

$$\widetilde{g}\Big(\xi(\delta(s)), \xi^*(s)\Big) = 1 - \frac{1}{2}\Big(g(A_p(u), A_p(u))\Big)^2 s^2 + O^{>2}(s), \tag{3}$$

since $\widetilde{\nabla}\widetilde{g} = 0$, $\widetilde{\nabla}_u\xi = -A(u)$, $\widetilde{\nabla}_u(A(u)) = \nabla_u(A(u)) + h(A(u),u)\xi(p)$ and $h(A(u),u) = g(A^2(u),u) = g(A(u),A(u))$. From (2) and (3), we obtain the following.

3 Theorem. *The Casorati curvature $c(u)$ of a hypersurface M^n in a Riemannian manifold $\widetilde{M}^{n+1}$ at a point p in a tangential direction u satisfies*

$$c(u) = g(A_p(u), A_p(u)) = g(A_p^2(u), u) = \left(\frac{\mathrm{d}\theta_u}{\mathrm{d}s}(0)\right)^2. \tag{4}$$

At p, of course, $\theta_u(0) = 0$, and $\left(\frac{\mathrm{d}\theta_u}{\mathrm{d}s}(0)\right)^2$ was *Casorati's measure* for the degree in which a surface M^2 in $\mathbb{E}^3$ is extrinsically curved at p in the direction u [1], which therefore may justify the definition for the Casorati curvature $c(u)$ of M^n in $\widetilde{M}^{n+1}$ at p in the direction $u \in T_pM^n$ given in Section 2.

Next, we consider "*the total Casorati curvature*" at the point p of M^n in $\widetilde{M}^{n+1}$, i.e. we consider the integral

$$\int_{S_p^{n-1}(1)} c(u)\,\mathrm{d}S_p^{n-1}(1) = \int_{S_p^{n-1}(1)} g(A_p(u), A_p(u))\,\mathrm{d}S_p^{n-1}(1), \tag{5}$$

whereby $S_p^{n-1}(1) = \{u \in T_pM^n : \|u\| = 1\}$ and $\mathrm{d}S_p^{n-1}(1)$ denotes the *volume element* of $S_p^{n-1}(1)$. Let $e_1, \ldots, e_n$ be the *principal directions* and $k_1, \ldots, k_n$ the *principal curvatures* of the shape-operator $A_p : T_pM^n \to T_pM^n$. By orthonormal expansion with respect to the Euler-Casorati principal directions $e_1, \dots, e_n$ at p, every vector $u \in S_p^{n-1}(1)$ can be written as $u = \sum_i x_i e_i$, whereby $\sum_i (x_i)^2 = 1$, and then $c(u) = g(A_p(u), A_p(u)) = \sum_i k_i^2 x_i^2$. Since, $N = (x_1, \dots, x_n)$ can be seen as a unit normal vector field on $S_p^{n-1}(1)$ in $T_pM^n = \mathbb{R}^n$, considering the vector field $V = (k_1^2 x_1, \dots, k_n^2 x_n)$ along $S_p^{n-1}(1)$ in $T_pM^n = \mathbb{R}^n$, (the used coordinates always referring to the above mentioned principal axes in T_pM^n), by the divergence theorem, we find that

$$\begin{aligned}
\int_{S_p^{n-1}(1)} g(A_p(u), A_p(u))\, \mathrm{d}S_p^{n-1}(1) &= \int_{S_p^{n-1}(1)} \left(\sum_{i=1}^{n} k_i^2 x_i^2 \right) \mathrm{d}S_p^{n-1}(1) \\
&= \int_{S_p^{n-1}(1)} g(N, V)\, \mathrm{d}S_p^{n-1}(1) \\
&= \int_{B^n} \mathrm{div}(V)\, \mathrm{d}B^n \\
&= \left(\sum_{i=1} k_i^2 \right) \mathrm{vol}(B^n),
\end{aligned}$$

whereby B^n is the *unit ball* in $\mathbb{R}^n$ *bounded by the unit sphere* $S_p^{n-1}(1)$ *centered at* p and $\mathrm{d}B^n$ denotes its *volume element*. Hence, recalling that $n \cdot \mathrm{vol}(B^n) = \mathrm{vol}(S_p^{n-1}(1))$, in full analogy with the fact that *the mean value of the normal curvatures* $k(u)$ at any point p of a hypersurface M^n in a Riemannian manifold $\widetilde{M}^{n+1}$ *over all tangent directions to* M^n at p equals *the mean curvature* $H(p)$ *of* M^n *in* $\widetilde{M}^{n+1}$, we have the following.

4 Theorem. *The Casorati curvature $C(p)$ of a hypersurface M^n in a Riemannian manifold $\widetilde{M}^{n+1}$ at any point $p \in M^n$ is the mean value of the Casorati curvature $c(u)$ of M^n in $\widetilde{M}^{n+1}$ at p in all directions $u \in T_pM^n$:*

$$C(p) := \frac{1}{n}\|h\|^2 = \frac{1}{n}\mathrm{tr}A_p^2 = \frac{\int_{S_p^{n-1}(1)} c(u)\, \mathrm{d}S_p^{n-1}(1)}{\int_{S_p^{n-1}(1)} \mathrm{d}S_p^{n-1}(1)}. \tag{6}$$

Now, we will do over the above geometrical observations on hypersurfaces in the more technical context of *general submanifolds M^n in arbitrary curved Riemannian spaces $\widetilde{M}^{n+m}$*, *for all dimensions* $n \geq 2$ and *for all codimensions* $m \geq 1$. Therefore we will make use of the notion of *the angle between subspaces of inner product spaces*; (for some general treatments on angles between subspaces in inner product spaces, see e.g. [6, 10, 13, 15]). Let γ be an *arclength parameterized curve* on M^n passing through $p = \gamma(0)$ with velocity $\gamma'(0) = u$. For an arbitrary local normal orthonormal frame field $\{\xi_1 \dots, \xi_m\}$ around p on M^n in

$\widetilde{M}^{n+m}$, its vectors at the points $\gamma(s)$ of this curve, corresponding to arclengths s, will be denoted by $\xi_\alpha(s)$. And, $\widetilde{\nabla}-$*parallel transported vectors of* $\xi_\alpha(p)$ *from* p *to* q along γ will be denoted by $\xi^*_\alpha(s)$. By the metric character of the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{M}^{n+m}$, the vectors $\xi^*_1(q), \ldots, \xi^*_m(q)$ form an *orthonormal basis of an* m*-dimensional subspace* of $T_q\widetilde{M}^{n+m}$. Hence, the *angle* $\theta_u(s) \in [0, \frac{\pi}{2}]$ between the m-dimensional subspaces of $T_q\widetilde{M}^{n+m} = \mathbb{R}^{n+m}$ spanned respectively by $\{\xi_1(s), \ldots, \xi_m(s)\}$ and by $\{\xi^*_1(s), \ldots, \xi^*_m(s)\}$ is determined by

$$\cos^2\theta_u(s) = (\det M(s))^2\,, \tag{7}$$

whereby $M(s)$ is the $m \times m$ matrix with general elements

$$[M(s)]_{\alpha\beta} = \widetilde{g}\Big(\xi_\alpha(s), \xi^*_\beta(s)\Big), \tag{8}$$

(see e.g. [6, 12]). As before, we recall the classical Maclaurin expansion

$$\cos^2\theta_u(s) = 1 - \left(\frac{\mathrm{d}\theta_u}{\mathrm{d}s}(0)\right)^2 s^2 + O^{>2}(s), \tag{9}$$

whereas a straightforward calculation, essentially only using the formula for taking the derivative of a determinant and the properties of parallel transport, gives the following Maclaurin expansion

$$(\det M(s))^2 = 1 - g\left(A^C_p(u), u\right)\, s^2 + O^{>2}(s). \tag{10}$$

Hence we can obtain the following.

5 Theorem. *Let* M^n *be a submanifold of a Riemannian manifold* $\widetilde{M}^{n+m}$. *Let* $c(u)$ *be the Casorati curvature of* M^n *in* $\widetilde{M}^{n+m}$ *at a point* $p \in M^n$ *in a direction* $u \in T_pM^n$, *i.e.* $c(u) = g(A^C_p(u), u)$, *whereby* A^C_p *is the Casorati curvature operator of* M^n *in* $\widetilde{M}^{n+m}$ *at* p. *Let* $\theta_u(s)$ *be the angle between the normal space* $T^\perp_{\gamma(s)}M^n = \mathbb{R}^n$ *of* M^n *in* $\widetilde{M}^{n+m}$ *at the point* $\gamma(s)$ *of* M^n *and the* $\widetilde{\nabla}-$*parallel along* γ *in* $\widetilde{M}^{n+m}$ *transported space, from* $p = \gamma(0)$ *to* $\gamma(s)$, *starting from the normal space* $T^\perp_pM^n = \mathbb{R}^m$ *at* $p = \gamma(0)$, *whereby* γ *is any arclength parameterized curve in* M^n *passing through* $p = \gamma(0)$ *with velocity* $\gamma'(0) = u$. *Then*

$$c(u) = \left(\frac{\mathrm{d}\theta_u}{\mathrm{d}s}(0)\right)^2.$$

6 Remark. We can also consider the angle $\phi_u(s)$ between the tangent space $T_{\gamma(s)}M^n$ of M^n at the point $\gamma(s)$ of M^n and the $\widetilde{\nabla}$-parallel along γ in $\widetilde{M}^{n+m}$ transported space, from $p = \gamma(0)$ to $\gamma(s)$, starting from the tangent space T_pM^n at $p = \gamma(0)$. Since the angle between arbitrary subspaces U and W of an inner product space V equals the angle between their orthogonal complements $U^\perp$ and $W^\perp$ in V [10], $\cos^2\phi_u(s) = \cos^2\theta_u(s)$, because of the metrical character of the connection $\widetilde{\nabla}$, and thus

$$c(u) = \left(\frac{d\theta_u}{ds}(0)\right)^2 = \left(\frac{d\phi_u}{ds}(0)\right)^2.$$

Like in the case of hypersurfaces, we can calculate "the total Casorati curvature" and then analogously obtain the following.

7 Theorem. *The Casorati curvature $C(p)$ of a submanifold M^n in a Riemannian manifold $\widetilde{M}^{n+m}$ at any point $p \in M^n$ is the mean value of the Casorati curvatures $c(u)$ of M^n in $\widetilde{M}^{n+m}$ at p in all directions $u \in T_pM^n$:*

$$C(p) := \frac{1}{n}\|h\|^2 = \frac{\int_{S_p^{n-1}(1)} c(u)\,\mathrm{d}S_p^{n-1}(1)}{\int_{S_p^{n-1}(1)} \mathrm{d}S_p^{n-1}(1)}.$$

4 Casorati and normal curvatures

It seems natural to compare the normal curvatures $k(u) := \|h(u,u)\|$ with the Casorati curvatures $c(u)$ of submanifolds M^n in a Euclidean space $\mathbb{E}^{n+m}$ for all dimensions $n \geq 2$ and codimensions $m \geq 1$. For hypersurfaces M^n in $\mathbb{E}^{n+1}$, *the Euler principal directions* and *the Casorati principal directions are the same* at all points, $e_i = f_i$, and *the Euler principal curvatures k_j* and *the Casorati principal curvatures c_j* are related by $k(e_j)^2 = c(e_j)$, but for submanifolds M^n in $\mathbb{E}^{n+m}$ with codimension $m > 1$, this in general is no longer so. In this respect, we have the following.

8 Theorem. *Let M^n be a submanifold of $\mathbb{E}^{n+m}$. Then, the following three statements are equivalent: (1) M^n has flat normal connection in $\mathbb{E}^{n+m}$, i.e. $R^\perp = 0$, (2) at every point p of M^n there exists an orthonormal basis $\{e_1,\ldots,e_n\}$ of the tangent space T_pM^n such that $k(e_i)^2 = c(e_i)$ for every $i \in \{1,\ldots,n\}$, and, (3) at every point p of M^n there exist an orthonormal basis $\{e_1,\ldots,e_n\}$ and a normal frame $\{\xi_1,\ldots,\xi_m\}$ in the neighborhood of p on M^n in $\mathbb{E}^{n+m}$ such that $\kappa^2_{\sigma_{i,\alpha}} = \|\widetilde{\nabla}_{e_i}\xi_\alpha\|^2$, whereby σ_i denotes the normal section of M^n in $\mathbb{E}^{n+m}$ at p in the direction e_i, and where $\sigma_{i,\alpha}$ denotes the projection of $\sigma_i \subset \mathbb{R}^{1+m} = \mathrm{vect}\{e_i, \xi_1(p),\ldots,\xi_m(p)\}$ onto the plane $\mathbb{R}^2 = \mathrm{vect}\{e_i,\xi_\alpha(p)\}$ and $\kappa_{\sigma_{i,\alpha}}(p)$ is the curvature of the Euclidean planar curve $\sigma_{i,\alpha}$ at p.*

Proof. Let M^n be a submanifold in a Euclidean space $\mathbb{E}^{n+m}$. The normal curvature can be defined as the (first) curvature of $\kappa_\sigma(p)$ of the 1-dimensional normal section σ of M^n in $\mathbb{E}^{n+m}$ at $p = \sigma(0)$ in the direction $u = \sigma'(0) \in T_pM^n$. Clearly, as normal section, σ is a curve lying in the submanifold M^n and at the same time σ is a curve lying in the Euclidean space $\mathbb{E}^{1+m}$ which is spanned at p by the tangent vector u at p and by any orthonormal basis $\{\xi_1(p), \ldots, \xi_m(p)\}$ of the normal space $T_p^\perp M^n = \mathbb{R}^m$ of M^n in $\mathbb{E}^{n+m}$ at p. Denote by σ_α the projection of $\sigma \subset \mathbb{R}^{1+m} = \mathrm{vect}\{u, \xi_1(p), \ldots, \xi_m(p)\}$ onto the plane $\mathbb{R}^2 = \mathrm{vect}\{u, \xi_\alpha(p)\}$. It follows easily that the curvature of the planar curve σ_α is given by

$$\kappa_{\sigma_\alpha} = g(A_\alpha(u), u). \tag{11}$$

(2) $\Rightarrow$ (1). Since at every point p of M^n there exists an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM^n such that $k(e_1)^2 = c(e_1), \ldots, k(e_n)^2 = c(e_n)$, it follows that

$$\sum_\alpha g(A_\alpha(e_i), e_i)^2 = \sum_{\alpha, j\neq i} g(A_\alpha(e_i), e_j)^2 + \sum_\alpha g(A_\alpha(e_i), e_i)^2.$$

Hence we obtain that

$$g(A_\alpha(e_i), e_j) = 0$$

for every α and every i, j with $i \neq j$. So $\{e_1, \ldots, e_n\}$ is an orthonormal basis that diagonalizes all the shape operators A_α simultaneously, and so $R^\perp = 0$ at every point p.

(1) $\Rightarrow$ (3). Since M^n is a submanifold with flat normal connection in $\mathbb{E}^{n+m}$, there exists a parallel orthonormal normal frame in the neighborhood of p on M^n, i.e. a normal frame $\{\xi_1, \ldots, \xi_m\}$ for which $\widetilde{g}(\xi_\alpha, \xi_\beta) = \delta_{\alpha\beta}$ and $\nabla^\perp \xi_\alpha = 0$, and at any point p there exists an orthonormal tangent basis $\{e_1, \ldots, e_n\}$ that diagonalizes simultaneously all the shape operators $A_1, \ldots, A_m$ with respect to $\xi_1(p), \ldots, \xi_m(p)$. Thus it follows from (11) and the formula of Weingarten that

$$\|\widetilde{\nabla}_{e_i}\xi_\alpha\|^2 = \|A_\alpha(e_i)\|^2 = \sum_j g(A_\alpha(e_i), e_j)^2 = g(A_\alpha(e_i), e_i)^2 = \kappa_{\sigma_{i,\alpha}}(p)^2,$$

whereby $\sigma_{i,\alpha}$ is the planar curve defined as above with curvature $\kappa_{\sigma_{i,\alpha}}(p)$ at p.

(3) $\Rightarrow$ (2). At a point p of M^n, we have an orthonormal tangent basis $\{e_1, \ldots, e_n\}$ and an orthonormal normal frame $\{\xi_1, \ldots, \xi_m\}$ in the neighborhood of p such that

$$\kappa_{\sigma_{i,\alpha}}(p)^2 = \|\widetilde{\nabla}_{e_i}\xi_\alpha\|^2.$$

Thus, we have that

$$g(A_\alpha(e_i), e_i)^2 = g(A_\alpha^2(e_i), e_i) + \|\nabla^\perp_{e_i}\xi_\alpha\|^2$$

for every α and every i. Hence, we obtain that $g(A_\alpha(e_i), e_j) = 0$ for every α and for every i, j with $i \neq j$, from which it easily follows that $k(e_i)^2 = c(e_i)$. $\boxed{QED}$

5 Some closing remarks

The n mutually orthogonal extrinsic principal directions as determined by the critical values of the Casorati curvatures on n-dimensional submanifolds M^n in Riemannian manifolds $\widetilde{M}^{n+m}$ may well deserve attention in future studies of the geometry of submanifolds, in our opinion. For instance, these directions turn out to be very relevant indeed in the study of the so-called ideal submanifolds; (for a recent survey on the latter essential topic in geometry and its applications, see [3]).
Already well before the present geometric study of the Casorati curvatures of submanifolds in general, for some surfaces M^2 occurring in various studies on computer and human vision, the above curvature $C : M^2 \to \mathbb{R}$ has proven its value e.g. in works of Koenderink-van Doorn [8,9] and of one of the authors [14].

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References

[1] F. Casorati: *Mesure de la courbure des surfaces suivant l'idée commune*, Acta Math. no. 1 **14** (1890), 95–110.

[2] B. Y. Chen: Geometry of submanifolds, Marcel Dekker, New York, (1974).

[3] B. Y. Chen: *δ-invariants, inequalities of submanifolds and their applications*, "Topics in Differential Geometry", eds. A. Mihai, I. Mihai and R. Miron, Editura Academiei Romane (Publishing House of the Romanian Academy), (2008).

[4] S. Decu, S. Haesen and L. Verstraelen: *Optimal inequalities involving Casorati curvatures*, Bulletin of the Transilvania University of Brasov **14**(49)-B, supplement (2007), 85–94.

[5] S. Decu, S. Haesen and L. Verstraelen: *Optimal inequalities characterising quasi-umbilical submanifolds*, to appear in J. Inequal. Pure and Appl. Math. (2008).

[6] H. Gunawan, O. Neswan and W. Setya-Budhi: *A formula for angles between subspaces of inner product spaces*, Beiträge Algebra Geom. no. 2 **46** (2005), 311–320.

[7] C. Jordan: *Généralisation du théoréma d'Euler sur la courbure des surfaces*, C.R. Acad. Sc. **79** (1874), 909–912.

[8] J. Koenderink: Solid shape, MIT Press, Cambridge, (1990).

[9] J. J. Koenderink and A. J. van Doorn: *Surface shape and curvature scales*, Image Vision Computing **10** (1992), 557–564.

[10] I. Risteski, K. Trenčevski: *Principal values and principal subspaces of two subspaces of vector spaces with inner product*, Beiträge Algebra Geom. no. 1 **42** (2001), 289–300.

[11] K. Trenčevski: *New approach for the submanifolds of the Euclidean space*, Balkan J. Geom. Appl. no. 1 **2** (1997), 117–127.

[12] K. Trenčevski *Principal directions for submanifolds imbedded in Euclidean spaces of arbitrary codimension*, Proceedings of the Third International Workshop on Differential Geometry and its Applications and the First German-Romanian Seminar on Geometry (Sibiu, 1997), Gen. Math. **5** (1997), 385–392.

[13] K. Trenčevski: *Geometrical interpretation of the principal directions and principal curvatures of submanifolds*, Differ. Geom. Dyn. Syst. no. 1 **2** (2000), 50–58.

[14] L. Verstraelen: *The geometry of eye and brain*, Soochow J. Math. no. 3 **30** (2004), 367–376.

[15] Y. C. Wong: Linear geometry in Euclidean 4-space, Southeast Asian Mathematical Society, Hong Kong, (1977).