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Groups with Large Centralizer Subgroups

Maria De Falco
Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, via Cintia, I - 80126 Napoli (Italy)
mdefalco@unina.it

Francesco de Giovanni
Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, via Cintia, I - 80126 Napoli (Italy)
degiovan@unina.it

Carmela Musella
Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II, via Cintia, I - 80126 Napoli (Italy)
cmusella@unina.it

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Abstract. This article describes the structure of locally graded groups in which every (infinite) proper self-centralizing subgroup is abelian.

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1 Introduction

We shall say that a subgroup X of a group G is *self-centralizing* (in G) if X contains its centralizer $C_G(X)$. Obvious examples of self-centralizing subgroups are provided by maximal abelian subgroups of arbitrary groups and by the Fitting subgroup of any soluble group. It follows immediately from Zorn's Lemma that if a group G does not contain proper self-centralizing subgroups, then G is abelian. The aim of this paper is to study groups for which the set of self-centralizing subgroups is small in some sense.

In Section 2 a full description will be given of locally graded groups in which every proper self-centralizing subgroup is abelian; here a group G is said to be *locally graded* if each finitely generated non-trivial subgroup of G contains a proper subgroup of finite index. We work within the universe of locally graded groups in order to avoid Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) and other similar pathological examples. The last section is devoted to the study of locally graded groups whose infinite proper self-centralizing subgroups are abelian.

Recall that a group is *metahamiltonian* if all its non-abelian subgroups are normal. Groups with such property were introduced and investigated by G.M. Romalis and N.F. Sesekin ([6], [7], [8]), who proved in particular that (generalized) soluble metahamiltonian groups have finite commutator subgroup. Metahamiltonian groups are naturally involved in the study of groups with few self-centralizing subgroups; in fact, it is easy to show that groups whose proper self-centralizing subgroups are abelian must be metahamiltonian. Note also that in Section 3 a result of S.N. Černikov [1] concerning locally graded groups whose infinite non-abelian subgroups are normal will be used.

Most of our notation is standard and can be found in [5].

2 Centralizers of non-abelian subgroups

Let $\mathfrak{Q}$ be the class consisting of all groups whose proper self-centralizing subgroups are abelian (i.e. a group G has the property $\mathfrak{Q}$ if and only if $C_G(X)$ is not contained in X for each proper non-abelian subgroup X of G). Of course, $\mathfrak{Q}$ contains all abelian groups, and also Tarski groups have the property $\mathfrak{Q}$. The main result of this section will characterize locally graded $\mathfrak{Q}$-groups.

We begin with the following obvious property.

1 Lemma. *Let G be a group and let X be a subgroup of G. Then the normalizer $N_G(X)$ is a self-centralizing subgroup of G.*

Proof. Clearly,

$$C_G(N_G(X)) \leq C_G(X) \leq N_G(X),$$

and hence the subgroup $N_G(X)$ is self-centralizing in G. QED

If $\mathfrak{X}$ is any class of groups, we will denote as usual by $\mathbf{L}\mathfrak{X}$ the class consisting of all groups with a local system by $\mathfrak{X}$-subgroups (i.e. $G \in \mathbf{L}\mathfrak{X}$ if and only if every finite subset of G is contained in some $\mathfrak{X}$-subgroup of G); the group class $\mathfrak{X}$ is *local* if $\mathbf{L}\mathfrak{X}$=$\mathfrak{X}$. We shall say that a local group class $\mathfrak{X}$ is *centrally stable* if it satisfies the following conditions:

- $\mathfrak{X}$ is closed with respect to normal subgroups (i.e. every normal subgroup of an arbitrary $\mathfrak{X}$-group belongs to $\mathfrak{X}$);
- if G is any group and X is an $\mathfrak{X}$-subgroup of G, then $\langle g, X\rangle \in \mathfrak{X}$ for each element $g \in C_G(X)$.

Of course, for each non-negative integer c the class $\mathfrak{N}_c$ of nilpotent groups with class at most c is centrally stable; in particular, the class $\mathfrak{A}$ of abelian groups has such property.

2 Lemma. *Let $\mathfrak{X}$ be a class of groups which is closed with respect to normal subgroups, and let G be a group whose proper self-centralizing subgroups belong to $\mathfrak{X}$. Then every non-normal subgroup of G is an $\mathfrak{X}$-group. Moreover, if the group class $\mathfrak{X}$ is centrally stable, then G contains a maximal subgroup which is an $\mathfrak{X}$-group.*

Proof. Let X be any subgroup of G which is not in $\mathfrak{X}$. As $\mathfrak{X}$ is closed with respect to normal subgroups, the normalizer $N_G(X)$ cannot belong to $\mathfrak{X}$; moreover, $N_G(X)$ is self-centralizing in G, and so it follows that $N_G(X) = G$ and X is normal in G. Suppose now that $\mathfrak{X}$ is also centrally stable, so that in particular by Zorn's Lemma G contains a maximal $\mathfrak{X}$-subgroup M and $C_G(M) \leq M$. Let H be any subgroup of G which properly contains M. Then

$$C_G(H) \leq C_G(M) \leq M < H,$$

and hence H is a self-centralizing subgroup of G which is not in $\mathfrak{X}$, so that $H = G$ and M is a maximal subgroup of G. QED

The above lemma provides information on the structure of groups whose proper self-centralizing subgroups belong to a given group class $\mathfrak{X}$, for several different choices of $\mathfrak{X}$. In particular, for $\mathfrak{X} = \mathfrak{A}$ we have the following consequence of Lemma 2.

3 Corollary. *Let G be a $\mathfrak{Q}$-group. Then G is metahamiltonian and contains a maximal subgroup which is abelian.*

Since it is well known that abelian-by-finite groups with finite commutator subgroup are central-by-finite, we also obtain the following result.

4 Corollary. *Let G be a locally graded $\mathfrak{Q}$-group. Then the factor group $G/Z(G)$ is finite.*

Proof. The group G is metahamiltonian by Corollary 3, so that in particular its commutator subgroup G' is finite. Moreover, G contains a maximal subgroup M which is abelian, and of course the index $|G : M|$ is finite. Thus G is abelian-by-finite and hence $G/Z(G)$ is finite. QED

5 Lemma. *A locally graded group G belongs to the class $\mathfrak{Q}$ if and only if $G = XZ(G)$ for each non-abelian subgroup X of G.*

Proof. Suppose first that G is a $\mathfrak{Q}$-group, and assume for a contradiction that G contains a non-abelian subgroup X such that $XZ(G) \neq G$. As $G/Z(G)$ is finite by Corollary 4, there exists a maximal subgroup M of G containing $XZ(G)$. By hypothesis, M is not self-centralizing and so we may consider an element g of $C_G(M) \setminus M$; then $G = \langle g, M\rangle$ and hence g belongs to $Z(G)$. This contradiction proves that $G = XZ(G)$ for every non-abelian subgroup X of G.

Conversely, suppose that G satisfies the condition of the statement, and let X be any proper non-abelian subgroup of G. Then $G = XZ(G)$, so that the centre $Z(G)$ is not contained in X and in particular X is not self-centralizing. Therefore G is a $\mathfrak{Q}$-group. QED

6 Corollary. *A locally graded group G belongs to the class $\mathfrak{Q}$ if and only if all proper subgroups of G containing $Z(G)$ are abelian.*

It is known that a locally graded group G is metahamiltonian if and only every non-abelian subgroup of G contains the commutator subgroup G' of G (see [3]). Thus the above corollary provides further evidence of the fact that the centre and the commutator subgroup of a group have dual behaviours. In fact, since any $\mathfrak{Q}$- group is metahamiltonian, we obtain the following information.

7 Corollary. *Let G be a locally graded group. If all proper subgroups of G containing the centre $Z(G)$ are abelian, then every non-abelian subgroup of G contains the commutator subgroup G' and in particular all proper subgroups of G' are abelian.*

We can now describe locally graded $\mathfrak{Q}$-groups, starting with the nilpotent case.

8 Theorem. *Let G be a nilpotent group. Then G belongs to the class $\mathfrak{Q}$ if and only either it is abelian or the factor group $G/Z(G)$ has order p^2 for some prime number p.*

Proof. Suppose first that G is a non-abelian $\mathfrak{Q}$-group. By Corollary 4 we have that $G/Z(G)$ is a finite (non-cyclic) group, and so it contains two distinct maximal subgroups $M_1/Z(G)$ and $M_2/Z(G)$. Moreover, it follows from Lemma 5 that M_1 and M_2 are abelian, so that $M_1 \cap M_2 = Z(G)$ and $G/Z(G)$ has order p^2 for some prime number p.

Conversely, assume that $G/Z(G)$ has order p^2 for some prime number p. If X is any non-abelian subgroup of G, the group $XZ(G)/Z(G)$ cannot be cyclic and hence $XZ(G) = G$. Therefore G belongs to $\mathfrak{Q}$ by Lemma 5. QED

9 Theorem. *Let G be a locally graded non-nilpotent group. Then G belongs to the class $\mathfrak{Q}$ if and only if $G = A \ltimes P$, where P is a finite abelian group of prime exponent p and $A = \langle a, Z(G)\rangle$ for some element a acting irreducibly on P, and $\langle a\rangle \cap Z(G) = \langle a^q\rangle$ for some prime number $q \neq p$.*

Proof. Suppose first that G is a $\mathfrak{Q}$-group, so that in particular $G/Z(G)$ is finite by Corollary 4. If Q is any Sylow subgroup of G, it follows that the product $QZ(G)$ is a proper subgroup of G and hence Q must be abelian by Lemma 5. Since G is metahamiltonian, an application of Theorem 2 of [3] yields that $G = A \ltimes P$, where P is a finite abelian group of prime exponent p and $A = \langle a, Z(G)\rangle$ with $\langle a\rangle \cap Z(G) = \langle a^q\rangle$ for some integer $q > 1$ which is prime to p; moreover, a^k acts irreducibly on P for each positive integer $k < q$. Assume for a contradiction that q is not a prime number, so that there exists a positive divisor r of q such that $\langle a^q\rangle < \langle a^r\rangle < \langle a\rangle$. Thus the subgroup $\langle a^r, P\rangle$ is not abelian and hence

$G = \langle a^r, P\rangle Z(G)$, a contradiction since $\langle a^r, Z(G)\rangle$ is properly contained in A. Therefore q is a prime number.

Conversely, suppose that $G = A \ltimes P$ has the structure described in the statement, so that in particular G is metahamiltonian (see [3], Theorem 2). Let X be any proper non-abelian subgroup of G. Since P is a minimal normal subgroup of G, it follows that $P = G'$ is contained in X (see [3], Theorem 3). Assume for a contradiction that X contains also $Z(G)$; then $PZ(G) \leq X$ and $|G : PZ(G)| = q$, so that $X = PZ(G)$ is abelian. This contradiction shows that $Z(G)$ is not contained in X, so that in particular X is not self-centralizing. Therefore G belongs to the class $\mathfrak{Q}$. QED

Finally, we note that Corollary 4 can be extended to the case of groups with finitely many self-centralizing non-abelian subgroups. Since every self-centralizing subgroup contains the centre, it is clear that if G is a central-by-finite group, then the set of all self-centralizing subgroups of G is finite.

10 Theorem. *Let G be a locally graded group with finitely many self-centralizing non-abelian subgroups. Then the factor group $G/Z(G)$ is finite.*

Proof. By Lemma 1 the group G has finitely many normalizers of non-abelian subgroups, so that its commutator subgroup G' is finite (see [2]). Let A be a maximal abelian subgroup of G. If X is any subgroup of G such that $A \leq X$, we have $C_G(X) \leq X$. It follows that the set of all subgroups of G containing A is finite. Then G/AG' is finite and hence the index $|G : A|$ is also finite. Therefore G is abelian-by-finite and so $G/Z(G)$ is finite. QED

Observe that the assumption that G is locally graded can be removed from the above statement, provided that G has finitely many self-centralizing subgroups. In fact, in this case G has finitely many normalizers of subgroups and a theorem of Y.D. Polovickiĭ [4] can be applied.

3 Centralizers of infinite non-abelian subgroups

The consideration of Tarski groups shows that the condition that the group is locally graded is necessary in our next result.

11 Lemma. *Let G be an infinite locally graded group whose proper self-centralizing subgroups are finite. Then G is abelian.*

Proof. Assume for a contradiction that G is not abelian. Let A be any maximal abelian subgroup of G; then $C_G(A) = A$ and hence A is finite. In particular, G is periodic. Moreover, it follows from the Hall-Kulatilaka-Kargapolov theorem (see [5] Part 1, Theorem 3.43) that G is not locally finite, so that it contains an infinite finitely generated subgroup E. If X is any subgroup of finite index of E, the normalizer $N_G(X)$ is an infinite self-centralizing subgroup, so that $N_G(X) = G$ and X is normal in G; in particular, all subgroups of finite index of E are normal. Let J be the finite residual of E; then E/J is nilpotent and so finite. Since G is locally graded, it follows that E itself is finite. This contradiction proves the statement. QED

Let $\mathfrak{Q}_\infty$ be the class consisting of all groups in which all infinite proper self-centralizing subgroups are abelian. Applying the argument used in the proof of the first part of Lemma 2 to the class $\mathfrak{A}_\infty$ of all infinite abelian groups, the following result can be proved.

12 Lemma. *Let G be an infinite $\mathfrak{Q}_\infty$-group. Then all infinite non-abelian subgroups of G are normal.*

Groups in which every infinite non-abelian subgroup is normal have been described by S.N. Černikov [1]. We state here his main result as a lemma; it will be used in order to describe (generalized soluble) $\mathfrak{Q}_\infty$-groups.

13 Lemma. *Let G be a locally graded group in which every infinite non-abelian subgroup is normal. Then either the commutator subgroup G' of G is finite or G is a Černikov group whose divisible part contains no infinite proper G-invariant subgroups.*

Our next theorem deals with the case of finite-by-abelian groups, and in particular it applies to metahamiltonian $\mathfrak{Q}_\infty$-groups which are locally graded.

14 Theorem. *Let G be an infinite $\mathfrak{Q}_\infty$-group with finite commutator subgroup. Then G belongs to $\mathfrak{Q}$.*

Proof. Assume for a contradiction that G is not a $\mathfrak{Q}$-group, so that it contains a proper non-abelian subgroup X such that $C_G(X) \leq X$. Thus X is finite, so that in particular the centre $Z(G)$ of G is finite and hence $Z_2(G)$ has finite exponent (see [5] Part 1, Theorem 2.23). On the other hand, as G' is finite, the index $|G : Z_2(G)|$ is likewise finite (see [5] Part 1, p.113). It follows that G has finite exponent and so the infinite abelian group G/G' contains a subgroup H/G' of finite index such that $|G/H| > |X|$. Then XH is an infinite proper non-abelian subgroup and

$$C_G(XH) \leq C_G(X) \leq X < XH,$$

and this contradiction proves the theorem. QED

We can now complete the description of locally graded $\mathfrak{Q}_\infty$-groups.

15 Theorem. *Let G be a locally graded group with infinite commutator subgroup. Then G has the property $\mathfrak{Q}_\infty$ if and only if G is a Černikov group whose divisible part J contains no infinite proper G-invariant subgroups and the factor group $G/JZ(G)$ has prime order.*

Proof. Suppose first that G is a $\mathfrak{Q}_\infty$-group. As G' is infinite, it follows from Lemma 12 and Lemma 13 that G is a Černikov group and its divisible part J has no infinite proper G-invariant subgroups. Moreover, J cannot be contained in $Z(G)$, and hence the centralizer $C_G(J)$ is an infinite proper subgroup of G. On the other hand,

$$C_G(C_G(J)) \leq C_G(J)$$

and so $C_G(J)$ must be abelian. Let X be any subgroup of G properly containing $C_G(J)$. Then X is not abelian and

$$C_G(X) \leq C_G(J) \leq X,$$

so that $X = G$. It follows that $C_G(J)$ is a maximal subgroup of G and the index $|G : C_G(J)|$ is a prime number. Let x be an element of G such that $G = \langle x, C_G(J)\rangle$ and consider the infinite non-abelian subgroup $\langle x, JZ(G)\rangle$. Then

$$C_G(\langle x, JZ(G)\rangle) \leq C_G(J) \cap C_G(x) = Z(G) \leq \langle x, JZ(G)\rangle,$$

and hence $G = \langle x, JZ(G)\rangle$ by the property $\mathfrak{Q}_\infty$. Therefore

$$C_G(J) = \langle x, JZ(G)\rangle \cap C_G(J) = JZ(G)(\langle x\rangle \cap C_G(J)) = JZ(G),$$

so that the group $G/JZ(G)$ has prime order.

Assume conversely that G is a Černikov group satisfying the conditions of the statement, and let X be any infinite non-abelian subgroup of G such that $C_G(X) \leq X$. Then $Z(G)$ lies in X and X is not contained in $JZ(G)$, so that

$$G = JZ(G)X = JX.$$

As X is infinite, its divisible part Y is likewise infinite and of course Y is a normal subgroup of G. It follows that $J = Y \leq X$ and hence $X = G$. Therefore all infinite proper self-centralizing subgroups of G are abelian and G has the property $\mathfrak{Q}_\infty$. QED

Our last result provides further information on the structure of locally graded $\mathfrak{Q}_\infty$-groups.

16 Corollary. *Let G be a locally graded $\mathfrak{Q}_\infty$-group with infinite commutator subgroup. Then G is a Černikov group and G' is the divisible part of G.*

Proof. By Theorem 15, G is a Černikov group and its divisible part J has no infinite proper G-invariant subgroups. Thus every infinite normal subgroup of G contains J and in particular $J \leq G'$. On the other hand, Theorem 15 also yields that the factor group $G/JZ(G)$ has prime order, so that G/J is central-by-cyclic and hence abelian. Therefore $G' = J$. QED

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