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Weakly compact composition operators between weighted spaces

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Abstract. Our aim in this paper is to study weak compactness of composition operators between weighted spaces of holomorphic functions on the unit ball of a Banach space.

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To the memory of our friend Klaus Floret

1 Weighted Fréchet spaces

Let X be a Banach space and B its open unit ball. We consider a countable family V of bounded and continuous functions $v : B \longrightarrow]0, +\infty[$. Any such function is called a weight. Weighted spaces of holomorphic functions defined by such families were first defined by Bierstedt, Bonet and Galbis in [3] for open subsets of $\mathbb{C}^n$ (see also [4–7,9]). García, Maestre and Rueda defined and studied in [12] analogous spaces of functions defined on Banach spaces. We recall now the basic definitions and results.

The space of all holomorphic functions $f : B \to \mathbb{C}$ is denoted by $H(B)$. We consider the space

$$HV(B) = \{f \in H(B) \ : \ p_v(f) = \sup_{x \in B} v(x)|f(x)| < \infty \ \text{ for all } \ v \in V\}$$

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We endow $HV(B)$ with the Fréchet topology τ_V generated by the family of seminorms $(p_v)_{v\in V}$. The family of weights $V = (v_n)_n$ can always be chosen to be increasing. When V consists only of one weight v, the corresponding space is denoted $H_v(B)$ and it is a Banach space whose open unit ball is denoted by B_v. We refer to [12] or [18] for a study of the properties of these spaces.

A set $A \subset B$ is said to be B-bounded if there exists $0 < r < 1$ such that $A \subset rB$. The subspace of $H(B)$ of those functions that are bounded on the B-bounded sets is denoted by $H_b(B)$. The space of bounded holomorphic functions is denoted by $H^\infty(B)$ and, as usual, for $h \in H^\infty(B)$ we write $\|h\|_\infty = \sup_{x\in B} |h(x)|$.

Following [12, Definition 1], we say that a family V of weights defined on B satisfies *Condition I* if for each B-bounded set $A \subseteq B$ there exists $v \in V$ such that $\inf\{v(x) : x \in A\} > 0$. If V satisfies Condition I, then $HV(B) \subseteq H_b(B)$ and τ_V is stronger than τ_b, the topology of the uniform convergence over the B-bounded sets.

A weight is *radial* if $v(\lambda x) = v(x)$ for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and every $x \in B$.

Following the standard notation if X is a Banach space we will denote its dual by X^*. If E is a Fréchet space, its dual will be denoted by E'.

In this article we continue our work [13] on compactness of composition operators between weighted spaces of holomorphic functions on the unit ball of a Banach space. We refer to the introduction of that paper for information and motivations. We want to emphasize here that our work is based upon [6,8]. Our main results on weak compactness of composition operators between Banach weighted spaces of holomorphic functions are Theorem 5, 6 and Corollary 7. We apply these results in Section 3 to obtain Theorem 17, a positive result of weak compactness of composition operators between Fréchet weighted spaces of holomorphic functions.

Throughout this paper X, Y will denote Banach spaces and B_X, B_Y their open unit balls. Note that in this setting, a weight v satisfies Condition I if $\inf_{x\in rB_X} v(x) > 0$ for every $0 < r < 1$. Given any weight v, following [4], we consider an associated growth condition $u : B_X \longrightarrow]0, +\infty[$ defined by $u(x) = \frac{1}{v(x)}$. From this, $\tilde{u} : B_X \longrightarrow]0, +\infty[$ is defined by

$$\tilde{u}(x) = \sup_{f\in B_v} |f(x)|$$

which produces an associated weight $\tilde{v} = 1/\tilde{u}$. All these functions were defined by Bierstedt, Bonet and Taskinen for open subsets of $\mathbb{C}^n$ in [4]. In [4, Proposition 1.2], the following relations between weights for open sets on $\mathbb{C}^n$ are proved. The same arguments work for the unit ball of a Banach space.

1 Proposition. *Let X be a Banach space and v a weight defined on B_X. The following hold,*

(i) $0 < v \leq \tilde{v}$ *and* $\tilde{v}$ *is bounded and continuous; i.e.* $\tilde{v}$ *is a weight.*

(ii) $\tilde{u}$ *(resp.* $\tilde{v}$*) is radial and decreasing or increasing with respect to the norm whenever* u *(resp.* v*) is so.*

(iii) $p_v(f) \leq 1 \Leftrightarrow p_{\tilde{v}}(f) \leq 1$.

(iv) For each $x \in B_X$ *there exists* $f_x \in B_v$ *such that* $\tilde{u}(x) = |f_x(x)|$.

A linear mapping between Banach spaces, $T : X \to Y$, is called *compact, weakly compact or Rosenthal* if $T(B_X)$ is, respectively, relatively compact, relatively weakly compact or conditionally weakly compact. A subset $A \subset X$ is called *conditionally weakly compact* if every sequence in A admits a weak Cauchy subsequence.

2 Weak compactness of composition operators on Banach spaces

Given two Banach spaces X, Y, let $\phi : B_Y \to B_X$ be a holomorphic mapping. The *composition operator* associated to ϕ is defined as

$$C_\phi : H(B_X) \longrightarrow H(B_Y) \quad , \quad f \rightsquigarrow C_\phi(f) = f \circ \phi.$$

This operator is linear and $\tau_0 - \tau_0$-continuous. Now, given any two weights v, w we consider the operator $C_\phi : H_v(B_X) \to H_w(B_Y)$ whenever this is well defined. This happens if and only if the operator is continuous [13, Remark 2.1].

Continuity and compactness of these operators have been studied in [7] when $X = Y = \mathbb{C}$, in [9] for arbitrary open sets in $\mathbb{C}$ and in [13,15] for the infinite dimensional case.

Weak compactness of composition operators was studied in [6] for the one dimensional case. There the following situation is considered; let G_1 and G_2 be two open connected domains in $\mathbb{C}$ such that $\mathbb{C}^* \setminus G_1$ has no one-point component and let $\phi : G_2 \to G_1$ be a holomorphic mapping. Given v and w weights on G_1 and G_2 respectively, if $C_\phi : H_v(G_1) \to H_w(G_2)$ is weakly compact or Rosenthal then C_ϕ is compact ([6, Theorem 1]).

Weakly compact composition operators on $H^\infty(B_X)$ were studied in [11]. In [11, Proposition 2] it is shown that if $C_\phi : H^\infty(B_X) \to H^\infty(B_Y)$ is Rosenthal or compact, then $\phi(B_Y)$ lies strictly inside B_X. The proof of this result is clearly inspired by the proof of [6, Theorem 1]. Following the same trends of ideas we will give an analogous result for general weights which strictly includes [11, Proposition 2].

The proof of the following lemma is very similar to that of [20, Section 2.4] and [7, Lemma 3.1].

2 Lemma. *Let $C_\phi : H_v(B_X) \longrightarrow H_w(B_Y)$ continuous. The following are equivalent,*

(i) C_ϕ is compact.

(ii) Each bounded net $(f_\alpha)_{\alpha\in A} \subseteq H_v(B_X)$ such that $\{f_\alpha : \alpha \in A\}$ is a countable set and $f_\alpha \xrightarrow{\tau_0} 0$ satisfies that $p_w(C_\phi f_\alpha) \longrightarrow 0$.

If, furthermore, X is separable, then (i) and (ii) are equivalent to

(iii) Each bounded sequence $(f_n)_n \subseteq H_v(B_X)$ such that $f_n \xrightarrow{\tau_0} 0$ satisfies that $p_w(C_\phi f_n) \longrightarrow 0$.

Proof. Let us suppose first that C_ϕ is compact. Then $C_\phi(B_v)$ is relatively compact in $H_w(B_Y)$. Let us take a bounded net $(f_\alpha)_{\alpha\in A} \subseteq H_v(B_X)$ with $\{f_\alpha : \alpha \in A\}$ countable such that $f_\alpha \longrightarrow 0$ in τ_0. Since C_ϕ is τ_0-τ_0-continuous, $C_\phi f_\alpha \xrightarrow{\tau_0} 0$. Convergence in p_w implies that of τ_0, hence each p_w-convergent subnet of $(C_\phi f_\alpha)_\alpha$ will converge to 0.

If $(p_w(C_\phi f_\alpha))_\alpha$ does not converge to 0, there exists a subnet $(f_\beta)_\beta$ and $c > 0$ such that $p_w(C_\phi f_\beta) \geq c$ for all β. But $(f_\beta)_\beta$ is bounded and C_ϕ is compact, therefore $(C_\phi f_\beta)_\beta$ is relatively compact and has a convergent subnet. This new subnet is also a subnet of $(C_\phi f_\alpha)_\alpha$ and it must converge to 0. This gives a contradiction. So, $\lim_\alpha p_w(C_\phi f_\alpha) = 0$.

Assume (ii) holds. Let $(f_n)_n \subseteq B_v$. By [18] B_v is τ_0-compact, in particular it is τ_0-bounded. Then, $(f_n)_n$ is τ_0-bounded and, by Montel's Theorem, there is a subnet $(g_\alpha)_{\alpha\in A}$ converging in τ_0 to some $g \in H(B_X)$. For each $x \in B_X$ and α we have $v(x)|g_\alpha(x)| \leq p_v(g_\alpha) \leq 1$. Hence

$$1 \geq \lim_\alpha v(x)|g_\alpha(x)| = v(x) \lim_\alpha |g_\alpha(x)| = v(x)|g(x)|.$$

This implies $\sup_{x\in B_X} v(x)|g(x)| < \infty$ and $g \in H_v(B_X)$.

Let us note that for each α there is n such that $g_\alpha = f_n$. This means that $\{g_\alpha : \alpha \in A\}$ is countable. Thus $(g_\alpha - g)_\alpha$ is a bounded net in $H_v(B_X)$ with $\{g_\alpha - g : \alpha \in A\}$ countable and $(g_\alpha - g) \longrightarrow 0$ in τ_0. By hypothesis $\lim_\alpha p_w(C_\phi(g_\alpha - g)) = 0$. This implies that $C_\phi(B_v)$ is relatively compact and C_ϕ is compact.

If X is separable, then Montel Theorem states that every τ_0-bounded sequence in $H(B_X)$ has a τ_0-convergent subsequence; this allows to show that in this case (iii) implies (i). QED

3 Proposition. *Let X, Y be Banach spaces and $\phi : B_Y \to B_X$ a holomorphic mapping such that $\phi(B_Y) \cap rB_X$ is relatively compact for every $0 < r < 1$. If the operator $C_\phi : H_v(B_X) \to H_w(B_Y)$ is not compact then it is neither weakly compact nor Rosenthal.*

PROOF. If the composition operator is not compact, by Lemma 2, there is a bounded net $(g_\alpha)_{\alpha\in A} \subseteq H_v(B_X)$ with $\{g_\alpha : \alpha \in A\}$ countable and τ_0-converging to 0 such that $(p_w(C_\phi g_\alpha))_\alpha$ does not converge to 0. We can find a subnet $(g_\beta)_\beta$ and $c > 0$ so that $p_w(C_\phi g_\beta) > c$ for all β. Note that $\{g_\beta\}$ is countable. Let us write $\{g_\beta\} = \{f_n : n \in \mathbb{N}\}$. Then we have $(f_n)_n$, bounded, such that $p_w(C_\phi f_n) > c$. For each n we can find $y_n \in B_Y$ such that

$$w(y_n)|f_n(\phi(y_n))| > c > 0. \tag{1}$$

Let us see that $\lim_n \|\phi(y_n)\| = 1$. If not, there are a subsequence $(y_{n_k})_k$ and $0 < r < 1$ such that $\phi(y_{n_k}) \in rB_X$ for all k. But $\phi(B_Y) \cap rB_X$ is relatively compact in B_X; hence we can extract a subsequence, which we denote in the same way, so that $(\phi(y_{n_k}))_k$ converges to some x_0. Since $K := \bigcup_k\{\phi(y_{n_k})\}\cup\{x_0\}$ is compact and $(g_\beta)_\beta$ is τ_0-null, there exists β_0 such that, for $\beta \geq \beta_0$,

$$\sup_{x\in K} |g_\beta(x)| < \frac{c}{\|w\|_\infty}. \tag{2}$$

From this, $w(y_{n_k})|g_\beta(x)| < c$ for every $x \in K$ and all k. But $g_\beta = f_{n_k}$ for some k; then $w(y_{n_k})|f_{n_k}(\phi(y_{n_k}))| < c$. This gives a contradiction. Therefore we have a bounded sequence $(f_n)_n \subseteq H_v(B_X)$ and $(y_n)_n \subseteq B_Y$ satisfying that $|f_n(\phi(y_n))|w(y_n) > c$ and $\lim_n \|\phi(y_n)\| = 1$. Now, by the proof of [1, Theorem 10.5], there is a $g \in H^\infty(B_X)$ and a subsequence $(\phi(y_{n_k}))_k$ so that $(g(\phi(y_{n_k})))_k$ is an interpolating sequence for $H^\infty(\mathbb{D})$. By [14, p. 294] we can find a sequence $(h_m)_m \subseteq H^\infty(\mathbb{D})$ and $M > 0$ so that, for all $z \in \mathbb{D}$,

$$\sum_m |h_m(z)| \leq M \quad , \quad h_m(g(\phi(y_{n_k}))) = \delta_{mk},$$

where δ_{mk} stands for the Dirac's delta. Let us define $T : \ell_\infty \to H_v(B_X)$ by

$$T((\xi_m)_m)(x) = \sum_{m=1}^{\infty} \xi_m f_{n_m}(x) h_m(g(x))$$

for every $\xi = (\xi_m)_m \in \ell_\infty$ and $x \in B_X$. This is clearly linear and continuous, since

$$\begin{aligned} p_v(T(\xi)) &= \sup_{x\in B_X} v(x)|T(\xi)(x)| \leq \sup_{x\in B_X} \sum_{m=1}^{\infty} \|\xi\|_\infty p_v(f_{n_m})|h_m(g(x))| \\ &\leq M\|\xi\|_\infty \sup_m p_v(f_{n_m}). \end{aligned}$$

Hence $\|T\| \leq M \sup_m p_v(f_{n_m})$.

We define now $S : H_w(B_Y) \to \ell_\infty$ by $S(h) = \left(\frac{h(y_{n_k})}{f_{n_k}(\phi(y_{n_k}))}\right)_k$. This is also linear and continuous. Indeed, using (1) we have

$$\|S(h)\| = \sup_k \frac{|h(y_{n_k})|w(y_{n_k})}{|f_{n_k}(\phi(y_{n_k}))|w(y_{n_k})} \le \frac{1}{c}\sup_k |h(y_{n_k})|w(y_{n_k}) \le \frac{1}{c} p_w(h)$$

and $\|S\| \le 1/c$.

These two mappings satisfy that $S \circ C_\phi \circ T = id_{\ell_\infty}$. For any $\xi = (\xi_k)_k \in \ell_\infty$ we have

$$\begin{aligned} S \circ C_\phi \circ T(\xi) &= \left(\frac{C_\phi \circ T(\xi)(y_{n_k})}{f_{n_k}(\phi(y_{n_k}))}\right)_k \\ &= \left(\frac{T(\xi)(\phi(y_{n_k}))}{f_{n_k}(\phi(y_{n_k}))}\right)_k \\ &= \left(\frac{\sum_{m=1}^\infty \xi_m f_{n_m}(\phi(y_{n_k}))h_m(g(\phi(y_{n_k})))}{f_{n_k}(\phi(y_{n_k}))}\right)_k \\ &= \left(\frac{\xi_k f_{n_k}(\phi(y_{n_k}))}{f_{n_k}(\phi(y_{n_k}))}\right)_k = (\xi_k)_k. \end{aligned}$$

Since S and T are continuous, they are weakly continuous. If C_ϕ were weakly compact, we would have that $S \circ C_\phi \circ T(B_{\ell_\infty})$ would be weakly compact, but this is not true. Hence C_ϕ cannot be weakly compact.

Similarly, C_ϕ cannot be Rosenthal since id_{ℓ_∞} is not so. QED

4 Proposition. *Let X, Y be Banach spaces and $\phi : B_Y \to B_X$ a holomorphic mapping such that $\phi(rB_Y)$ is relatively compact for every $0 < r < 1$. Let v, w be weights defined on X and Y respectively, satisfying $\lim_{\|y\|\to 1^-} w(y) = 0$. If the operator $C_\phi : H_v(B_X) \to H_w(B_Y)$ is not compact then it is neither weakly compact nor Rosenthal.*

Proof. If C_ϕ is not compact, proceeding as in Proposition 3, we can obtain a bounded, τ_0-null net $(g_\beta)_\beta$ such that $\{g_\beta\}$ is a countable set, which we write $\{f_n : n \in \mathbb{N}\}$ and a sequence $(y_n)_n \subseteq B_Y$ so that

$$w(y_n)|f_n(\phi(y_n))| \ge c > 0.$$

Let us see now that $\lim_n \|y_n\| = 1$. If not, there is a subsequence $(y_{n_k})_k$ such that $(y_{n_k})_k \subseteq rB_Y$ for some $0 < r < 1$. Since $\phi(rB_Y)$ is relatively compact, $(\phi(y_{n_k}))_k$ is relatively compact. We consider now $K = \overline{(\phi(y_{n_k}))_k}$, which is compact. Hence there is β_0 such that, for all $\beta \ge \beta_0$,

$$\sup_{x\in K} g_\beta(x) < \frac{c}{\|w\|_\infty}.$$

This is the same inequality as in (2); proceeding in the same way we get a contradiction that shows that $\lim_n \|y_n\| = 1$.

From this we can show that $\lim_n \|\phi(y_n)\| = 1$. If this is not true, we can find a subsequence $(y_{n_k})_k$ such that $\|\phi(y_{n_k})\| \leq \lambda < 1$. Now, on the one hand, $(f_n)_n$ is bounded in $H_v(B_X)$; so let us write $\sup_n \|f_n\|_v = M$. On the other hand, v satisfies Condition I; from this, $\inf_{x \in \lambda B_X} v(x) = K > 0$. Hence

$$c \leq w(y_n)|f_n(\phi(y_n))| = \frac{w(y_n)}{v(\phi(y_n))} v(\phi(y_n))|f_n(\phi(y_n))| \leq \frac{M}{K} w(y_n).$$

Since $\lim_n \|y_n\| = 1$, we have $\lim_n w(y_n) = 0$. This gives a contradiction that shows that $\lim_n \|\phi(y_n)\| = 1$. From this point, following the same steps as in Proposition 3 we get that C_ϕ is neither weakly compact nor Rosenthal. QED

With these results we can prove the following ones.

5 Theorem. *Let v, w be weights satisfying Condition I such that $w(y)$ converges to 0 as $\|y\| \to 1^-$ and $\phi : B_Y \to B_X$ be a holomorphic mapping. The following are equivalent,*

(i) C_ϕ is compact.

(ii) C_ϕ is weakly compact and $\phi(rB_Y)$ is relatively compact for all $0 < r < 1$.

(iii) $\displaystyle\lim_{\|y\| \to 1^-} \frac{w(y)}{\tilde{v}(\phi(y))} = 0$ and $\phi(rB_Y)$ is relatively compact for all $0 < r < 1$.

Proof. The equivalence between (*i*) and (*ii*) is a straightforward consequence of Proposition 4 and [13, Proposition 3.2]. The fact that (*i*) and (*iii*) are equivalent is [13, Proposition 3.2]. But in the proof of that result it is actually used Lemma 2 (*iii*). Hence that proof is formally true only if X is separable. Nevertheless, an easy adaption to nets and Lemma 2 give that the result remains true for any Banach space X. For the sake of completeness we show here the adapted proof.

The proof given in [13, Proposition 3.2] that (*i*) implies (*iii*) does not use the characterization of Lemma 2 and it is valid for any Banach space.

Now, let us assume that (*iii*) holds. Following the same steps as in [13, Proposition 3.2] we get that C_ϕ is continuous. Let us suppose that C_ϕ is not compact; then by Lemma 2 there is a net $(f_\alpha)_\alpha \subseteq B_v$ that τ_0-converges to 0 such that $(C_\phi(f_\alpha))_\alpha$ does not p_w-converge to 0. Taking a subnet if necessary we can assume that there is $\lambda > 0$ with $p_w(C_\phi(f_\alpha)) > \lambda > 0$ for all α. For each α, let $y_\alpha \in B_Y$ be such that $w(y_\alpha)|f_\alpha(\phi(y_\alpha))| \geq \lambda$. If $1 \in \overline{\{\|y_\alpha\|\}_\alpha}$ then there is a sequence of points in $\{\|y_\alpha\|\}_\alpha$ converging to 1. Let us denote this sequence by $\{\|y_n\|\}_n$. Given any $\varepsilon > 0$ there is n_0 so that, for any $n \geq n_0$,

$$w(y_n) \leq \varepsilon \tilde{v}(\phi(y_n)).$$

Thus we have

$$\lambda \leq w(y_n)|f_n(\phi(y_n))| \leq \varepsilon\tilde{v}(\phi(y_n))|f_n(\phi(y_n))| \leq \varepsilon.$$

This gives a contradiction and shows that there is $0 < r < 1$ such that $\|y_\alpha\| < r$ for all α. Now, $\phi(rB_Y)$ is relatively compact and $(\phi(y_\alpha)) \subseteq \phi(rB_Y)$; this implies that given any $\varepsilon > 0$ there exists α_0 with

$$\sup_{x\in\phi(rB_Y)} |f_\alpha(x)| < \frac{\varepsilon}{\sup\limits_{y\in B_Y} w(y)}$$

for every $\alpha \geq \alpha_0$. From this, $|f_\alpha(\phi(y_\alpha))| < \varepsilon/\sup_{y\in B_Y} w(y)$ for every $\alpha \geq \alpha_0$ and $\lambda \leq w(y_\alpha)|f_\alpha(\phi(y_\alpha))| < \varepsilon$. This is again a contradiction that finally shows that C_ϕ is compact. QED

6 Theorem. *Let v, w be weights satisfying Condition I and $\phi : B_Y \to B_X$ be a holomorphic mapping such that $\phi(B_Y) \cap rB_X$ is relatively compact for all $0 < r < 1$. The following are equivalent,*

(i) C_ϕ is compact.

(ii) C_ϕ is weakly compact.

(iii) C_ϕ is Rosenthal.

(iv) $\lim\limits_{r\to 1^-} \sup\limits_{\|\phi(y)\|>r} \dfrac{w(y)}{\tilde{v}(\phi(y))} = 0$.

(If $\|\phi\|_\infty < 1$, the above limit is taken as zero by definition).

Proof. The equivalence between (*i*), (*ii*) and (*iii*) follows from Proposition 3. Statements (*i*) and (*iv*) are equivalent by [13, Theorem 3.3]. In this case, as in Theorem 5, it is also necessary to make a slight change in the original proof for the case of a non separable Banach space X. QED

7 Corollary. *Let v, w be weights satisfying Condition I such that $w(y)$ does not converges to 0 as $\|y\| \to 1^-$ and $\phi : B_Y \to B_X$ be a holomorphic mapping. The following are equivalent,*

(i) C_ϕ is compact.

(ii) $\phi(B_Y)$ is relatively compact and $\|\phi\|_\infty < 1$.

(iii) C_ϕ is weakly compact and $\phi(B_Y)$ is relatively compact.

Proof. The fact that (*i*) and (*ii*) are equivalent is proved in [13, Corollary 3.5 (b)]. Trivially (*i*) implies (*iii*). Theorem 6 gives that (*iii*) implies (*i*). QED

A particular case of this is when $v(x) = w(x) = 1$; this gives H^∞. Compact composition operators between $H^\infty(B_Y)$ and $H^\infty(B_X)$ have been studied in [2,11]. The following result is proved there (see also [11, Preliminaries]).

8 Proposition. *[2, Proposition 2.2] Consider the composition operator* $C_\phi : H^\infty(B_X) \to H^\infty(B_Y)$. *The following statements are equivalent,*

(i) C_ϕ *is compact.*

(ii) C_ϕ *is weakly compact and* $\phi(B_Y)$ *is relatively compact in* X.

(iii) $\phi(B_Y)$ *lies strictly inside* B_X *and* $\phi(B_Y)$ *is relatively compact in* X.

This was first proved by Maestre [16] for the space $A_u(B_X)$, the algebra of the holomorphic functions on the open unit ball of X which are uniformly continuous on the closed unit ball of X. That proof for $A_u(B_X)$ can be easily adapted to obtain the result for $H^\infty(B_X)$. The difficult part in the characterization of compactness of C_ϕ is to prove necessity. The proof of $\phi(B_Y) \subseteq sB_X$ for some $0 < s < 1$ in [2] goes through weak compactness of C_ϕ. We present in the following remark a very easy proof based on [16].

9 Remark. Let us assume that $C_\phi : H^\infty(B_X) \to H^\infty(B_Y)$ is compact and let us show that $\phi(B_Y) \subseteq sB_X$ for some $0 < s < 1$. Suppose that this is not true. Then there would exist a sequence $(y_n)_n \subseteq B_Y$ such that $\lim_n \|\phi(y_n)\| = 1$. Without loss of generality we can assume that

$$\|\phi(y_n)\| > \sqrt[n]{1 - \frac{1}{n}}.$$

For each $n \in \mathbb{N}$ we choose $x_n^* \in X^*$ such that $\|x_n^*\| = 1$ and $x_n^*(\phi(y_n)) > \sqrt[n]{1 - 1/n}$. We consider the family

$$\mathcal{F} = \{(x_n^*)^n : n = 1, 2, \dots\}.$$

We have

$$1 \geq \|(x_n^* \circ \phi)^n\|_\infty > 1 - \frac{1}{n} \tag{3}$$

for all $n \in \mathbb{N}$. As $\mathcal{F}$ is a bounded set, $C_\phi(\mathcal{F})$ is relatively compact, i.e. there exists a subsequence $(\big(x_{n_k}^* \circ \phi\big)^{n_k})_k$ that $\|\cdot\|_\infty$-converges to some $f \in H^\infty(B_Y)$. By (3) $\|f\|_\infty = 1$. But for $y \in B_Y$ we have $|x_{n_k}^* \circ \phi(y)|^{n_k} \leq \|\phi(y)\|^{n_k}$ for all $k \in \mathbb{N}$ and $\|\phi(y)\|^{n_k}$ goes to 0 as k tends to infinity. Hence $f(y) = 0$ for all $y \in B_Y$. This gives a contradiction and completes the proof.

3 Composition Operators on Fréchet spaces

Given two countable families of weights V, W we consider now the composition operator $C_\phi : HV(B_X) \to HW(B_Y)$. In [8] these operators are defined and studied when $B_X = B_Y = \mathbb{D}$. Bonet and Friz prove a general result [8, Proposition 4.2] which allows them to give conditions on the continuity and compactness of the composition operator [8, Proposition 4.1]. We use a slight modification of their general result to find conditions characterizing continuity and compactness of the operator in our case. Let us state now this general result. Let (H,τ), (G,τ') be Hausdorff locally convex spaces. For each n, let E_n and F_n be Banach spaces with closed unit balls B_n and C_n and norms $\|\cdot\|_n$ and $|\cdot|_n$. Suppose that $E_{n+1} \subseteq E_n \subseteq E_1 \subseteq H$, $B_{n+1} \subseteq B_n$ and $F_{n+1} \subseteq F_n \subseteq F_1 \subseteq G$, $C_{n+1} \subseteq C_n$ for every n. Suppose that for each n, both B_n and C_n are compact in (H,τ) and (G,τ') respectively.

Let E be the projective limit of $(E_n)_n$ and F the projective limit of $(F_n)_n$. Let us assume that for every $n \in \mathbb{N}$ and all $x \in E_n$ there exists a sequence $(y_k)_k \subseteq E$ converging to x in (H,τ) such that $\|y_k\|_n \leq \|x\|_n$ for all k. In [8] the case $(H,\tau) = (G,\tau')$ is considered; the same proof of [8, Proposition 4.2] gives the following proposition. Let us recall that a linear mapping $T : E \longrightarrow F$ between two locally convex spaces is said to be *compact* (resp. *weakly compact* or *bounded*) if there exists a 0-neighborhood in E such that its image by T is relatively compact (resp. relatively weakly compact or bounded) in F.

10 Proposition. *Let $T : (H,\tau) \longrightarrow (G,\tau')$ be a continuous, linear operator.*

(a) The following are equivalent,

(i) *$TE \subseteq F$.*

(ii) *$T \in \mathcal{L}(E;F)$.*

(iii) *For each m, there is n such that $TE_n \subseteq F_m$.*

(iv) *For each m, there is n such that $T : E_n \longrightarrow F_m$ is well defined and continuous.*

(b) The following are equivalent,

(i) *$T : E \longrightarrow F$ is bounded.*

(ii) *There exists n such that for all m, $TE_n \subseteq F_m$.*

(iii) *There exists n such that for all m, $T : E_n \longrightarrow F_m$ is well defined and continuous.*

(c) The following are equivalent,

(i) *$T : E \longrightarrow F$ is compact (resp. weakly compact).*

(ii) *There exists n such that for all m, $T : E_n \longrightarrow F_m$ is compact (resp. weakly compact).*

As an application of this result we characterize continuity and compactness composition operators. Let (H,τ) be $(H(B_X),\tau_0)$ and (G,τ') be $(H(B_Y),\tau_0)$. The operator $C_\phi : (H(B_X),\tau_0) \to (H(B_Y),\tau_0)$ is linear and continuous. Let $V = (v_n)_{n=1}^\infty$ and $W = (w_n)_{n=1}^\infty$ be two increasing families of weights satisfying Condition I defined on B_X and B_Y respectively. We put $E_n = H_{v_n}(B_X)$ and $F_n = H_{w_n}(B_Y)$. Each one of these is a Banach space. They satisfy $H_{v_{n+1}}(B_X) \subseteq H_{v_n}(B_X) \subseteq H_{v_1}(B_X) \subseteq H(B_X)$, the closed unit ball $\overline{B}_{v_n}$ is τ_0-compact ([7], [18, page 349]) and $\overline{B}_{v_{n+1}} \subseteq \overline{B}_{v_n}$ for all n (the same happens for $H_{w_n}(B_Y)$). Let us take $E = HV(B_X)$ and $F = HW(B_Y)$.

Let $f \in H(B_X)$ and consider its Taylor series expansion at 0, $f = \sum_{m=0}^{\infty} P_m f$. For each $k \in \mathbb{N}$, the k-th Cesàro mean is defined by (see [3, Section 1] or [12, Proposition 4])

$$C_k f(x) = \frac{1}{k+1} \sum_{l=0}^{k} \left(\sum_{m=0}^{l} P_m f(x) \right) = \sum_{m=0}^{k} \left(1 - \frac{m}{k+1} \right) P_m f(x).$$

Since every weight is bounded on B_X, every polynomial belongs to $HV(B_X)$. In particular, for every $f \in H(B_X)$, the sequence $(C_k f)_k$ is in $HV(B_X)$. Also, $C_k f \longrightarrow f$ in τ_0 (see [3], also [12]). If v is a radial weight then for all $f \in H_v(B_X)$,

$$\sup_{x \in B_X} v(x)|C_k f(x)| \leq \sup_{x \in B_X} v(x)|f(x)|$$

(see [3, Proposition 1.2(b)], also [12]). Hence, if every $v \in V$ is radial, then the spaces and the composition operator satisfy all the above conditions to apply Proposition 10 in a very similar way to that used by Bonet and Friz to obtain the following generalizations of [8, Proposition 4.1].

11 Proposition. *Let $\phi : B_Y \longrightarrow B_X$ be holomorphic and $V = (v_n)_n$ and $W = (w_n)_n$ increasing countable families of weights satisfying Condition I defined on B_X and B_Y respectively such that every v_n is radial. The following statements are equivalent,*

(i) $C_\phi : HV(B_X) \longrightarrow HW(B_Y)$ is continuous.

(ii) For each $w \in W$ there exists $v \in V$ such that $C_\phi : H_v(B_X) \longrightarrow H_w(B_Y)$ is continuous.

12 Proposition. *Let $\phi : B_Y \longrightarrow B_X$ be holomorphic and $V = (v_n)_n$ and $W = (w_n)_n$ increasing countable families of weights such that every v_n is radial. The following statements are equivalent,*

(i) $C_\phi : HV(B_X) \longrightarrow HW(B_Y)$ is (weakly) compact.

(ii) There exists $v \in V$ such that $C_\phi : H_v(B_X) \longrightarrow H_w(B_Y)$ is (weakly) compact for every $w \in W$.

We draw now our attention to vector valued holomorphic functions. Following [8], given any countable family of weights V and a Banach space Z, we consider the space

$$HV(B_X, Z) = \{f : B_X \to Z \text{ holomorphic } : \sup_{x \in B_X} v(x)\|f(x)\| < \infty, \ v \in V\}$$

We are interested in composition operators $C_\phi : HV(B_X, Z) \to HW(B_Y, Z)$, where V and W are countable families of weights satisfying Condition I defined on B_X and B_Y, respectively. In particular we are interested in when such an operator is weakly compact. We study this case using wedge operators.

If E and F are locally convex spaces, $L_b(E,F)$ denotes the space of continuous linear mappings from E into F endowed with the topology of uniform convergence on bounded subsets of E. Now, given E_1, E_2, E_3, E_4, complete locally convex spaces, and $L : E_3 \to E_4$, $R : E_1 \to E_2$ continuous linear mappings, the wedge operator

$$R \wedge L : L_b(E_2, E_3) \to L_b(E_1, E_4)$$

is defined by $(R \wedge L)(T) = LTR$ for $T \in L(E_2, E_3)$. We refer to [8,17,19] for a study of wedge operators. In [8, Section 2], several results are proved regarding weak compactness of wedge operators.

It is known that given any Banach space and any countable family of weights $V = (v_n)_n$ with Condition I,

$$GV(B_X) = \{\psi \in HV(B_X)' \ : \psi|_{D_\alpha} \text{ is } \tau_0\text{-continuous for all } \alpha = (\alpha_n)_n, \alpha_n > 0\}$$

where

$$D_\alpha = \{f \in HV(B_X) \ : \ p_{v_n}(f) \leq \alpha_n \text{ for all } n \in \mathbb{N}\}$$

is a complete, barrelled (DF)-space such that its strong dual is topologically isomorphic to $(HV(B_X), \tau_V)$ (see [12, Section 3] for details). By using this predual we obtain a linearization result for $HV(B_X, Z)$, compare with [8, Theorem 3.3].

Since any weakly holomorphic mapping on an open set of a Banach space is holomorphic [10, Example 3.8 (g)] the following Lemma holds.

13 Lemma. *The mapping $\Delta : B_X \to GV(B_X)$ given by $x \mapsto \delta_x$ is holomorphic and the set $\{\delta_x \ : \ x \in B_X\}$ is total in $GV(B_X)$.*

14 Theorem. *Let X and Z be Banach spaces and V a countable family of weights defined on B_X satisfying Condition I. Then*

$$HV(B_X, Z) = L_b(GV(B_X), Z)$$

holds algebraically and topologically.

Proof. Let us consider the mapping $\chi : L_b(GV(B_X), Z) \to HV(B_X, Z)$, defined by $\chi(T) = T \circ \Delta$. By Lemma 13 $\chi(T) \in H(B_X, Z)$. Let us take any $v \in V$ and consider

$$A_v = \{v(x)\delta_x \ : \ x \in B_X\}. \tag{4}$$

This is a bounded set in $GV(B_X)$. Since $T \in L(GV(B_X), Z)$, the set

$$T(A_v) = \{v(x)T(\delta_x) \ : \ x \in B_X\}$$

is bounded in Z and

$$\sup_{x\in B_X} \|v(x)T(\delta_x)\| = \sup_{x\in B_X} v(x)\|\chi(T)(x)\| < \infty.$$

Hence $\chi(T) \in HV(B_X, Z)$. So we have that χ is well defined. It is clearly linear and the continuity follows in a natural way.

Now, for each $f \in HV(B_X, Z)$ we define $\psi(f) : GV(B_X) \to Z^{**}$ by the equality $(\psi(f)(u))(z^*) = u(z^* \circ f)$ for all $z^* \in Z^*$. Since $z^* \circ f \in HV(B_X)$ for each $z^* \in Z^*$, the mapping $\psi(f)(u) : Z^* \to \mathbb{C}$ is well defined and is linear for each $u \in GV(B_X) \subset HV(B_X)'$. It is not difficult to see that $\psi(f)(u) \in Z^{**}$ for every $u \in GV(B_X)$. By the definition of $\psi(f)$ it is linear. Let us see now that $\psi(f)$ is also continuous. Since $f \in HV(B_X, Z)$, the set $\{v(x)f(x) \ : \ x \in B_X\}$ is bounded in Z for all $v \in V$ and

$$\sup_{x\in B_X} v(x)\|f(x)\| = \sup_{x\in B_X} \sup_{z^*\in B_{Z^*}} v(x)|z^*(f(x))|.$$

Hence $D = \{z^* \circ f \ : \ z^* \in B_{Z^*}\}$ is bounded in $HV(B_X)$. Let us consider a neighbourhood of zero given by $U = \overset{\circ}{D} \cap GV(B_X)$, where $\overset{\circ}{D}$ is the polar of D in $HV(B_X)'$. Now, by definition of ψ, $\|\psi(f)(u)\| \le 1$ for all $u \in U$.

On the other hand $(\psi(f)(\delta_x))(z^*) = \delta_x(z^* \circ f) = z^*(f(x))$ for every $x \in B_X$ and $z^* \in Z^*$. Since $\{\delta_x \ : \ x \in B_X\}$ is total in $GV(B_X)$ we conclude that $\psi(f) \in L(GV(B_X), Z)$. The mapping $\psi : HV(B_X, Z) \to L_b(GV(B_X), Z)$ is clearly linear and by the Closed Graph Theorem it is also continuous.

An easy computation shows that $\psi \circ \chi$ is the identity on $L_b(GV(B_X), Z)$ and $\chi \circ \psi$ is the identity on $HV(B_X, Z)$. This completes the proof. QED

15 Proposition. *If $C_\phi : HV(B_X) \to HW(B_Y)$ is continuous, then*

(1) *For any Banach space Z, the vector-valued composition operator*

$$C_\phi : HV(B_X, Z) \to HW(B_Y, Z)$$

is also continuous.

(2) *The transpose C_ϕ^t of C_ϕ satisfies $C_\phi^t(GW(B_Y)) \subset GV(B_X)$. In particular the restriction of C_ϕ^t to $GW(B_Y)$, which we denote by C_ϕ', satisfies*

$$C_\phi' \in L(GW(B_Y), GV(B_X))$$

and $(C_\phi')^t = C_\phi$.

Proof. Clearly we only need to prove (2). If $g \in HV(B_X)$, then

$$(C_\phi^t)(\delta_y)(g) = \delta_y(C_\phi(g)) = \delta_y(g \circ \phi) = g(\phi(y)) = \delta_{\phi(y)}(g).$$

Hence $(C_\phi^t)(\delta_y) = \delta_{\phi(y)}$ for all $y \in B_Y$. Since $C_\phi^t : HW(B_Y)'_b \to HV(B_X)'_b$ is continuous, the set $\{\delta_y : y \in B_Y\}$ is total in $GW(B_Y)$ and $GV(B_X)$ is complete, we have $C_\phi^t(GW(B_Y)) \subset GV(B_X)$. Now, $C'_\phi : GW(B_Y) \to GV(B_X)$ is clearly continuous since both preduals are endowed with the restriction of the corresponding strong topologies.

By using that $HV(B_X)$ (respectively $HW(B_Y)$) is the strong dual of $GV(B_X)$ (respectively $GW(B_Y)$), we get $(C'_\phi)^t = C_\phi$. QED

With all the mappings we have considered so far we have the following diagram

$$\begin{array}{ccc} HV(B_X,Z) & \xrightarrow{C_\phi} & HW(B_Y,Z) \\ \chi_V \uparrow & & \downarrow \psi_W \\ L_b(GV(B_X),Z) & \xrightarrow[C'_\phi \wedge id_Z]{} & L_b(GW(B_Y),Z) \end{array} \tag{5}$$

16 Proposition. *The diagram* (5) *is commutative; that is* $C'_\phi \wedge id_Z = \psi_W \circ C_\phi \circ \chi_V$.

Proof. Let $S \in L(GV(B_X),Z)$, we first have $(C'_\phi \wedge id_Z)(S) = S \circ C'_\phi$. Let us see that this coincides with $(\psi_W \circ C_\phi \circ \chi_V)(S)$. By Lemma 13 it is enough to see that they coincide on $\{\delta_y : y \in B_Y\}$. For each $y \in B_Y$ we have

$$(S \circ C'_\phi)(\delta_y) = S(C'_\phi(\delta_y)) = S(\delta_{\phi(y)})$$

and

$$(\psi_W \circ C_\phi \circ \chi_V)(S)(\delta_y) = \left((C_\phi \circ \chi_V)(S)\right)(y) = \chi_V(S)(\phi(y)) = S(\delta_{\phi(y)}).$$

QED

Note that as an immediate consequence of this result we have that the vector-valued composition operator C_ϕ is reflexive or weakly compact if and only if the wedge operator $C'_\phi \wedge id_Z$ is of the same type. We use the results on wedge operators given in [8] to obtain the following.

17 Theorem. *Let X, Y, Z be Banach spaces and $\phi : B_Y \to B_X$ be a holomorphic mapping. Let V, W be countable families of weights defined on B_X and B_Y respectively such that each weight satisfies Condition I. If any of the two following conditions hold*

(a) $\phi(B_Y) \cap rB_X$ is relatively compact for every $0 < r < 1$,

(b) $\lim_{\|y\| \to 1^-} w(y) = 0$ for every $w \in W$ and $\phi(rB_Y)$ is relatively compact for every $0 < r < 1$, then the operator

$$C_\phi : HV(B_X,Z) \to HW(B_Y,Z)$$

is weakly compact if and only if Z is reflexive and $C_\phi : HV(B_X) \to HW(B_Y)$ is weakly compact.

PROOF. If $C_\phi : HV(B_X, Z) \to HW(B_Y, Z)$ is weakly compact then, by [8, Proposition 2.1], $(C'_\phi)^t$ is weakly compact. But by Proposition 15 $(C'_\phi)^t = C_\phi$; hence $C_\phi : HV(B_X) \to HW(B_Y)$ is weakly compact. On the other hand [8, Proposition 2.1] implies that $id_Z : Z \to Z$ is also weakly compact; hence Z is reflexive.

Let us assume now that Z is reflexive and $C_\phi : HV(B_X) \to HW(B_Y)$ is weakly compact. By Proposition 15 $(C'_\phi)^t = C_\phi$; hence $(C'_\phi)^t : HV(B_X) \to HW(B_Y)$ is weakly compact. By Proposition 12 there exists $v \in V$ such that for every $w \in W$ the composition operator $C_\phi : H_v(B_X) \to H_w(B_Y)$ is weakly compact. If (a)(resp. (b)) holds, using Theorem 6 (resp. Theorem 5) we have that $C_\phi : H_v(B_X) \to H_w(B_Y)$ is compact. Applying again Proposition 12 $(C'_\phi)^t : HV(B_X) \to HW(B_Y)$ is compact. Moreover $id_Z : Z \to Z$ is weakly compact since Z is reflexive. By [8, Theorem 2.15] $C'_\phi \wedge id_Z$ is weakly compact. QED

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