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# LOCALLY CONVEX ALGEBRAS

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## Preface

Given an associative algebra $A$ over the field $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ endowed with a locally convex ($lc$) topology, it is a natural question whether multiplication is continuous on $A$. In general, continuity of all the maps $A \to A, x \mapsto yx$, and $x \mapsto xy$, respectively, $(y \in A)$ does not imply that multiplication is (jointly) continuous. An algebra with jointly continuous multiplication with respect to an $lc$ topology will be called $lc$ algebra.
However, in view of the well-known representation of Hausdorff $lc$ spaces as dense subspaces of projective limits of Banach spaces, continuity of multiplication on an algebra $A$ is not a satisfying requirement. This leads to the concept of locally multiplicatively convex ($lmc$) algebras which have, in addition, a zero-neighbourhood-basis (0nbhd-basis) consisting of multiplicative sets. For an $lmc$ algebra it is easy to see that it is topologically isomorphic to a dense subalgebra of a projective limit of Banach algebras. Now there is a vast amount of applications of the classical theory of Banach algebras to $lmc$ algebras. For instance, almost immediately one obtains that the quasiinversion map is always continuous on an $lmc$ algebra. In fact there are not too many examples of $lc$ algebras which are not $lmc$ (see e.g. [31]). The most prominent one is due to R. Arens (see [2]). The Arens algebra is a Fréchet algebra where the inversion map fails to be continuous. In chapter 1 of the thesis there will be two examples of this type which both have continuous inversion map.

Moreover, we are going to investigate some permanence properties of $l(m)c$ algebras in chapter 1. Especially, we will discuss the question whether the algebra of polynomials with coefficients in an $lc$ algebra $A$, $A[X]$, has again continuous multiplication with respect to the direct sum topology. It will turn out that multiplication is continuous on $A[X]$, if $A$ satisfies the countable neihgbourhood condition. On the other hand, $\omega[X]$ is not an $lc$ algebra. Moreover, even in the scalar case $\mathbb{K}[X] = \varphi$ fails to be an $lmc$ algebra. However, at the beginning of chapter 2 a description of the characters (i.e. the linear multiplicative functionals) on $A[X]$ will be given for any algebra $A$.

It is very easy to give an example of an algebra which contains a non-trivial $l(m)c$ ideal but which is not $l(m)c$ itself. Thus one faces the following three-space-problem: Given an algebra $A$ which contains an $l(m)c$ ideal $I \subset A$ such that $A/_I$ is an $l(m)c$ algebra we are going to investigate conditions such that $A$ is $l(m)c$. Motivated by the notion of a topological group, which is the semidirect product of a normal subgroup and a subgroup (see [32]), which turned out to be a rich source for examples and counterexamples, an analogous notion of a locally convex algebra $A$, which is the semidirect product of an ideal $C$ and a subalgebra $B$ will be introduced. In this case the quotient algebra $A/_C$ is (topologically) isomorphic to the subalgebra $B$. It will turn out that any *lc* algebra which is the topological semidirect product of an *lmc* ideal and an *lmc* subalgebra is *lmc* itself. Moreover, a general method of constructing such semidirect products, which contains the direct products and the adjunction of a unit element as special cases, will be presented. As an application one obtains an example of an algebra $A$ provided with a Banach space topology $\mathcal{T}$ and containing an ideal $C$ such that both $(C, \mathcal{T} \cap C)$ and $(A/_C, \mathcal{T}/_C)$ are Banach algebras but $(A, \mathcal{T})$ is not. (For these results see also [10].)

Moreover, chapter 1 yields that on any normal Banach sequence space $\lambda$ a multiplication $\odot$ can be defined, such that $(\lambda, \odot)$ becomes a Banach algebra. In case $\lambda \in \{c, c_0, l^p\}$ $(1 \leq p \leq \infty)$, $\odot$ coincides with pointwise multiplication on $\lambda$.
It will turn out that for any $l(m)c$ algebra $A$ $(\lambda(A), \odot)$ is an $l(m)c$ algebra. This can be generalized to the projective limits of Moscatelli type. (For a close investigation of vector-valued sequence spaces and of projective limits of Moscatelli type, respectively, see e.g. [16] and [25].)

Chapter 2 deals with characters on *lc* algebras. It is a well-known result of the classical theory of Banach algebras that every character on a Banach algebra is continuous (e.g. [21, Satz 125.2]). As a consequence of this fundamental theorem we will obtain a characterization of the characters on $(\lambda, \odot)$ for any normal Banach sequence space $\lambda$ containing $\varphi$ as a dense subspace, and of the characters on $(l^1, *)$, $* : l^1 \times l^1 \to l^1$ denoting the convolution on $l^1$.

For many years it has been an open question (the so-called Michael problem), whether the characters on an *lmc* Fréchet algebra are continuous (see [28, p. 53]). Now it seems that the Michael problem obtained a positive answer (see [35]).
However, it still seems desirable, to describe the characters on certain algebras. Now, chapter 2 gives characterizations of the linear, multiplicative functionals on some classes of algebras. These can still be regarded as results on automatic continuity, because the algebras under consideration are, in general, not Fréchet algebras. Nor need they be *lmc*.
To begin with, there is a thorough examination of the characters on the cartesian product of algebras provided with pointwise multiplication. We will characterize those index sets $S$ which admit for any family of algebras $(A_s)_{s \in S}$ a description of all the charaters on $\prod_{s \in S} A_s$ as the application of some character on $A_t$ to the projection $pr_t : \prod_{s \in S} A_s \to A_t$ for some $t \in S$.

We will for instance characterize the characters on vector-valued sequence spaces $\lambda(A)$ for certain *lc* algebras $A$ and normal Banach sequence spaces $\lambda$ containing $\varphi$ as a dense subspace thus generalizing the representation of the charcters on $\lambda$. Moreover a description of the characters on the algebra of continuous functions of a topological space to an *lc* algebra can be given in some important cases. There is also a characterization of the characters on the algebra of holomorphic functions of an open subset of the complex plane to a locally complete *lc* algebra. (For the latter results see also [11].)
The representation results for the characters on certain algebras can also be interpreted as results on permanence properties of the properties 'functionally continuous' (i.e. all the characters on a certain algebra are continuous) and 'functionally bounded' (i.e. all characters are

bounded on bounded sets), respectively.

Chapter 3 studies conditions such that the *lc* inductive limit of an inductive sequence of $l(m)c$ algebras is again an $l(m)c$ algebra. It has been proved in [1] that the *lc* inductive limit of an inductive sequence $(A_n)_{n\in\mathbb{N}}$ of seminormed algebras is an *lmc* algebra. In the category of *lc* algebras, the requirement each $A_n$ being a seminormed algebra can be weakened to the countable neighbourhood condition.
In [12] the authors prove that the inductive limit of a sequence of commutative *lmc* algebras each satisfying the countable neighbourhood condition is an *lmc* algebra thus generalizing the result in [1] for the commutative case.
For the inductive limit of Moscatelli type $A := \operatorname{ind}(C \hookrightarrow B, \lambda)$, where $C$ and $B$ are $l(m)c$ algebras, we will state several conditions yielding that $A$ is again an $l(m)c$ algebra.
In [12] the authors give an example of a strict inductive limit of *lmc* Fréchet algebras on which multiplication fails to be continuous. There is a whole class of such examples in chapter 3. For the inductive limit of Moscatelli type $A := \operatorname{ind}(C \hookrightarrow B)$, where $C$ and $B$ are $l(m)c$ algebras, we will characterize when $A$ is an $l(m)c$ algebra. It turns out that for this class the inductive limit of Moscatelli type of *lmc* algebras is either an *lmc* algebra itself or multiplication fails to be continuous. Moreover, there is a characterization of the $l(m)c$ inductive topology on $\operatorname{ind}(C \hookrightarrow B)$.

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## Chapter 0

## Notations and Terminology

For any set $T$ and any linear space $E$ the space $\prod_{s\in T} E$ will be denoted by $E^T$, whereas we will refer to

$$\bigoplus_{s\in T} E := \{(x_s)_{s\in T} \in E^T : \{s \in T : x_s \neq 0\} \text{ is finite}\}$$

as $E^{(T)}$. If, in addition, $E$ carries a linear topology, we will always endow $E^T$ with the corresponding product topology, and $E^{(T)}$ with the direct sum topology, respectively, unless other announcement is made.
Especially, for the scalar case, we define $\omega := \mathbb{K}^{\mathbb{N}}$, and $\varphi := \mathbb{K}^{(\mathbb{N})}$.
Given a set $B \subset E$ the linear span of $B$ will be denoted by $[B]$, the convex hull of $B$ by $\operatorname{Conv}(B)$, the circled hull of $B$ by $\operatorname{Circ}(B)$, and the absolutely convex hull of $B$ by $\Gamma(B)$.

For an $F$-norm $\|\cdot\|$ on $E$, $x \in E$, and $\varepsilon > 0$ we define

$$B(x,\varepsilon) := \{y \in E : \|x-y\| < \varepsilon\}$$

and

$$B[x,\varepsilon] := \{y \in E : \|x-y\| \leq \varepsilon\},$$

respectively.

Considering any sequence space such as $\omega$, $\varphi$, $E^{\mathbb{N}}$, or a normal Banach sequence space $\lambda$, it does not matter, whether one takes either $\mathbb{N}$ or $\mathbb{N} \cup \{0\}$ as index set. For convenience sake, there will be deliberate shifting from $\mathbb{N}$ to $\mathbb{N} \cup \{0\}$ as corresponding index set without explicit mentioning. Especially, this will be so, whenever convolution is considered on such a space.
In abbreviation of '$n \in \mathbb{N} \cup \{0\}$' we will use the notation '$n \geq 0$'.

Moreover, 'topologogical vector space' and 'locally convex space' will be abbreviated to '*tvs*' and to '*lcs*', respectively.
Given a *tvs* $E$ its algebraic dual (i.e. the space of all linear functionals on $E$) will be denoted by $E^*$, and its dual (i.e. the space of all continuous linear functionals on $E$) by $E'$.

For a topological space $X$ and any element $x \in X$ the nbhd-filter of $x$ will be denoted by $\mathcal{U}_x(X)$.
The closure of any subset $T \subset X$ will be denoted by $\overline{T}$, and its open kernel by $\mathring{T}$.

## Chapter 1
## Locally Convex Algebras

In this section we are going to investigate (semi)normed, locally convex (*lc*), and locally *m*–convex (*lmc*) algebras. An algebra $A = (A,+,\cdot)$ is a linear space $A$ over the field $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ with an associative and bilinear map

$$\cdot : A \times A \longrightarrow A, (x,y) \longmapsto xy := x \cdot y,$$

called multiplication (on $A$). An algebra $A$ is called commutative, if $xy = yx$ for all $x,y \in A$.
$B \subset A$ is called a subalgebra of $A$, if $B$ is a subspace of the linear space $A$ and $B \cdot B \subset B$. If, in addition, $(B \cdot A) \cup (A \cdot B) \subset B, B$ is called an ideal (in $A$). For a subspace $B \subset A$ and the corresponding quotient map $q : A \to A/_B$ t.f.a.e.

*i*) $A/_B$ is an algebra and $q$ is multiplicative.

*ii*) $B$ is an ideal in $A$.

$e \in A$ is called a unit (in $A$), if $ex = xe = x$ for all $x \in A$. Obviously, there can be at most one unit in $A$. An element $x \in A$ is called quasiinvertible, if there is a so-called quasiinverse (element) $\tilde{x} \in A$ such that $x\tilde{x} = \tilde{x}x = x + \tilde{x}$. $\tilde{G}(A)$ denotes the set of all quasiinvertible elements of $A$. Immediately, one verifies that an element of an ideal $I \subset A$ is quasiinvertible in $I$, iff it

is quasiinvertible in $A$.
$x \in A$ is called invertible, if $A$ has a unit $e$ and there is a so-called inverse (element) $x^{-1} \in A$ such that $xx^{-1} = x^{-1}x = e$. $G(A)$ denotes the set of all invertible elements in $A$. It is easy to see that $(G(A), \cdot)$ is a group, if $A$ has a unit. Hence the inverse element $x^{-1}$ is uniquely determined. If $I \cap G(A) \neq \emptyset$ for an ideal $I \subset A$, then $I = A$.
For $x \in A$ the spectrum of $x$ in $A$ is denoted by $\sigma_A(x)$ which is defined as

$$\sigma_A(x) := \{\lambda \in I\!K : x - \lambda e \notin G(A)\}$$

if $A$ has a unit and

$$\sigma_A(x) := \{0\} \cup \left\{\lambda \in I\!K \setminus \{0\} : \frac{1}{\lambda}x \text{ is not quasiinvertible in } A\right\},$$

if $A$ has no unit. In the latter case, $\sigma_A(x)$ and $\sigma_{A_e}((x,0))$ coincide ($A_e$ denoting the algebra $A \times I\!K$ provided with the multiplication $((x,\lambda),(\tilde{x},\tilde{\lambda})) \longmapsto (x\tilde{x} + \lambda\tilde{x} + \tilde{\lambda}x, \lambda\tilde{\lambda})$ which has the unit $(0,1)$ and contains $A = A \times \{0\}$ as an ideal). In both cases, the following holds:

$$\sigma_A(x) = \left\{\lambda \in I\!K \setminus \{0\} : \frac{1}{\lambda}x \text{ is not quasiinvertible in } A\right\},$$

if $x$ is invertible in $A$, and

$$\sigma_A(x) = \{0\} \cup \left\{\lambda \in I\!K \setminus \{0\} : \frac{1}{\lambda}x \text{ is not quasiinvertible in } A\right\},$$

if $x$ is not invertible. Again, in the latter case, $\sigma_A(x)$ and $\sigma_{A_e}(x)$ coincide.
As quasiinvertibility of an element $x \in A$ with quasiinverse $\tilde{x} \in A$ is the same as invertibility of $x - e$ in $A_e$ with inverse element $\tilde{x} - e$, also the quasiinverse $\tilde{x}$ is uniquely determined.

**Definition 1.1** *Let $A$ be an algebra and $\|\cdot\|$ be a (semi)norm on $A$ such that*

*i)* $\|\cdot\|$ *is submultiplicative, i.e.* $\|x \cdot y\| \leq \|x\| \cdot \|y\|$ *for all* $x, y \in A$ *and*

*ii)* $\|e\| = 1$, *if $e$ is a unit in $A$,*

*then $(A, \|\cdot\|)$ is called a (semi)normed algebra.*

**Remarks:** (For the sequel see e.g. [17, p. 64ff].) Let $A$ be an algebra and $\|\cdot\|$ a (semi)norm on $A$. Let us denote by $B_{\|\cdot\|}$ the closed unit ball of $A$ with respect to $\|\cdot\|$.

1. T.f.a.e.:

   *i)* $\|\cdot\|$ is submultiplicative.

   *ii)* $\varepsilon B_{\|\cdot\|} \cdot \varepsilon B_{\|\cdot\|} \subset \varepsilon B_{\|\cdot\|}$ for all $\varepsilon \in (0,1]$.

   *iii)* $B_{\|\cdot\|} \cdot B_{\|\cdot\|} \subset B_{\|\cdot\|}$.

Moreover, in case of *i)*, *ii)*, or *iii)* multiplication

$$\cdot : (A, \|\cdot\|) \times (A, \|\cdot\|) \longrightarrow (A, \|\cdot\|), (x,y) \longmapsto xy$$

is continuous.

**Proof:** *i)* $\Rightarrow$ *ii)* $\Leftrightarrow$ *iii)* is obvious. It is also clear that *ii)* implies continuity of $\cdot : (A, \|\cdot\|) \times (A, \|\cdot\|) \to (A, \|\cdot\|)$ at $(0,0)$. Now, by [22, proposition 5.1.3], a bilinear map is continuous, if it is continuous at $(0,0)$.
It only remains to prove *iii)* $\Rightarrow$ *i)*. Let $x, y \in A$. If $\|x\| \cdot \|y\| \neq 0$, then $\left\| \frac{x}{\|x\|} \cdot \frac{y}{\|y\|} \right\| \leq 1$. If $\|x\| \cdot \|y\| = 0$, then $x \in \overline{\{0\}}$ or $y \in \overline{\{0\}}$. Hence $x \cdot y \in \overline{\{0\}}$, i.e. $\|x \cdot y\| = 0$. □

**Observation**: In the above proof one makes use of (separate) continuity in the following way: Continuity of $A \to A, x \mapsto xy$ implies $\overline{\{0\}}y \subset \overline{\{0\}}$. In other words: If for all $y \in A$ the linear maps $A \to A, x \mapsto xy$ and $x \mapsto yx$ are continuous with respect to a linear topology $\mathcal{T}$ on $A$, then $\overline{\{0\}}^{\mathcal{T}}$ is an ideal in $A$.

2. Let, on the other hand, $A$ be different from $\overline{\{0\}}$, such that multiplication $\cdot : (A, \|\cdot\|) \times (A, \|\cdot\|) \to (A, \|\cdot\|)$ is continuous, then one can find a (semi)norm $|||\cdot|||$ on $A$ which is equivalent to $\|\cdot\|$, such that $(A, |||\cdot|||)$ is a (semi)normed algebra.

   **Proof:** One can find $\varepsilon > 0$ such that $\varepsilon B_{\|\cdot\|} \cdot B_{\|\cdot\|} \subset B_{\|\cdot\|}$. $|||\cdot||| := \varepsilon^{-1}\|\cdot\|$ is equivalent to $\|\cdot\|$ and submultiplicative. If $A$ has no unit, we are finished.
   Let us now assume that $A$ has a unit $e$. Since $A$ is different from $\overline{\{0\}}$, $|||e||| \neq 0$. Thus

   $$p : A \longrightarrow [0,\infty), x \longmapsto \sup\{|||xu||| : u \in A, |||u||| = 1\} \quad (\leq |||x|||)$$

   is well-defined. $p$ is a (semi)norm on $A$ satisfying $p(e) = 1$ and $|||x||| \leq |||e||| \cdot p(x)$ for all $x \in A$. Thus, $p$ is equivalent to $\|\cdot\|$.
   Let now $x, y, u \in A$ be given such that $|||u||| = 1$. If $|||yu||| = 0$, then $|||xyu||| = 0 \leq p(x)p(y)$. If $|||yu||| \neq 0$, then $|||x\frac{yu}{|||yu|||}||| \leq p(x)$, hence $|||xyu||| \leq p(x)p(y)$. Thus $p$ is submultiplicative. □

**Examples 1.1.**

1. Let $\lambda \in \{c_0, c, l^\infty\}$ be given. Then $(\lambda, \|\cdot\|_\infty)$ is a normed algebra with respect to componentwise multiplication (see proposition 1.7). As a normed space $(\lambda, \|\cdot\|_\infty)$ is a Banach space. Such algebras are called Banach algebras. $c_0$ is an ideal in $l^\infty$. But $c$ is not an ideal in $l^\infty$. As for $\lambda \in \{c, l^\infty\}, \lambda$ has a unit (namely the sequence $(1)_{n\geq 0}$), whereas $c_0$ has none. $x = (x_n)_{n\geq 0} \in \lambda$ is invertible, iff there is $\varepsilon > 0$ such that $|x_n| > \varepsilon$ for all $n \geq 0$. Hence

   $$\sigma_\lambda(x) = \overline{\{x_n : n \geq 0\}}$$

   for all $x = (x_n)_{n\geq 0} \in \lambda$.
   An element $x = (x_n)_{n\geq 0} \in c_0$ is quasiinvertible, iff $x_n \neq 1$ for all $n \geq 0$. Then $(\frac{x_n}{x_n - 1})_{n\geq 0} \in$

$c_0$ is the quasiinverse of $x$. Thus, we have

$$\sigma_{c_0}(x) = \{0\} \cup \{x_n : n \geq 0\} = \sigma_\lambda(x),$$

for all $x = (x_n)_{n\geq 0} \in c_0$.

2. Let $p \in [1,\infty)$, then $(l^p, \|\cdot\|_p)$ becomes a Banach algebra by componentwise multiplication, as well (see proposition 1.7). Clearly, $(l^p, \|\cdot\|_p)$ has no unit. It is easy to see that $x = (x_n)_{n\geq 0} \in l^p$ is quasiinvertible, iff $x_n \neq 1$ for all $n \geq 0$. Then $(\frac{x_n}{x_n-1})_{n\geq 0} \in l^p$ is the quasiinverse of $x$.

3. On $l^1$ also the multiplication

$$* : l^1 \times l^1 \longrightarrow l^1, ((x_n)_{n\geq 0}, (y_n)_{n\geq 0}) \longmapsto \left(\sum_{k=0}^{n} x_k y_{n-k}\right)_{n\geq 0},$$

which is called convolution, is well-defined and makes $(l^1, \|\cdot\|_1)$ a Banach algebra with the unit $(\delta_{0n})_{n\geq 0}$. (See e.g. [20, p. 24].)

4. Each of the above mentioned algebras contains $\varphi$ as an ideal with the exception of $(l^1, *)$. However, $\varphi$ is a subalgebra of $(l^1, *)$ which contains the unit.
Clearly, $G((\varphi, *)) = (\mathbb{K} \setminus \{0\}) \times \{0\}^{\mathbb{N}}$. However, an element $x \in \varphi \setminus G((\varphi, *))$ may be invertible in $(l^1, *)$ (take e.g. $x = (1, \frac{1}{2}, 0, \ldots) \in \varphi$, which has the inverse $((-\frac{1}{2})^n)_{n\geq 0} \in l^1$; whereas $y = (\frac{1}{2}, 1, 0, \ldots)$ is not invertible in $l^1$, because invertibility of $y$ would require $y^{-1} = ((-1)^n 2^{n+1})_{n\geq 0} \in \omega \setminus l^1$).
On $(\varphi, \|\cdot\|_\infty)$, both

$$R_y : \varphi \longrightarrow \varphi, x \longmapsto x * y \quad \text{and} \quad L_y : \varphi \longrightarrow \varphi, x \longmapsto y * x$$

are continuous for all $y \in \varphi$, whereas

$$* : (\varphi, \|\cdot\|_\infty) \times (\varphi, \|\cdot\|_\infty) \longrightarrow (\varphi, \|\cdot\|_\infty), (x, y) \longmapsto x * y$$

is not continuous.

**Proof:** Let $y = (y_n)_{n\geq 0} \in \varphi \setminus \{0\}$ and $\varepsilon > 0$. There is $n \geq 0$ such that $y_k = 0$ for all $k > n$. Defining $\rho := \|y\|_\infty$ and $\delta := \frac{\varepsilon}{(n+1)\rho}$ we obtain

$$\|L_y(x)\|_\infty = \|R_y(x)\|_\infty = \max_{k\geq 0} \left| \sum_{j=0}^{k} y_j x_{k-j} \right| \leq (n+1)\rho\delta = \varepsilon$$

for all $x = (x_n)_{n\geq 0} \in B_\varphi[0, \delta]$. On the other hand,

$$x_\varepsilon^{(n)} := \sum_{k=0}^{n} (\varepsilon \delta_{kl})_{l\geq 0} \in B_\varphi[0, \varepsilon]$$

for all $\varepsilon > 0$ and for all $n \geq 0$. But

$$||x_\varepsilon^{(n)} * x_\varepsilon^{(n)}||_\infty = (n+1)\varepsilon^2 \to \infty \; (n \to \infty).$$ □

A multiplication $\cdot : A \times A \to A, (x,y) \mapsto xy$ on an algebra $A$ which is endowed with a linear topology is called separately continuous, if for all $y \in A$ both $R_y$ and $L_y$ are continuous. So we have seen that separate continuity of multiplication on an algebra does not imply continuity.

However, by [36, p. 112], on an algebra with a metrizable Baire topology multiplication is continuous, iff it is separately continuous (see also [22, propostion 5.1.4]). Thus, by Baire's theorem (see e.g. [22, theorem 5.1.1]), on an algebra with a Fréchet topology continuity and separate continuity of multiplication coincide.

More generally, if $E$ is a Baire *tvs*, and $F$ is a metrizable *tvs*, then for any *tvs* $G$ a bilinear map $b : E \times F \to G$ is continuous, iff it is separately continuous (see [36, p. 112]).

**Definition-Remark 1.1.** *Let $A$ be an algebra and $X, Y \subset A$.*

1. *$X$ will be called multiplicative or idempotent, if $X^2 := X \cdot X \subset X$.*
2. *As arbitrary intersections of multiplicative sets are again multiplicative and, especially, $A$ is multiplicative,*

$$\mathcal{M}(Y) := \bigcap \{Z \subset A : Y \cup Z^2 \subset Z\}$$

*is the smallest multiplicative subset of $A$ containing $Y$ and equal to $\bigcup_{k \in \mathbb{N}} Y^k$, where $Y^{k+1} := YY^k$ is defined inductively for all $k \in \mathbb{N}$.*

**Remarks:** Let $A, X$, and $Y$ be given as above. Let, furthermore, $\mathcal{T}$ be any topology on $A$.

1. It is easy to see that $\mathrm{Circ}(X)\,\mathrm{Circ}(Y) \subset \mathrm{Circ}(XY)$, and $\mathrm{Conv}(X)\,\mathrm{Conv}(Y) \subset \mathrm{Conv}(XY)$, and $\Gamma(X)\Gamma(Y) \subset \Gamma(XY)$. Hence, if $X$ is multiplicative, so will be $\mathrm{Circ}(X)$, $\mathrm{Conv}(X)$, and $\Gamma(X)$ (see e.g. [28, lemma 1.3]).
   Moreover, if $X$ is multiplicative, then $[X]$ is a subalgebra of $A$. Thus,

   $$< Z > := [\mathcal{M}(Z)]$$

   is the smallest subalgebra of $A$ containing $Z$ for any subset $Z \subset A$.

2. If multiplication is separately continuous on $(A, \mathcal{T})$, then $\overline{X} \cdot \overline{Y} \subset \overline{X \cdot Y}$, hence the closure of a multiplicative set is again multiplicative. This generalizes [28, lemma 1.4, b)].

   **Proof:** Since $L_x : A \to A, y \mapsto xy$ is continuous for all $x \in X$, we obtain $X\overline{Y} \subset \overline{XY}$. Now, continuity of $R_y : A \to A, x \mapsto xy$ for all $y \in \overline{Y}$ yields $\overline{X} \cdot \overline{Y} \subset \overline{X\overline{Y}} \subset \overline{XY}$. □

3. If $U \subset A$ is an absorbant, absolutey convex, idempotent subset, then the Minkowski functional $p_U$ is a submultiplicative seminorm on $A$ (see e.g. [17, p. 61], and [28, lemma 1.2]). We have already mentioned that, on the other hand, the closed unit ball of a submultiplicative seminorm $\|\cdot\|$ on $A$ is an idempotent subset of $A$.

**Definition 1.2** *Let $A$ be an algebra and $\mathcal{T}$ be an lc or a linear topology on $A$. $A = (A, \mathcal{T})$ will be called*

1. *an lc or a topological (top) algebra, respectively, if $\cdot : A \times A \to A$ is continuous.*

2. *an lmc or an m-topological (mtop) algebra, respectively, if $A$ has a 0nbhd-basis consisting of multiplicative sets.*

*In the sequel a 0nbhd-basis consisting of multiplicative sets will be called an $m-$basis.*

Note that in literature often '$top$ ($lc$) algebra' means that multiplication is separately continuous only.

**Remarks:**

1. Clearly, the existence of an $m-$basis implies continuity of $\cdot : A \times A \to A$ at $(0,0)$, hence, by [22, proposition 5.1.3] continuity.

2. For an algebra $A$ and a linear topology on $A$ t.f.a.e.

   *i)* $A$ is an $mtop$ algebra.
   *ii)* $\forall U \in \mathcal{U}_0(A)\ \exists V \in \mathcal{U}_0(A)\ \exists \alpha > 0 : \ \alpha V^2 \subset V \subset U$.
   *iii)* $\forall U \in \mathcal{U}_0(A)\ \exists V \in \mathcal{U}_0(A) : \ \mathcal{M}(V) \subset U$.
   *iv)* $\forall U \in \mathcal{U}_0(A)\ \exists V \in \mathcal{U}_0(A)\ \exists \alpha > 0 : \ \alpha \mathcal{M}(V) \subset U$.

   The equivalence of *i)* $-$ *iv)* will still hold, if we replace $\mathcal{U}_0(A)$ by any 0nbhd-basis $\mathcal{V}$ of $A$.
   Moreover, the sets of an $m-$basis in an $mtop$ algebra $A$ can be assumed as circled and closed or as absolutely convex and closed, respectively, if $A$ is in fact $lmc$ (see remark 1.1.1 and 1.1.2). In the sequel a multiplicative and (absolutely) convex set will be called (absolutely) $m$-convex

3. Immediately, one checks, that $A_e$ is an $(m)top$ algebra, if $A$ is an $(m)top$ algebra.

4. Note that, in general, multiplication on a (commutative) $top$ algebra is not uniformly continuous (cf. [26, p. 23], where the contrary is claimed). Even in the scalar case multiplication
   $$\cdot : \mathbb{K} \times \mathbb{K} \longrightarrow \mathbb{K}, (\alpha, \beta) \longmapsto \alpha\beta$$
   fails to be uniformly continuous, as can be verified easily (see also [22, p. 56ff]).
   However, our first proposition yields that the completion of a Hausdorff $(m)top$ algebra is an $(m)top$ algebra again.

5. Given algebras $A$ and $B$, and a multiplicative map $f : A \to B$, and subsets $X, Y, Z$ of $A$ or of $B$, respectively, the relation $XY \subset Z$ will be preserved under the formation of images and preimages of $f$ (see e.g. [28, lemma 1.4, a)]). Thus, an algebra provided with the initial topology with respect to an arbitrary family of linear, multiplicative maps into $(m)top$ algebras is again an $(m)top$ algebra. Now we may conclude that subalgebras and products of $(m)top$ algebras (the latter with respect to pointwise multiplication) are $(m)top$ algebras.

It is also obvious that the projective limit of a projective spectrum

$$((A_s)_{s\in T}, (p_{rs} : A_r \longrightarrow A_s)_{r,s\in T, r\geq s}),$$

where $(T, \leq)$ is a directed set, $A_s$ is an algebra for all $s \in T$, and $p_{rs} : A_r \to A_s$ is a linear, multiplicative map for all $r, s \in T, r \geq s$, is a subalgebra of $\prod_{s\in S} A_s$. Thus, if, in addition, each $A_s$ is an $(m)top$ algebra and each $p_{rs}$ is continuous, then the projective limit is $(m)top$ as well.
For inductive topologies, the situation is much more complicated (see chapter 3). However, if $A$ is an $(m)top$ algebra and $I \subset A$ is an ideal, then it is quite clear that $A/_I$ is an $(m)top$ algebra.

**Examples 1.2.**

1. There are always most trivial examples of $mtop$, and of $lmc$ algebras, namely the algebra $E_{nil}$ (i. e. $E$ provided with zero-multiplication) which is an $mtop$ algebra for any $tvs$ $E$.
   In case $A = E_{nil}$ we have $\tilde{x} = -x$ and $\sigma_A(x) = \{0\}$ for all $x \in A$.
2. Each seminormed algebra $(A, \|\cdot\|)$ is an $lmc$ algebra with the $m-$basis

$$\{\varepsilon B_{\|\cdot\|} : 0 < \varepsilon \leq 1\}$$

   (see e.g. [28, proposition 2.4, a)] or [26, p. 13]).
   More generally, a locally bounded $top$ algebra is $mtop$ and has an $m$-basis of the form $\{\varepsilon B : \varepsilon \in (0,1]\}$, where $B$ is a bounded and circled 0nbhd.

   **Proof:** Let $\tilde{B}$ be a bounded and circled 0nbhd. Continuity of multiplication implies that there is $\delta > 0$ such that $\delta\tilde{B}^2 \subset \tilde{B}$. Now, $B := \delta\tilde{B}$ is the desired bounded, circled, and idempotent 0nbhd. □

3. $\omega$ is an $lmc$ algebra which is not a normed algebra. As an $lcs$, $\omega$ is a Fréchet space. Such algebras are called $lmc$ Fréchet algebras. An algebra with a Fréchet topology and continuous multiplication will be called Fréchet algebra.
4. Let $A$ be a $top$ algebra. On $A^{\mathbb{N}\cup\{0\}}$ convolution is well-defined and $(A^{\mathbb{N}\cup\{0\}}, *)$ is an algebra. Let $U \in \mathcal{U}_0(A)$ be given. Then

$$\left(\prod_{k=0}^{n} U \times \prod_{k>n} A\right) * \left(\prod_{k=0}^{n} U \times \prod_{k>n} A\right) \subset \prod_{k=0}^{n} \sum_{j=0}^{k} U^2 \times \prod_{k>n} A.$$

Hence $(A^{\mathbb{N}\cup\{0\}},*)$ is a *top* algebra. If $A$ is even an *lmc* algebra, then $(A^{\mathbb{N}\cup\{0\}},*)$ is *lmc*, because $U \in \mathcal{U}_0(A)$ can be assumed as absolutely $m$-convex and one obtains

$$\left(\prod_{k=0}^{n}\frac{1}{n+1}U\times\prod_{k>n}A\right) \;*\; \left(\prod_{k=0}^{n}\frac{1}{n+1}U\times\prod_{k>n}A\right) \;\subset\; \prod_{k=0}^{n}\frac{1}{n+1}U\times\prod_{k>n}A.$$

5. Let $p \in (0,1)$. $l^p := \left\{(x_n)_{n\geq 0} \in \omega : \sum_{n\geq 0} |x_n|^p < \infty\right\}$ is a complete, metrizable *tvs* which is not *lc*, its topology being defined by the $F$-norm

$$||\cdot|| : l^p \longrightarrow [0,\infty), (x_n)_{n\geq 0} \longmapsto \sum_{n\geq 0} |x_n|^p$$

(see [22, example 6.10 F]). Now we claim that componentwise multiplication '$\cdot$' as well as convolution '$*$' are well-defined on $l^p$ and both $(l^p,\cdot)$ and $(l^p,*)$ are *mtop* algebras. (Note that $l^p$ is locally bounded.)

**Proof:** $l^p$ has a 0nbhd-basis consisting of all sets

$$B[0,\varepsilon] := \{x \in l^p : ||x|| \leq \varepsilon\} \quad (0 < \varepsilon \leq 1).$$

Now, for all $x = (x_n)_{n\geq 0}, y = (y_n)_{n\geq 0} \in B[0,\varepsilon]$ we have

$$||x\cdot y|| = \sum_{n\geq 0} |x_n y_n|^p \leq \min\{||x||,||y||\},$$

because $\varepsilon \leq 1$. Thus, we obtain $B[0,\varepsilon]\cdot B[0,\varepsilon] \subset B[0,\varepsilon]$.
As for '$*$', one easily computes that $(\alpha+\beta)^p \leq \alpha^p + \beta^p$ for all $\alpha,\beta \geq 0$.
Thus, for all $x = (x_n)_{n\geq 0}, y = (y_n)_{n\geq 0} \in l^p$ we have

$$||x * y|| = \sum_{n\geq 0}\left|\sum_{k=0}^{n} x_k y_{n-k}\right|^p \leq \sum_{n\geq 0}\sum_{k=0}^{n} |x_k y_{n-k}|^p = ||x||\,||y||.$$

That is to say $||\cdot||$ is submultiplicative and we get

$$B[0,\varepsilon] * B[0,\varepsilon] \subset B[0,\varepsilon].$$

□

Now, we are going to introduce an example of a *top* algebra which is not *mtop*.

6. For a measurable space $(X,\mathcal{S})$, and a *top* algebra $C$ we call a map $f : X \to C$ $\mathcal{S}$-simple (or 'simple', in abbreviation), if there are pairwise disjoint sets $S_1,\ldots,S_n \in \mathcal{S}$, and $\gamma_1,\ldots,\gamma_n \in C$ for some natural number $n$ such that

$$\bigcup_{1\leq k\leq n} S_k = X \quad\text{and}\quad f = \sum_{k=1}^{n} \chi_{S_k}\gamma_k.$$

It is easy to see that the set of all simple maps $f : X \to C$, which will be denoted by $S(X,C)$, is an algebra with respect to pointwise operations.
Now, we call $f : X \to C$ $\mathcal{S}$-measurable (or 'measurable', for short), if it is the pointwise limit of a sequence of simple maps $(f_n)_{n\in\mathbb{N}} \in S(X,C)^{\mathbb{N}}$. $A := \{f : X \to C : f \text{ measurable}\}$ is an algebra with respect to pointwise operations, because addition, and multiplication, and scalar multiplication are continuous on $C$.
Let us now denote by $\mathcal{B}$ the Borel $\sigma$-algebra on $C$, generated by its linear topology $\mathcal{T}$. Furthermore, we assume that each open subset $T$ of $C$ is is the union of countably many open sets such that the closure of each is contained in $T$. We will refer to this property as ($\sharp$). Then $f^{-1}(\mathcal{B}) \subset \mathcal{S}$, if $f : X \to C$ is measurable. Indeed, it suffices to prove $f^{-1}(\mathcal{T}) \subset \mathcal{S}$. This is obvious for $f \in S(X,C)$. Let

$$f = \lim_{n\to\infty} f_n, \quad (f_n)_{n\in\mathbb{N}} \in S(X,C)^{\mathbb{N}}.$$

For each $T = \overset{\circ}{T} \subset C$ one can find a sequence $(T_n)_{n\in\mathbb{N}}$ of open subsets of $C$, such that $T = \bigcup_{n\in\mathbb{N}} T_n$ and $\overline{T}_n \subset T$ for all $n \in \mathbb{N}$. Now we can conclude for any $x \in X$:

$$f(x) \in T \iff \exists m \in \mathbb{N} : f(x) \in T_m \iff$$

$$\iff \exists m \in \mathbb{N}\, \exists n \in \mathbb{N}\, \forall k \geq n : f_k(x) \in T_m.$$

That is to say

$$f^{-1}(T) = \bigcup_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} f_k^{-1}(T_m) \in \mathcal{S}.$$

Note that each semimetrizable topological space has property ($\sharp$). The above considerations could as well have been undertaken for any topological space instead of a topological algebra (see [13, p. 94]).

7. Let $C$ be a *top* algebra with property ($\sharp$) and $A$ the algebra of all measurable functions $f : [0,1) \to C$. A 0nbhd-basis of a linear topology on $A$ (the so-called topology of convergence in measure) can be defined by $\mathcal{V} := \left\{U(V,\varepsilon) : V = \overline{V} \in \mathcal{U}_0(C), 0 < \varepsilon \leq 1\right\}$, where

$$U(V,\varepsilon) := \left\{f \in A : \lambda\left(f^{-1}(C \setminus V)\right) \leq \varepsilon\right\},$$

$\lambda$ denoting the Lebesgue-measure on $[0,1)$. The topology thus achieved is the coarsest topology, iff $C$ carries the coarsest topology.
It is well known for the case $C = \mathbb{K}$ that $A' = \{0\}$ (see [22, example 6.10 J]). Now, the same reasoning as in the scalar case yields $\mathrm{Conv}(U(V,\varepsilon)) = A$ for all closed sets $V \in \mathcal{U}_0(C)$ and for all $\varepsilon > 0$.
Indeed, let $f \in A$, $\varepsilon > 0$, and $V = \overline{V} \in \mathcal{U}_0(C)$ be given. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. Moreover, for all $0 \leq j \leq n$ we set $t_j := \frac{j}{n}$ and

$$f_j(t) := \begin{cases} f(t), & t \in [t_{j-1}, t_j) \\ 0, & t \in [0,1) \setminus [t_{j-1}, t_j) \end{cases}$$

for all $1 \le j \le n$. Clearly, $nf_j \in U(V,\varepsilon)$ and $f = \sum_{j=1}^{n} f_j$.
Now we claim that $A$ is a *top* algebra with respect to pointwise multiplication.

**Proof:** Let a closed 0nbhd $V \in \mathcal{U}_0(C)$ and $\varepsilon > 0$ be given. Since multiplication is continuous on $C$, we can choose a closed set $W \in \mathcal{U}_0(C)$ satisfying $W^2 \subset U$. But this yields

$$(fg)^{-1}(C\setminus V) \subset f^{-1}(C\setminus W) \cup g^{-1}(C\setminus W),$$

for all $f,g \in A$. Hence

$$\left(U\left(W,\frac{\varepsilon}{2}\right)\right)^2 \subset U(V,\varepsilon).$$

□

Now we claim that $A$ is not *mtop* if $C$ has a unit $e$ and $C \neq \overline{\{0\}}$. We prove

$$\mathcal{M}(U(V,\varepsilon)) = A$$

for all $V \in \mathcal{U}_0(C)$ and for all $\varepsilon > 0$.

**Proof:** One can find $\delta > 0$ such that $\delta \cdot e \in V$. For $f \in A$ choose $n \in \mathbb{N}$ and $0 = t_0 < \ldots < t_n = 1$, such that $t_j - t_{j-1} < \varepsilon$, for all $1 \le j \le n$. Now, we set

$$f_j(t) := \begin{cases} \delta e, & t \in [0,1)\setminus[t_{j-1},t_j) \\ \delta^{-n+1} f(t), & t \in [t_{j-1},t_j) \end{cases}$$

for all $1 \le j \le n$. Then, clearly, $f_j \in U(V,\varepsilon)$ and $\prod_{j=1}^{n} f_j = f$. □

The requirement $C$ having a unit $e$ is indeed necessary, because in case $C = C_{nil}$ we obtain $A = A_{nil}$, which is an *mtop* algebra.

8. Let now $C$ be any *top* algebra satisfying (♯) with a unit $e$. Then $(f_e : [0,1) \to C, x \mapsto e) \in S([0,1),C)$ is a unit in $A$. Invertibility of $f \in A$ implies that all its values are invertible. We claim that inversion is continuous in $A$, if it is continuous in $C$. (See also [38, p. 731], where continuity of the inversion map is proved for the special case $C = \mathbb{K}$.)

   **Proof:** Note that for multiplicative groups $(G,\cdot),(H,\cdot)$ each provided with a topology making multiplication separately continuous, continuity of a map $\varphi : G \to H$ which is multiplicative or antimultiplicative, respectively, is the same as continuity of $\varphi$ at the unit of $G$.
   Let now $V = \overline{V} \in \mathcal{U}_0(C)$ be given. We can find $W = \overline{W} \in \mathcal{U}_0(C)$, such that $x^{-1} \in e + V$ for all $x \in (e+W) \cap G(C)$. Now for any $f \in G(A)$ we obtain

$$\{x \in [0,1) : (f - f_e)(x) \in W\} \subset \left\{x \in [0,1) : (f^{-1} - f_e)(x) \in V\right\}.$$

Hence

$$\lambda\left(\{x \in [0,1) : (f^{-1} - f_e)(x) \in C \setminus V\}\right) \leq$$
$$\leq \lambda(\{x \in [0,1) : (f - f_e)(x) \in C \setminus W\}).$$

□

**Proposition 1.1** *Let Hausdorff tv-spaces $E$, $\tilde{E}$, $F$, $\tilde{F}$, and $G$ be given such that $E$ is a dense subspace of $\tilde{E}$, and $F$ is a dense subspace of $\tilde{F}$, respectively, and such that $G$ is complete. Then any continuous, bilinear map*

$$b : E \times F \longrightarrow G, (x,y) \longmapsto b(x,y)$$

*has a unique continuous and bilinear extension*

$$\tilde{b} : \tilde{E} \times \tilde{F} \longrightarrow G, (x,y) \longmapsto \tilde{b}(x,y).$$

(See also [6, chap. 3, p. 50], where the assertion is proved for complete topological groups $\tilde{E}$, $\tilde{F}$, $G$, and dense subgroups $E \subset \tilde{E}$, and $F \subset \tilde{F}$, respectively.)

**Proof:** First observe that it suffices to prove that there is a unique, bilinear extension $\tilde{b} : \tilde{E} \times F \to G, (x,y) \mapsto \tilde{b}(x,y)$.
Since $b_y : E \to G, x \mapsto b(x,y)$ is a linear, continuous map to a complete Hausdorff $tvs$ for all $y \in F$ and $E$ is a dense subspace of $\tilde{E}$, there is by [22, theorem 3.4.2] a unique extension of $b_y$ to a linear, continuous map $\tilde{b}_y : \tilde{E} \to G, x \mapsto \tilde{b}_y(x)$. Now we claim that

$$\tilde{b} : \tilde{E} \times F \longrightarrow G, (x,y) \longmapsto \tilde{b}_y(x)$$

satisfies the above requirements.
$\tilde{b}$ is bilinear, because $\mu\tilde{b}_y + \tilde{b}_z$ is a linear, continuous extension of $b_{\mu y+z}$ for all $y, z \in F$ and $\mu \in I\!K$ which yields $\mu\tilde{b}_y + \tilde{b}_z = \tilde{b}_{\mu y+z}$.
It remains to prove that $\tilde{b}$ is continuous. Let therefore $U = \overline{U} \in \mathcal{U}_0(G)$ be given. Since $b$ is continuous, one can find $V \in \mathcal{U}_0(E)$ and $W \in \mathcal{U}_0(F)$ satisfying $b(V \times W) \subset U$. Now, $\overline{V}^{\tilde{E}}$ is a 0nbhd in $\tilde{E}$. We show that $\tilde{b}\left(\overline{V}^{\tilde{E}} \times W\right) \subset U$. For all $w \in W$ we obtain

$$\tilde{b}\left(\overline{V}^{\tilde{E}}, w\right) = \tilde{b}_w\left(\overline{V}^{\tilde{E}}\right) \subset \overline{\tilde{b}_w(V)} = \overline{b_w(V)} \subset U$$

because $\tilde{b}_w$ is continuous on $\tilde{E}$ for each $w$. □

As a corollary, we obtain an impotant permanence property of $(m)top$ algebras:

**Corollary 1.1** *Let $A$ be a Hausdorff $(m)top$ algebra. Then there is a unique extension of multiplication on $A$ to a multiplication on the completion of $A$, $\tilde{A}$, such that $\tilde{A}$ is an $(m)top$ algebra. Moreover, $\tilde{A}$ is commutative, if $A$ is commutative.*

**Proof:** Multiplication on $A$ can be extended to a uniquely determined continuous, bilinear map

$$\odot : \tilde{A} \times \tilde{A} \longrightarrow \tilde{A}, (x,y) \longmapsto xy := x \odot y.$$

Since $A$ is dense in $\tilde{A}$, and $\odot : \tilde{A} \times \tilde{A} \to \tilde{A}$ is a continuous extension of $\cdot : A \times A \to A$, and since $\cdot : A \times A \to A$ is associative, $\odot$ is associative as well. The same reasoning yields that $(A, \odot)$ is commutative, if $A$ is commutative.
Let finally $\mathcal{V}$ be an $m$-basis in $A$. Then, naturally, $\left\{ \overline{V}^{\tilde{A}} : V \in \mathcal{V} \right\}$ is an $m$-basis in $\tilde{A}$. □

In analogy to the theory of *lc* spaces there is a well-known representation of *lmc* algebras as dense subalgebras of projective limits of Banach algebras (see [28, p. 19f], also [22, p. 42f]).

**Definition-Remark 1.2.** *A projective system*

$$\mathcal{E} = ((E_s)_{s \in T}, (p_{rs} : E_r \longrightarrow E_s)_{r,s \in T, r \geq s})$$

*of tv-spaces $E_s$ with linear, continuous linking maps $p_{rs}$ and projective limit*

$$proj\, \mathcal{E} = \left\{ (x_s)_{s \in T} \in \prod_{s \in T} E_s : \forall r,s \in T, r \geq s : \; p_{rs}(x_r) = x_s \right\}$$

*is called reduced, if the canonical projections $p_t : proj\mathcal{E} \to A_t, (x_s)_{s\in T} \mapsto x_t$ have dense range for all $t \in T$.*
*Clearly, $p_s = p_{rs} \circ p_r$ implies range$(p_s) \subset$ range$(p_{rs})$ $(r,s \in T, r \geq s)$. Thus, all the linking maps $p_{rs}$ have dense range, if the projective system is reduced. The converse is, in general, not true (see [3, example 2.4]).*

The following proposition sums up the various representation theorems for *lmc* algebras stated in literature (e.g. [28, theorem 5.1], [17, theorem 3.3.7]).

**Proposition 1.2** *Let $A$ be an algebra and $\mathcal{T}$ a Hausdorff linear topology on $A$. Then in 1)-3) i) and ii) are equivalent:*

*1)* *i) $A$ is lmc.*

*ii) $A = (A, \mathcal{T})$ is isomorphic to a dense subalgebra of the projective limit of a reduced projective system $((A_s)_{s\in T}, (p_{rs} : A_r \to A_s)_{r,s\in T, r\geq s})$ of Banach algebras $A_s$, where $(T, \leq)$ is a directed set, and $p_{rs} : A_r \to A_s$ is a linear, multiplicative, and continuous map for all $r,s \in T$, $r \geq s$.*

*2)* *i) $A$ is a complete lmc algebra.*

*ii) $A = (A, \mathcal{T})$ is isomorphic to the projective limit of a reduced projective system $((A_s)_{s\in T}, (p_{rs} : A_r \to A_s)_{r,s\in T, r\geq s})$ of Banach algebras $A_s$, where $(T, \leq)$ is a directed set, and $p_{rs} : A_r \to A_s$ is a linear, multiplicative, and continuous map for all $r,s \in T$, $r \geq s$.*

*3)* *i) A is an lmc Fréchet algebra.*

*ii)* $A = (A, \mathcal{T})$ *is isomorphic to the projective limit of a reduced projective sequence* $((A_n)_{n\in\mathbb{N}}, (p_{n+1,n} : A_{n+1} \to A_n)_{n\in\mathbb{N}})$ *of Banach algebras* $A_n$, *where* $p_{n+1,n} : A_{n+1} \to A_n$ *is a linear, multiplicative, and continuous map for all* $n \in \mathbb{N}$.

**Proof:**

1) It only remains to prove $i) \Rightarrow ii)$. Let $\mathcal{V}$ be a 0nbhd-basis in $A$ consisting of absolutely $m$-convex sets. For each $U \in \mathcal{V}$ the Minkowski functional $p_U$ is a submultiplicative seminorm on $A$ and the kernel $\text{kern}(p_U)$ is an ideal in $A$. The completion $A_U$ of $A/_{\text{kern}(p_U)}$ is a Banach algebra and for all $U, V \in \mathcal{V}, U \subset V$ the linking map

$$q_{UV} : A/_{\text{kern}(p_U)} \to A_V, x + \text{kern}(p_U) \mapsto x + \text{kern}(p_V)$$

is linear, multiplicative, and continuous and has a unique continuous extension $\widetilde{q_{UV}}$ to $A_U$ which is again linear and multiplicative. From the theory of *lc* spaces one knows that $A$ is (linearly) topologically isomorphic to a dense subspace (which is in fact a subalgebra) of the projective limit of the reducedx projective spectrum $\left((A_U)_{U\in\mathcal{V}}, \widetilde{(q_{UV})}_{U,V\in\mathcal{V},U\subset V}\right)$ (which is an *lmc* algebra) and one easily checks that the canonical (linear) isomorphism is also multiplicative.

2) This is a direct consequence of 1), because the projective limit in 1) is a completion of A.

3) This is obvious, because we may assume $\mathcal{V}$ in 1) as countable, i.e.

$$\mathcal{V} = \{U_n : n \in \mathbb{N}\}.$$

□

Note that every projective system of *top* algebras

$$((A_s)_{s\in T}, (p_{rs} : A_r \longrightarrow A_s)_{r,s\in T, r\geq s}),$$

where every $p_{rs}$ is linear, multiplicative, and continuous, is equivalent to a reduced one, namely the projective system

$$\left((\overline{p_s(A)})_{s\in T}, (p_{rs}|\overline{p_r(A)})_{r,s\in T, r\geq s}\right),$$

$p_s : A \to A_s$ denoting the canonical projection. This means, that the corresponding projective limits

$$A := \text{proj}\,((A_s)_{s\in T}, (p_{rs} : A_r \longrightarrow A_s)_{r,s\in T, r\geq s})$$

and

$$\tilde{A} := \text{proj}\left((\overline{p_s(A)})_{s\in T}, (p_{rs}|\overline{p_r(A)})_{r,s\in T, r\geq s}\right)$$

are topologically isomorphic. The corresponding isomorphism is in fact the identity map. (See e.g. [22, proposition 2.6.2].)

It is well known that in a Banach algebra $A$ which has a unit $G(A)$ is an open subset of $A$ and that the inversion map

$$\cdot^{-1} : G(A) \longrightarrow G(A), x \longmapsto x^{-1}$$

is a homeomorphism (see e.g. [27, Satz 17.3]). Hence also $\tilde{G}(A)$ is an open subset of $A$ and the quasiinversion map

$$\tilde{\cdot} : \tilde{G}(A) \longrightarrow \tilde{G}(A), x \longmapsto \tilde{x}$$

is a homeomorphism as well, because $A \to A_e, x \mapsto x - e$ is a homeomorphism onto its range. On the other hand, if $A$ is an arbitrary topological algebra with a unit $e$, continuity of the quasiinversion map is equivalent to continuity of the inversion map, because quasiinvertibility of an element $x \in A$ with quasiinverse $\tilde{x}$ is the same as invertibility of $x - e$ with inverse $\tilde{x} - e$.

We will now see that the quasiinversion map is always continuous on *lmc* algebras (see also [28, proposition 2.8]). For this purpose we prove the next lemma, which generalizes [28, theorem 5.2]

**Lemma 1.1** *Let $((A_s)_{s\in T}, (p_{rs} : A_r \to A_s)_{r,s\in T, r\geq s})$ be a projective spectrum of algebras with linear and multiplicative linking maps, and denote by $A$ its projective limit.*

1. *An element $x = (x_s)_{s\in T}$ of $A$ is quasiinvertible in $A$, iff it is quasiinvertible in $\prod_{s\in T} A_s$, iff each $x_s$ is quasiinvertible in $A_s$. In this case $\widetilde{(x_s)_{s\in T}} = (\tilde{x}_s)_{s\in T}$ holds.*
2. *Let now each $A_s$ be an algebra with a Hausdorff linear topology such that multiplication is separately continuous on $A_s$ for all $s \in T$ and let $p_{rs}$ be continuous for all $r \geq s$ and let us assume that the projective system is reduced. Then $A$ has a unit $e$, iff each $A_s$ has a unit $e_s$. In this case $e = (e_s)_{s\in T}$ and $p_{rs}|G(A_r) : G(A_r) \to G(A_s)$ is a group homomorphism for all $r, s \in T$, $r \geq s$. Consequently, $x = (x_s)_{s\in T} \in A$ is invertible in $A$, iff $x_s$ is invertible in $A_s$ for all $s \in T$, in which case we have $((x_s)_{s\in T})^{-1} = (x_s^{-1})_{s\in T}$.*

**Proof:**

1. Let $(x_s)_{s\in T} \in A$ be given. Quasiinvertibility of $(x_s)_{s\in T}$ in $\prod_{s\in T} A_s$ means that one can find $(y_s)_{s\in T} \in \prod_{s\in T} A_s$, such that $x_s y_s = y_s x_s = x_s + y_s$ for all $s \in T$. Now, $p_{rs}$ is linear and multiplicative for all $r, s \in T$ with $r \geq s$. This implies
$$p_{rs}(y_r) = (p_{rs}(x_r))^{\sim} = \tilde{x}_s = y_s.$$
2. If $e = (e_s)_{s\in T}$ is a unit in $A$, then for all $s \in T$:
$$L_{e_s}|p_s(A) = R_{e_s}|p_s(A) = id_{p_s(A)} \Longrightarrow L_{e_s}|A_s = R_{e_s}|A_s = id_{A_s}.$$

Let now $e_s$ be a unit in $A_s$ for all $s \in T$ and $x_s = p_{rs}(x_r) \in \text{range}(p_{rs})$ for some $x_r \in A_r$. Then

$$p_{rs}(e_r)x_s = p_{rs}(e_r x_r) = x_s \text{ and } x_s p_{rs}(e_r) = x_s.$$

As $p_{rs}$ has dense range, $p_{rs}(e_r)$ is a unit in $A_s$, hence $p_{rs}(e_r) = e_s$.

□

The next corollary generalizes [28, corollary 5.3, a)].

**Corollary 1.2** *Let $((A_s)_{s\in T}, (p_{rs} : A_r \to A_s)_{r,s\in T, r\geq s})$ be a projective spectrum of algebras with linear and multiplicative linking maps and A denote its projective limit. Then we have*

*i)* $\sigma_A(x) \cup \{0\} = \bigcup_{s\in T} \sigma_{A_s}(x_s) \cup \{0\}$ *for all* $x = (x_s)_{s\in T} \in A$.

*ii) If each $A_s$ carries a Hausdorff linear topology such that multiplication is separately continous on $A_s$, and if the projective spectrum is reduced, then* $\sigma_A(x) = \bigcup_{s\in T} \sigma_{A_s}(x_s)$ *for all* $x = (x_s)_{s\in T} \in A$.

The proof is an immediate a konsequence of lemma 2.1.

**Corollary 1.3** *Let A be a Hausdorff lmc algebra, then the quasiinversion map*

$$\tilde{\ } : \tilde{G}(A) \longrightarrow \tilde{G}(A), x \longmapsto \tilde{x}$$

*is continuous.*

**Proof:** (For an alternative proof see [28, proposition 2.8].) There is a projective spectrum of Banach algebras

$$\mathcal{A} = ((A_s)_{s\in T}, (p_{rs} : A_r \longrightarrow A_s)_{r,s\in T, r\geq s})$$

such that $A$ is a subalgebra of proj$\mathcal{A}$. Now, an element $(x_s)_{s\in T}$ of the projective limit is quasiinvertible, iff $x_s$ is quasiinvertible in $A_s$ with quasiinverse $\tilde{x}_s \in A_s$ for all $s \in T$. Hence (quasi)inversion is continuous, because $\tilde{G}(A_s) \to \tilde{G}(A_s), x_s \mapsto \tilde{x}_s$ is continuous for all $s \in T$. □

**Corollary 1.4** *(See [28, proposition 2.9, a)].) For a Hausdorff lmc algebra A over the field of complex numbers and any* $x \in A$ *we have* $\sigma_A(x) \neq \emptyset$.

**Proof:** By [24, p. 811], for every algebra $B$ with a unit and with an *lc* topology such that multiplication is separately continuous and such that the inversion map is continuous the above assertion is true. If $A$ has no unit, then $0 \in \sigma_A(x)$ for all $x \in A$. □

**Examples 1.3.**

1. In general, continuity of the inversion map does not imply continuity of multiplication even on an algebra where multiplication is separately continuous. Take e.g. $(\varphi, \|\cdot\|_\infty, *)$ (cf. example 1.1.4). Of course, the inversion map is continuous on $(G(\varphi,*), \|\cdot\|_\infty|G(\varphi,*),*)$, since $(G(\varphi,*), \|\cdot\|_\infty|G(\varphi,*),*) \cong (\mathbb{K}\setminus\{0\}, |\cdot|, \cdot)$. Furthermore, for an *lc* algebra $A$ continuity of the inversion map does not imply that $A$ is *lmc* (see example 1.4.3).

2. There are only a few examples of *lc* algebras which are not *lmc*. In [2] Arens presented the Fréchet algebra

$$L^\omega := L^\omega([0,1]) := \bigcap_{p\in[1,\infty)} L^p([0,1])$$

provided with pointwise multiplication and with the initial topology on $L^\omega$ with respect to $(L^\omega \hookrightarrow L^p)_{p\in\mathbb{N}}$, which may as well be generated by the metric

$$d : L^\omega \times L^\omega \longrightarrow L^\omega,\ (f,g) \longmapsto \sum_{p=1}^{\infty} 2^{-p} \frac{\|f-g\|_p}{1+\|f-g\|_p}.$$

$L^\omega$ is an *lc* algebra, because Hölder's inequality implies

$$\|fg\|_p^p \leq \|f^p\|_2 \cdot \|g^p\|_2 = \|f\|_{2p}^p \|g\|_{2p}^p$$

for all $f,g \in L^\omega$ and for all $p \in \mathbb{N}$ (which also proves that pointwise multiplication is well-defined on $L^\omega$). The inversion map is not continuous on $L^\omega([0,1])$, hence $L^\omega([0,1])$ cannot be an *lmc* algebra. Take

$$f_n(t) := \begin{cases} \frac{1}{n}, & 0 \leq t \leq \frac{1}{n} \\ 1, & \frac{1}{n} < t \leq 1 \end{cases}$$

for all $n \in \mathbb{N}$. This implies

$$f_n^{-1}(t) := \begin{cases} n, & 0 \leq t \leq \frac{1}{n} \\ 1, & \frac{1}{n} < t \leq 1 \end{cases}$$

for all $n \in \mathbb{N}$. Now

$$\|f_n - 1\|_p^p = \frac{1}{n} \cdot \left(\frac{n-1}{n}\right)^p \to 0 \ \ (n \to \infty)$$

for all $p \in \mathbb{N}$. But

$$\|f_n^{-1} - 1\|_p^p = \frac{(n-1)^p}{n} \to 1 \ \ (n \to \infty)$$

if $p = 1$ and

$$\|f_n^{-1} - 1\|_p^p \to \infty \ \ (n \to \infty)$$

if $p > 1$. Another example of this type will be presented in this section.

Let us denote the algebra of polynomials on an algebra $A$,

$$\left\{\sum_{k=0}^{n} \alpha_k X^k : n \in \mathbb{N},\ \alpha_1,\dots,\alpha_n \in A\right\},$$

which is canonically isomorphic to $(A^{(\mathbb{N}\cup\{0\})}, *)$, by $A[X]$. In the sequel $A[X]$ will always be endowed with the direct sum topology. We are now going to investigate conditions such that $A[X]$ is a *top* algebra, if $A$ is a *top* algebra. Later on we will see that even in the scalar case the algebra $\mathbb{K}[X] = \varphi$ is not *lmc*.

**Definition 1.3** *Let $E$ be a tvs. If for all $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(E)^{\mathbb{N}}$ there is a sequence $(\rho_n)_{n\in\mathbb{N}}$ of positive real numbers such that*

$$\bigcap_{n\in\mathbb{N}} \rho_n U_n \in \mathcal{U}_0(E),$$

*we will say that E satisfies the countable neighbourhood condition, which we will abbreviate to $(cnc)$.*

The property $(cnc)$ is usually defined for *lc* spaces only. Some of the results for *lc* spaces concerning $(cnc)$ (see [8]) can easily be transferred to *tv*-spaces, as will be done in the sequel.

**Remarks:**

1. It is clear that each locally bounded *tvs* $E$ satisfies $(cnc)$. The converse is also true, if $E$ is, in addition, semimetrizable.

2. Moreover it is easy to check that each $gDF$-space $E$ (an *lc* space endowed with the finest *lc* topology coinciding with itself on all the sets of a fundamental sequence of bounded sets) satisfies $(cnc)$.

   **Proof:** (See [8].) Let $(B_n)_{n\in\mathbb{N}}$ be an increasing fundamental sequence of absolutely convex, bounded sets and $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(E)^{\mathbb{N}}$. We can find a sequence $(\rho_n)_{n\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ such that $B_n \subset \rho_n U_n$ for all $n \in \mathbb{N}$. Now, it suffices to prove that

$$B_m \cap \bigcap_{n\in\mathbb{N}} \rho_n U_n$$

   is a 0nbhd in $B_m$ for every $m \in \mathbb{N}$. This is obvious, because for all $n \geq m$ $B_m \subset B_n \subset \rho_n U_n$, hence

$$B_m \cap \bigcap_{n\in\mathbb{N}} \rho_n U_n = B_m \cap \bigcap_{n\leq m} \rho_n U_n.$$

□

3. If $(E_n)_{n\in\mathbb{N}}$ is a sequence of $tv$-spaces with $(cnc)$ then their direct sum $\bigoplus_{n\in\mathbb{N}} E_n$ also satisfies $(cnc)$.

**Proof:** Let $(U_n^{(m)})_{m\in\mathbb{N}} \in \mathcal{U}_0(E_n)^{\mathbb{N}}$ be a sequence of circled 0nbhds for all $n \in \mathbb{N}$. We define

$$V_m := \bigoplus_{n\in\mathbb{N}} U_n^{(m)}$$

for all $m \in \mathbb{N}$. One can find $(\rho_n^{(m)})_{m\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ such that

$$\bigcap_{m\in\mathbb{N}} \rho_n^{(m)} U_n^{(m)} \in \mathcal{U}_0(E_n)$$

for every natural number $n$. Furthermore, for all $m \in \mathbb{N}$ we define $\rho^{(m)} := \max\{\rho_n^{(m)} : n \le m\}$. Then we obtain

$$U_n := \bigcap_{m\in\mathbb{N}} \rho^{(m)} U_n^{(m)} \supset \bigcap_{m<n} \rho^{(m)} U_n^{(m)} \cap \bigcap_{m\ge n} \rho_n^{(m)} U_n^{(m)} \in \mathcal{U}_0(E_n)$$

for all $n \in \mathbb{N}$ and

$$\bigcap_{m\in\mathbb{N}} \rho^{(m)} V_m = \bigoplus_{n\in\mathbb{N}} U_n.$$

□

**Lemma 1.2** *Let $E$ be a tvs satisfying $(cnc)$. Then for every sequence $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(E)^{\mathbb{N}}$ there is $U \in \mathcal{U}_0(E)$ and a sequence $(\rho_n)_{n\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ such that*

$$\sum_{k=1}^{n} U \subset \rho_n U_n$$

*for all $n \in \mathbb{N}$.*

**Proof:** For all $n \in \mathbb{N}$ one can find $V_n \in \mathcal{U}_0(E)$ such that $\sum_{k=1}^{n} V_n \subset U_n$.
Now, the countable neighbourhood condition implies that there is a sequence $(\rho_n)_{n\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ such that

$$U := \bigcap_{n\in\mathbb{N}} \rho_n V_n \in \mathcal{U}_0(E).$$

Clearly, this yields $\sum_{k=1}^{n} U \subset \rho_n U_n$ for all $n \in \mathbb{N}$. □

**Proposition 1.3** *Let $A$ be a top algebra satisfying (cnc). For $n \in \mathbb{N}$ we define inductively*

$$A[X_1,\ldots,X_{n+1}] := (A[X_1,\ldots,X_n])[X_{n+1}].$$

*Then $A[X_1,\ldots,X_n]$ is a top algebra for all $n \in \mathbb{N}$.*

**Proof:** Remark 1.3.3 implies that it suffices to prove that $A[X]$ is a *top* algebra. Let therefore $(U_n)_{n\geq 0} \in \mathcal{U}_0(A)^{\mathbb{N}\cup\{0\}}$ be given. Now, by the above lemma, there is $V \in \mathcal{U}_0(A)$ and $(\rho_n)_{n\geq 0} \in (0,\infty)^{\mathbb{N}\cup\{0\}}$ such that $\sum_{k=0}^{n} V \subset \rho_n U_n$ for all $n \geq 0$. Inductively, one finds a sequence $(\delta_n)_{n\geq 0} \in (0,\infty)^{\mathbb{N}}$, satisfying $\delta_k\delta_l \leq \rho_{k+l}^{-1}$ for all $k,l \geq 0$. Finally, take $W \in \mathcal{U}_0(A)$, $W = \mathrm{Circ}(W)$, such that $W^2 \subset V$. This implies

$$\bigoplus_{n\geq 0} \delta_n W * \bigoplus_{n\geq 0} \delta_n W \subset \bigoplus_{n\geq 0} \left( \sum_{k=0}^{n} (\delta_k \delta_{n-k} W)\, W \right) \subset$$

$$\bigoplus_{n\geq 0} \left( \sum_{k=0}^{n} \rho_n^{-1} W W \right) \subset \bigoplus_{n\geq 0} \left( \sum_{k=0}^{n} \rho_n^{-1} V \right) \subset \bigoplus_{n\geq 0} U_n.$$

□

**Examples 1.4.**

1. The requirement (*cnc*) for $A$ is necessary in the above proposition. Take e.g. $(\omega,\cdot)$, which is an *lmc* Fréchet algebra which does not satisfy (*cnc*). We claim that

$$\bigoplus_{n\geq 0} (\omega,\cdot) * \bigoplus_{n\geq 0} (\omega,\cdot) \longrightarrow \bigoplus_{n\geq 0} (\omega,\cdot)$$

is not continuous. (Of course, it is separately continuous.)

**Proof:** For all $m \in \mathbb{N}$ we define $U_m := \prod_{k=0}^{m} B[0,\frac{1}{m}] \times \prod_{k>m} \mathbb{K} \in \mathcal{U}_0(\omega)$. Now, a 0nbhd-basis in $\bigoplus_{n\geq 0} (\omega,\cdot)$ is given by

$$\left\{ \bigoplus_{n\geq 0} U_{k_n} : (k_n)_{n\geq 0} \in \mathbb{N}^{\mathbb{N}\cup\{0\}} \text{ strictly increasing} \right\}.$$

For no such 0nbhd $U = \bigoplus_{n\geq 0} U_{k_n}$, $U^2$ is contained in $\bigoplus_{n\geq 0} U_n$, because this would imply

$$\left( \prod_{j=0}^{k_n} B\left[0,\frac{1}{k_n}\right] \times \prod_{j>k_n} \mathbb{K} \right) \cdot \left( \prod_{j=0}^{k_m} B\left[0,\frac{1}{k_m}\right] \times \prod_{j>k_m} \mathbb{K} \right) =$$

$$\prod_{j=0}^{k_n} B\left[0, \frac{1}{k_n \cdot k_m}\right] \times \prod_{j>k_n} I\!K \subset \prod_{j=0}^{n+m} B\left[0, \frac{1}{n+m}\right] \times \prod_{j>n+m} I\!K,$$

for all $n \geq 0$ and for all $m \geq n$, which is a contradiction, because $n+m \to \infty$ $(m \to \infty)$.
□

2. On the other hand, it is easy to see that continuity of multiplication on $A[X]$ for a *top* algebra $A$ does not imply that $A$ satisfies $(cnc)$. $A = \omega_{nil}$ yields $A[X] = \left(\omega^{(I\!N \cup \{0\})}\right)_{nil}$.

3. We know by proposition 1.3 that $(\varphi, *)$ is an *lc* algebra. We will now see that it is not *lmc*.

**Proof:** We define $U_n := B[0, \frac{1}{n!}] \subset I\!K$ and $e_n := (\delta_{nk})_{k \geq 0}$ for all $n \geq 0$. For each $V \in \mathcal{U}_0(\varphi)$ one can find $\varepsilon > 0$ such that $\varepsilon\, e_1 \in V$. Let us assume

$$\mathcal{M}(V) \subset \bigoplus_{n \geq 0} U_n.$$

This implies $(\varepsilon\, e_1)^n = \varepsilon^n e_n \in U_n$ for all $n \geq 0$ and we obtain $\frac{\varepsilon^{-n}}{n!} \geq 1$ for all $n \geq 0$, which is a contradiction. □

Nevertheless, the inversion map

$$\cdot^{-1} : \left(G(\varphi, *),\ \bigoplus_{n \geq 0} \mathcal{T}_{|\cdot|} \bigcap G(\varphi, *),\ *\right) \longrightarrow \left(G(\varphi, *),\ \bigoplus_{n \geq 0} \mathcal{T}_{|\cdot|} \bigcap G(\varphi, *),\ *\right)$$

is continuous, because $\left(G(\varphi, *),\ \bigoplus_{n \geq 0} \mathcal{T}_{|\cdot|} \cap G(\varphi, *),\ *\right) \cong (I\!K \setminus \{0\},\ |\cdot|,\ \cdot\ )$.

We have already mentioned that a subalgebra of an *mtop* algebra is again *mtop*. On the other hand, $\overline{\{0\}}$ is an *lmc* ideal in any algebra $A$. One may as well take algebras $A$ and $B$, where $A$ is *mtop* and $B$ is not. Then $A \times B$ is not *mtop* neither, although it contains a non-trivial ideal, which is *mtop*. So we are facing the following three-space-problem: Does the existence of an *mtop* ideal $I \subset A$ such that $A/_I$ is *mtop* imply that $A$ is itself *mtop*?
Now, one has to distinguish two cases, namely whether multiplication on the algebra under consideration (endowed with a linear topology) is already continuous or not. For the first case, there will be a partial positive result. For the second case there will be a counterexample. (These results are due to S. Dierolf, Khin Aye Aye, Schröder; see also [10].)

Motivated by the notion of a topological group, which is the semidirect product of a normal subgroup and a subgroup (see [32]), which turned out to be a rich source for examples and counterexamples, an analogous notion of a locally convex algebra, which is the semidirect product of an ideal and a subalgebra will be introduced. Moreover, a general method of constructing such semidirect products, which contains the direct products and the adjunction of a unit element as special cases, will be presented. As an application one obtains an example of an algebra $A$ provided with a Banach space topology $\mathcal{T}$ and containing an ideal $C$ such that both $(C, \mathcal{T} \cap C)$ and $(A/_C, \mathcal{T}/_C)$ are Banach algebras but $(A, \mathcal{T})$ is not.

**Definition 1.4** *Let $A$ be an algebra containing an ideal $C$ and a subalgebra $B$ such that $A = C + B$ and $C \cap B = \{0\}$. Then we call $A$ the semidirect product of $C$ and $B$ and use the notation $A = C \rtimes_S B$.*

**Remarks:** Let $A = C \rtimes_S B$.

1. Clearly, the quotient algebra $A/_C$ is canonically isomorphic to $B$, since $q : A \to B, c + b \mapsto b$ is an algebra-epimorphism with $\text{kern}(q) = C$.

2. Quasiinvertibility of elements $c + b \in A$, where $c \in C$, and $b \in B$, can be characterized in terms of qusiinvertibility in $C$ and in $B$. In fact, formal adjunction of a unit element to $A$ yields $A_e = (C \times \{0\}) \rtimes_S B_e$ which we may abbreviate to $A_e = C \rtimes_S B_e$.
Let $c \in C$ and $b \in B$ be given. Then $c + b$ is quasiinvertible in $A$ if and only if $b$ is quasiinvertible in $B$ and $c(e-b)^{-1}$ is quasiinvertible in $C$. In fact, if $e - (c + b) = -c + (e - b) \in G(A_e)$, then $e - b \in G(B_e) \subset G(A_e)$, thus $-c(e-b)^{-1} + e \in G(A_e)$ and $c(e-b)^{-1}$ is quasiinvertible in $A_e$ and hence in $C$, as $C$ is an ideal in $A_e$.
Conversely, if $e - b \in G(B_e) \subset G(A_e)$, and $e - c(e-b)^{-1} \in G(A_e)$, then $e - (c + b) \in G(A_e)$. Thus, $c + b$ is quasiinvertible in $A_e$ and hence also in $A$.
Now, consequently, a complex number $\lambda \in \mathbb{K} \setminus \{0\}$ belongs to $\sigma_A(c+b)$ iff either $\lambda \in \sigma_B(b)$ or $\frac{1}{\lambda} b$ is quasiinvertible in $B$ with quasiinverse element $d \in B$ and $\lambda \in \sigma_C(c - cd)$.

**Definition-Remark 1.3.** *Let $(A, \mathcal{T})$ be a top algebra, and let $C \subset A$ be an ideal, and $B \subset A$ a subalgebra, such that $A = C \rtimes_S B$. Then we call $(A, \mathcal{T})$ the topological semidirect product of $C$ and $B$, if the canonical linear bijection*

$$(C, \mathcal{T} \cap C) \times (B, \mathcal{T} \cap B) \longrightarrow (A, \mathcal{T}), (c, b) \longmapsto c + b$$

*is a homeomorphism. In this case, the quotient algebra $(A, \mathcal{T})/_C$ is canonically topologically isomorphic to the algebra $(B, \mathcal{T} \cap B)$.*

**Proposition 1.4** *Let $A$ be an lc algebra, which is the topological semidirect product of an ideal $C$ and a subalgebra $B$ such that both $C$ and $B$ are lmc. Then $A$ is also lmc.*

**Proof:** Let $U$ be an absolutely convex 0-nbhd in $A$; we may assume that $(U \cap C)^2 \subset U \cap C$. As $A$ is an *lc* algebra, there are absolutely $m$-convex 0nbhds $V$ and $W$ in $C$ and in $B$, respectively, such that

$$(\Gamma(V \cup W))^2 \cup (\Gamma(V \cup W))^3 \subset U, \quad V \subset U, \quad W \subset U.$$

We will show that $(V \cup W)^k \subset U$ for all $k \in \mathbb{N}$, which proves that

$$\mathcal{M}(\Gamma(V \cup W)) \subset U.$$

Now, $(V \cup W)^k$ is the union of $V^k(\subset V \subset U), W^k(\subset W \subset U)$ and of finite products of sets of the form $VW$, $WVW$, $WV$, $VWV$. The latter sets are all contained in $U \cap C$ and $(U \cap C)^m \subset U$ for all $m \in \mathbb{N}$. □

Our next aim is to present a method to construct algebras which are semidirect products.

**Proposition 1.5**

1. *Let $C$ and $B$ be algebras and assume that there is a linear multiplicative map*

$$l : B \longrightarrow L(C) := \{f : C \to C \text{ linear}\}, b \longmapsto l_b$$

*and a linear antimultiplicative map*

$$r : B \longrightarrow L(C), b \longmapsto r_b$$

*such that $r_b \circ l_{\tilde{b}} = l_{\tilde{b}} \circ r_b$ for all $b, \tilde{b} \in B$ and such that $l_b(ac) = l_b(a)c$, $r_b(ac) = ar_b(c)$, $al_b(c) = r_b(a)c$ for all $b \in B, a, c \in C$. Then the multiplication*

$$(C \times B) \times (C \times B) \longrightarrow (C \times B),$$

$$((c_1, b_1), (c_2, b_2)) \longmapsto (c_1c_2 + l_{b_1}(c_2) + r_{b_2}(c_1), b_1b_2)$$

*makes $A := C \times B$ (provided with componentwise addition and scalar multiplication) an algebra, such that $A$ is the semidirect product of the ideal $C \times \{0\}$ and the subalgebra $\{0\} \times B$. We will use the notation $A = C \times_S B$.*

2. *Moreover, let $\mathcal{T}$ and $\mathcal{S}$ be linear topologies on $C$ and on $B$, respectively, such that $(C, \mathcal{T})$ and $(B, \mathcal{S})$ are top algebras. Then $A$ provided with the product topology $\mathcal{T} \times \mathcal{S}$ is a top algebra, iff the two bilinear maps*

$$\varphi : (B, \mathcal{S}) \times (C, \mathcal{T}) \longrightarrow (C, \mathcal{T}),\ (b, c) \longmapsto l_b(c)$$

*and*

$$\psi : (C, \mathcal{T}) \times (B, \mathcal{S}) \longrightarrow (C, \mathcal{T}),\ (c, b) \longmapsto r_b(c)$$

*are continuous.*

**Proof:** One directly computes that the above multiplication is associative and distributive. For the last assertion, $(A, \mathcal{T} \times \mathcal{S})$ is a *top* algebra, iff

$$\forall U \in \mathcal{U}_0(C)\ \forall V \in \mathcal{U}_0(B)\ \exists \tilde{U} \in \mathcal{U}_0(C)\ \exists \tilde{V} \in \mathcal{U}_0(B) :\ \varphi(\tilde{V} \times \tilde{U}) + \psi(\tilde{U} \times \tilde{V}) \subset U,$$

because every such $\tilde{U}$ and $\tilde{V}$, can be assumed to satisfy $\tilde{U}^2 \subset U$ and $\tilde{V}^2 \subset V$, respectively. □

**Examples 1.5.**

1. Let $C$ be an algebra, $B := \mathbb{K}$ and for all $\lambda \in \mathbb{K}$ let $l_\lambda := r_\lambda : C \to C, c \mapsto \lambda c$. Then $C \times_S B$ coincides with $C_e$ (formal adjunction of a unit element). If $(C, \mathcal{T})$ is a *top* algebra, then $(C \times_S B, \mathcal{T} \times \mathcal{T}_{|.|})$ and $(C, \mathcal{T})_e$ coincide.

2. Let $B, C$ be *top* algebras and for all $b \in B$ let $l_b := r_b :=$ 0-map. Then $C \times_S B$ is equal to the direct product $C \times B$ with componentwise multiplication.

3. Let $D$ be an algebra and let $B, C \subset D$ be subalgebras such that $BC \cup CB \subset C$. For each $b \in B$ let $l_b : c \mapsto bc$ and $r_b : c \mapsto cb$. Then $l, r$ satisfy the requirements of proposition 1.5, hence $A = C \times_S B$ is a well-defined algebra. Moreover, $C + B$ is a subalgebra of $D$ and $q : C \times_S B \to C + B, (c, b) \mapsto c + b$ is an algebra-epimorphism.
   Thus a semidirect product $A = C \times_S B$ need not be commutative, even if $C$ and $B$ are commutative. Take e.g.
   $$B := \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{K} \right\}, C := \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} : c \in \mathbb{K} \right\} \subset M_2(\mathbb{K}).$$

4. Let $C$ be an algebra with unit element $e$, and let $\mathcal{T}$, $\mathcal{S}$ be linear topologies on $C$ such that $(C, \mathcal{T})$ and $(C, \mathcal{S})$ are *top* algebras. Form $A := C \times_S C$ according to 3. by putting $B := D := C$. Then t.f.a.e.:

   *i)* $(A, \mathcal{T} \times \mathcal{S})$ is a *top* algebra.

   *ii)* Multiplication on $(A, \mathcal{T} \times \mathcal{S})$ is separately continuous.

   *iii)* $\mathcal{S} \supset \mathcal{T}$.

   If in addition $(C, \mathcal{T}), (C, \mathcal{S})$ are *mtop*, then *i)* – *iii)* are equivalent to

   *iv)* $(A, \mathcal{T} \times \mathcal{S})$ is an *mtop* algebra.

   **Proof:** '*ii)* $\Rightarrow$ *iii)*': As left multiplication with $(e, 0)$ on $(A, \mathcal{T} \times \mathcal{S})$ is continuous and as $(e, 0) \cdot (0, a) = (a, 0)$ for all $a \in C$, the identity map $(C, \mathcal{S}) \to (C, \mathcal{T})$ is continuous.

   '*iii)* $\Rightarrow$ *i)*' is true by the last assertion in proposition 1.5, because multiplication as a map $(C, \mathcal{T}) \times (C, \mathcal{T}) \longrightarrow (C, \mathcal{T})$ is continuous, thus also continuous on $(C, \mathcal{T}) \times (C, \mathcal{S})$ and on $(C, \mathcal{S}) \times (C, \mathcal{T})$. The last part follows from the above proposition, as well. □

5. Let $C$ be an algebra and let $\mathcal{T}, \mathcal{S}$ be Hausdorff linear topologies on $C$, such that $\mathcal{S} \supset \mathcal{T}$ and $\mathcal{S} \neq \mathcal{T}$. Then $(A, \mathcal{R}) := (C \times_S C, \mathcal{T} \times \mathcal{S})$ is a *top* algebra. As $\Delta := \{(-c, c) : c \in C\}$ is an ideal in $A$ ($\Delta$ is the kernel of the algebra-epimorphism $q : C \times_S C \to C, (c_1, c_2) \mapsto c_1 + c_2$ in example 3), we obtain that $(A, \mathcal{R})$ is the semidirect product of $\Delta$ and $\{0\} \times C$ which are both closed, but not the topological semidirect product of $\Delta$ and $\{0\} \times C$.
   On the other hand $(A, \mathcal{R})$ is topologically isomorphic as an algebra to the direct topological product of $(\Delta, \mathcal{R} \cap \Delta)$ and $(C \times \{0\}, \mathcal{R} \cap (C \times \{0\}))$.

**Proposition 1.6** *There exists an algebra A provided with a Banach space topology $\mathcal{R}$ which contains an ideal C such that both $(C, \mathcal{R} \cap C)$ and $(A/_C, \mathcal{R}/_C)$ are Banach algebras and such that on $(A, \mathcal{R})$ multiplication is not continuous.*

**Proof:** Let $X$ be a linear space of dimension $2^{\mathbb{N}}$, then by [5, chap II, §5, exercise 24], $X$ is linearly isomorphic to $l^1$ and to $l^2$. Consequently, there exist two different Banach space topologies $\mathcal{T}$ and $\mathcal{S}$ on $X$. Providing $X$ with zero-multiplication we obtain the algebra $X_{nil}$; formal adjunction of a unit element yields the algebra $C := (X_{nil})_e$, which carries the two different Banach algebra topologies $\tilde{\mathcal{S}}$ and $\tilde{\mathcal{T}}$ generated by $\mathcal{S}$ and $\mathcal{T}$, respectively
Now, by example 1.5.4, $A := C \times_{\mathcal{S}} C$ provided with $\tilde{\mathcal{T}} \times \tilde{\mathcal{S}}$ satisfies the requirements, as $\mathcal{S} \not\supset \mathcal{T}$ and $(A, \tilde{\mathcal{T}} \times \tilde{\mathcal{S}})/(C \times \{0\})$ is isomorphic to the Banach algebra $(C, \tilde{\mathcal{S}})$. □

Now we are going to examine algebra structure on vector-valued sequence spaces $\lambda(A)$, where $A$ is an $l(m)c$ algebra and $\lambda$ is a normal Banach sequence space.

**Definition 1.5** *Let $(\lambda, \|\cdot\|_\lambda)$ be a Banach space with closed unit ball $B_\lambda$, such that*

1. *$\varphi \subset \lambda \subset \omega$ and*
2. *$B_\lambda$ is normal, i.e.*
   $B_\lambda = \{(x_n)_{n\geq 0} \in \omega : \exists (y_n)_{n\geq 0} \in B_\lambda \; \forall n \geq 0 : \; |x_n| \leq |y_n|\}$.

*Then $\lambda$ will be called a normal Banach sequence space. For the sequel, we define $\rho_n := \|e_n\|_\lambda, e_n = (\delta_{nk})_{k\geq 0}$ denoting the nth unitvector for all $n \geq 0$.*

**Remarks:**

1. For a normed space $(\lambda, \|\cdot\|), \varphi \subset \lambda \subset \omega$ with closed unit ball $B_\lambda$ t.f.a.e.
   *i)* $B_\lambda$ is normal.
   *ii)* If, for any $\mathbb{K}$-valued sequence $(x_n)_{n\geq 0} \in \omega$, there is $(y_n)_{n\geq 0} \in \lambda$ such that $|x_n| \leq |y_n|$ for all $n \geq 0$, then $(x_n)_{n\geq 0} \in \lambda$ and $\|(x_n)_{n\geq 0}\|_\lambda \leq \|(y_n)_{n\geq 0}\|_\lambda$.
2. For a normal Banach sequence space $\lambda$ the inclusions $\varphi \hookrightarrow \lambda \hookrightarrow \omega$ are continuous, $\varphi$ being endowed with the finest $lc$ topology and because $\rho_k |x_k| \leq \|(x_n)_{n\geq 0}\|_\lambda$ for all $(x_n)_{n\geq 0} \in \lambda$ and for all $k \geq 0$.
3. For a sequence of positive numbers $\alpha = (\alpha_n)_{n\geq 0}$ the so called diagonal transformation of $\lambda$
   $$\alpha\lambda := \{(\alpha_n x_n)_{n\geq 0} : (x_n)_{n\geq 0} \in \lambda\}$$
   is a normal Banach sequence space with respect to the norm
   $$\|\cdot\|_{\alpha\lambda} : \alpha\lambda \longrightarrow [0, \infty), (y_n)_{n\geq 0} \longmapsto \|(\alpha_n^{-1} y_n)_{n\geq 0}\|_\lambda.$$
   Now we can state a sharper version of 2., which is due to S. Dierolf and Fernandez (see [9]), namely:
   $$B_{l^1} \subset B_{\rho\lambda} \subset B_{l^\infty}$$

$\rho$ denoting the sequence $(\rho_n)_{n\geq 0}$. The latter inclusion has already been proved in 2. The proof of the former inclusion requires completeness of $\lambda$.
In case $\varphi$ is dense in $\lambda$, one even obtains:

$$B_{\rho\lambda} \subset B_{c_0},$$

because for all $(x_n)_{n\geq 0} \in \lambda$ and for all $k \geq 0$ we have $|\rho_k x_k| = \|x_k e_k\|_\lambda \to 0 \ (k \to \infty)$. The condition $\lambda$ containing $\varphi$ as dense subspace is often refered to as $(sc)$ (in abbreviation for 'sectional convergence'), because it is equivalent to

$$\sum_{k=0}^{n} x_k e_k \to x \ (n \to \infty) \text{ in } (\lambda, \|\cdot\|_\lambda)$$

for all $x = (x_n)_{n\geq 0} \in \lambda$, which can be verified easily.

**Definition-Remark 1.4.** *(see e.g. [16].) Let E be an lcs and $\lambda$ be a normal Banach sequence space.*

1. *On*
$$\lambda(E) := \left\{(x_n)_{n\geq 0} \in E^{\mathbb{N}\cup\{0\}} : \forall p \in cs(E) : (p(x_n))_{n\geq 0} \in \lambda\right\}$$
*($cs(E)$ denoting the set of all continuous seminorms on E) a 0nbhd-basis of an lc topology can be defined by $\{\lambda(U) : U = \Gamma U \in \mathcal{U}_0(E)\}$, where*
$$\lambda(U) := \{(x_n)_{n\geq 0} \in \lambda(E) : (p_U(x_n))_{n\geq 0} \in B_\lambda\}.$$
*The topology thus achieved may as well be generated by the family of seminorms $(\tilde{p} : \lambda(E) \to [0,\infty), (x_n)_{n\geq 0} \mapsto \|(p(x_n))_{n\geq 0}\|_\lambda)_{p\in cs(E)}$.*
2. *Furthermore we define for all $n \geq 0$:*
$$\lambda((E)_{k\geq n}) := \left\{(x_k)_{k\geq n} \in \prod_{k=n}^{\infty} E : ((0)_{k<n}, (x_k)_{k\geq n}) \in \lambda(E)\right\}$$
*and supply $\lambda((E)_{k\geq n})$ with the initial topology with respect to the embedding*
$$\lambda((E)_{k\geq n}) \hookrightarrow \lambda(E), (x_k)_{k\geq n} \longmapsto ((0)_{k<n}, (x_k)_{k\geq n}).$$

Note that one can always change the multiplication on an algebra $(A,\cdot)$ in the following way: $\odot : A \times A \to A, (x,y) \mapsto x \odot y := \sigma xy$, where $\sigma \in \mathbb{K}$. In case $\sigma \neq 0$, $\psi : (A,\cdot) \to (A,\odot), x \mapsto \sigma x$ is an algebra-isomorphism which is a homeomorphism if $A$ is a *top* algebra.

**Proposition 1.7** *Let an $l(m)c$ algebra A and a normal Banach sequence space $\lambda$ be given. Then $\lambda((A)_{k\geq n})$ is an $l(m)c$ algebra with respect to the multiplication*

$$\odot : \lambda((A)_{k\geq n}) \times \lambda((A)_{k\geq n}) \longrightarrow \lambda((A)_{k\geq n}),$$
$$((x_k)_{k\geq n}, (y_k)_{k\geq n}) \longmapsto (\rho_k x_k y_k)_{k\geq n}$$

*for all $n \geq 0$. (Note that in case $n = 0$ and $A = \mathbb{K}, \lambda((A)_{k\geq n}) = \lambda$.)*

**Proof:** It is sufficient to prove that $\lambda(A)$ is $l(m)c$, if $A$ is $l(m)c$. Let $U \in \mathcal{U}_0(A)$ such that $U = \Gamma U$. One can find $V \in \mathcal{U}_0(A)$ satisfying $V = \Gamma V$ and $V^2 \subset U$. This implies $p_U(xy) \leq p_V(x)p_V(y)$ for all $x, y \in A$. ¿From the above remark we obtain:

$$B_\lambda \odot B_\lambda = B_{\rho\lambda} \cdot B_\lambda \subset B_{l^\infty} \cdot B_\lambda \subset B_\lambda.$$

This implies $\lambda(V) \odot \lambda(V) \subset \lambda(U)$. In case $A$ is in fact $lmc$ choose (w.l.o.g.) $V = U$. □

**Corollary 1.5** *For any lc algebra A and any normal Banach sequence space* $\lambda$

$$\phi : \lambda(A) \longrightarrow l^\infty(A), (x_n)_{n\geq 0} \longmapsto (\rho_n x_n)_{n\geq 0}$$

*is an algebra-isomorphism onto an ideal in* $l^\infty(A)$*. in case* $\lambda$ *satisfies (sc),* $\phi$ *is an algebra-isomorphism onto an ideal in* $c_0(A)$*.*

The proof is immediate.

**Corollary 1.6** *Let A and B be* $l(m)c$ *algebras,* $f : B \to A$ *a linear, multiplicative, and continuous map, and* $\lambda$ *a normal Banach sequence space. For all* $n \geq 0$ *we define*

$$A_n := \prod_{0\leq k<n} B \times \lambda((A)_{k\geq n}))$$

*which is an* $l(m)c$ *algebra with respect to the multiplication*

$$\odot : A_n \times A_n \longrightarrow A_n, ((x_k)_{k\geq 0}, (y_k)_{k\geq 0}) \longmapsto (\rho_k x_k y_k)_{k\geq 0}.$$

*Moreover* $f_n : A_{n+1} \to A_n, (x_k)_{k\geq 0} \mapsto ((x_k)_{k<n}, f(x_n), (x_k)_{k>n})$ *is linear, multiplicative and continuous for all* $n \geq 0$*. Hence*

$$proj\left(B \overset{f}{\to} A, \lambda\right) := proj((A_n)_{n\geq 0}, (f_n)_{n\in\mathbb{N}})$$

*is an* $l(m)c$ *algebra.* $proj\left(B \overset{f}{\to} A, \lambda\right)$ *is called a projective limit of Moscatelli type (cf. [25, p. 21]).*

We do not know whether convolution is well-defined on any normal Banach sequence space other than $l^1$. Anyhow, we have:

**Proposition 1.8** *For any normal Banach sequence space* $\lambda$ *such that* $(\lambda, *)$ *is a Banach algebra and any l(m)c algebra A,* $(\lambda(A), *)$ *is an l(m)c algebra.*

**Proof:** Proceed as in the proof of proposition 1.7. Let $(x_n)_{n\geq 0}$, $(y_n)_{n\geq 0} \in \lambda(V)$. Then we get:

$$\left\| \left( p_U\left( \sum_{k=0}^{n} x_k y_{n-k} \right) \right)_{n\geq 0} \right\|_\lambda \leq \left\| \left( \sum_{k=0}^{n} p_V(x_k) p_V(y_{n-k}) \right)_{n\geq 0} \right\|_\lambda =$$

$$\|(p_V(x_n))_{n\geq 0} * (p_V(y_n))_{n\geq 0}\|_\lambda \leq \|(p_V(x_n))_{n\geq 0}\|_\lambda \cdot \|(p_V(y_n))_{n\geq 0}\|_\lambda.$$

We have just shown $\lambda(V) * \lambda(V) \subset \lambda(U)$. □

In [25] Khin Aye Aye proved that the projective Limit of Moscatelli type proj $\left(B \stackrel{f}{\rightarrow} A, \lambda\right)$ is (linearly) topologically isomorphic to

$$C := \left\{ (x_k)_{k\geq 0} \in B^{\mathbb{N}\cup\{0\}} : (f(x_k))_{k\geq 0} \in \lambda(A) \right\}$$

provided with the initial topology with respect to $C \hookrightarrow B^{\mathbb{N}\cup\{0\}}$ and $C \to \lambda(A), (x_k)_{k\geq 0} \mapsto (f(x_k))_{k\in\mathbb{N}}$. Obviously, $C$ is a subalgebra of $(B^{\mathbb{N}\cup\{0\}}, \odot)$, where

$$\odot : B^{\mathbb{N}\cup\{0\}} \times B^{\mathbb{N}\cup\{0\}} \longrightarrow B^{\mathbb{N}\cup\{0\}}, \left((x_k)_{k\geq n}, (y_k)_{k\geq n}\right) \longmapsto (\rho_k x_k y_k)_{k\geq n}.$$

Immediately, one checks that the topological isomorphism

$$\phi : \text{proj}\left(B \stackrel{f}{\rightarrow} A, \lambda\right) \longrightarrow C, \left(\left(x_l^{(k)}\right)_{l\geq 0}\right)_{k\geq 0} \longmapsto \left(x_k^{(k+1)}\right)_{k\geq 0}$$

is also multiplicative.
Hence, in case $B = A$ and $f = id$, we obtain proj $\left(B \stackrel{f}{\rightarrow} A, \lambda\right) = \lambda(A)$. In the sequel we will always identify proj $\left(B \stackrel{f}{\rightarrow} A, \lambda\right)$ and $C$ without explicit mentioning.

We are now going to conclude this chapter with a sharper version of propostion 1.4, which is in fact a corollary of proposition 1.4 and proposition 1.5 (cf. example 1.5.3).

**Proposition 1.9** *Let $A = (A, \mathcal{T})$ be an lc algebra containing subalgebras $C$ and $B$ satisfying $BC \cup CB \subset C$ such that both $(C, \mathcal{T} \cap C)$ and $(B, \mathcal{T} \cap B)$ are lmc algebras. Then $(C + B, \mathcal{T} \cap (C + B))$ is an lmc algebra.*

This proposition is a slight improvement of what is stated in proposition 1.4, because proposition 1.4 requires $C \cap B = \{0\}$.

**Proof:** Immediately, one checks that $C + B$ is a subalgebra of $A$. According to example 1.5.3, $C \times_S B$ is a well-defined algebra which is *lmc* by proposition 1.4 and proposition 1.5. The addition map

$$q : C \times_S B \longrightarrow C + B, (c, b) \longmapsto c + b$$

is a continuous, open algebra-epimorphism. Now, $C + B$ is topologically isomorphic to the quotient algebra $(C \times_S B)/_{\text{kern}(q)}$ which is an *lmc* algebra. □

# Chapter 2
## Characters on Locally Convex Algebras

**Definition 2.1** *For an algebra A we define*

$$\sigma(A) := \{\chi \in A^* \setminus \{0\} : \chi \text{ is multiplicative}\}$$

*and call* $\chi \in \sigma(A)$ *a character on A. Moreover we define*

$$\sigma_c(A) := \sigma(A) \cap A',$$

*in case A is endowed with a linear topology, and*

$$\tilde{\sigma}(A) := \sigma(A) \cup \{0\}, \;\; \tilde{\sigma}_c(A) := \sigma_c(A) \cup \{0\}.$$

Here are some well-known results about characters on algebras; the proofs are only included for the reader's convenience and to help the understanding of further conclusions.

**Remarks:** Let $A$ be an algebra.

1. If $A$ has a unit and $\chi \in \sigma(A)$, then

$$\chi|_{G(A)} : G(A) \longmapsto I\!K \setminus \{0\}$$

   is a group homomorphism.

2. (See [28, theorem C.1].) Let $I \subset A$ be an ideal in $A$ and $\chi \in \sigma(I)$. Then there is a unique extension of $\chi$ to a character $\psi$ on $A$. Moreover, in case $A$ is endowed with a linear topology such that multiplication is separately continuous on $A$, $\psi$ is continuous, iff $\chi$ is continuous.

   **Proof:** One can find $u \in I$ such that $\chi(u) = 1$. We define:

$$\psi(x) := \chi(ux)$$

   for all $x \in A$. Then, clearly, $\psi$ is linear. Now for all $x, z \in A$ we have $\psi(xz) = \chi(uxz) = \chi(uxz)\chi(u) = \chi(ux)\chi(zu) = \chi(ux)\chi(u)\chi(zu) = \chi(ux)\chi(uz) = \psi(x)\psi(z)$.
   If $\varphi \in \sigma(A)$ is any extension of $\chi$, we obtain $\varphi(x) = \chi(u)\varphi(x) = \chi(ux) = \psi(x)$ for all $x \in A$.
   It remains to prove that $\psi$ is continuous, if $\chi$ is continuous. One can find $U \in \mathcal{U}_0(A)$ such that $\chi(U \cap I) \subset \{\lambda \in I\!K : |\lambda| \leq 1\}$. Since multiplication is separately continuous on $A$, there is $V \in \mathcal{U}_0(A)$ satisfying $uV \subset U$. This yields $\psi(V) = \chi(uV) \subset \chi(U \cap I)$. □

   Observation: One might as well have taken any algebra $B$ with unit $e$ instead of $I\!K$ on the right side, if, in addtion, $\chi(I) \ni e$. For the continuity part of the above remark it suffices that $B$ is endowed with a linear topology.
   However, we obtain for a $(top)$ algebra $A$ and an ideal $I \subset A$:

$$\sigma(I) \subset \sigma(A) \quad \text{and} \quad \sigma_c(I) \subset \sigma_c(A),$$

respectively. If $I$ is a maximal ideal, we even get

$$\sigma(I) \subset \sigma(A) \subset \tilde{\sigma}(I) \text{ and } \sigma_c(I) \subset \sigma_c(A) \subset \tilde{\sigma}_c(I)$$

(see remark 8.).

3. Considering the characters on $A$, it is no restriction to assume that $A$ has a unit, because we have the bijective maps

$$\phi : \tilde{\sigma}(A) \longrightarrow \sigma(A_e), \chi \longmapsto (A_e \to I\!K, (x,\lambda) \mapsto \chi(x) + \lambda),$$

(with inverse map $\sigma(A_e) \to \tilde{\sigma}(A), \chi \mapsto \chi|A$) and

$$\phi_c : \tilde{\sigma}_c(A) \longrightarrow \sigma_c(A_e), \chi \longmapsto (A_e \to I\!K, (x,\lambda) \mapsto \chi(x) + \lambda),$$

respectively, if $A$ is provided with a linear topology (with inverse map $\sigma_c(A_e) \to \tilde{\sigma}_c(A), \chi \mapsto \chi|A$). (See e.g. [17, p. 75].) It is easy to see that both $\phi$ and $\phi_c$ are homeomorphisms with respect to the relative topologies induced by $(A^*, \sigma(A^*, A))$ and by $(A_e^*, \sigma(A_e^*, A_e))$, respectively (the so-called Gelfand topology).
Thus, in this special case of 2. we obtain:

$$\sigma(A_e) = \tilde{\sigma}(A) \text{ and } \sigma_c(A_e) = \tilde{\sigma}_c(A),$$

respectively.
Note that in 2. the map $\sigma(A) \to \tilde{\sigma}(I), \chi \mapsto \chi|I$ is generally not injective, if $I$ is not a maximal ideal.

4. (See [28, lemma 6.1, a)].) For all $x \in A$ and any character $\chi \in \sigma(A)$ we have $\chi(x) \in \sigma_A(x)$. Hence, $\sigma_A(x) \neq \emptyset$ for all $x \in A$, if $\sigma(A) \neq \emptyset$.

   **Proof:** If $\chi(x) = 0$, then $x$ is not invertible in $A$. This yields $0 \in \sigma_A(x)$.
   For $\chi(x) \neq 0$, we may assume w.l.o.g. that $A$ has a unit $e$. Clearly, $\chi(x - \chi(x)e) = 0$. Thus, $x - \chi(x)e$ cannot be invertible in $A$. □

5. In [28, corollary 5.6, a)] the following result is proved: If $A$ is a commutative, complete, *lmc* algebra, then:

$$\sigma_A(x) \cup \{0\} = \{\chi(x) : \chi \in \sigma_c(A)\}$$

for all $x \in A$. Hence, by the above remark, for a commutative, complete, *lmc* algebra we obtain:

$$\{\chi(x) : \chi \in \sigma(A)\} = \{\chi(x) : \chi \in \sigma_c(A)\}$$

for all $x \in A$.

6. $\tilde{\sigma}(A)$ is always closed in $(A^*, \sigma(A^*, A))$, because

$\tilde{\sigma}(A) = \bigcap_{x,y \in A} \{f \in A^* : f(xy) - f(x)f(y) = 0\}$. Accordingly, if $A$ is endowed with any linear topology, then $\tilde{\sigma}_c(A)$ is closed in $(A', \sigma(A', A))$ (see [28, lemma 6.2, a)]).

7. Let $\chi \in \sigma(A)$. For $I := \operatorname{kern}(\chi)$ the following holds:

   *i*) I is a modular ideal in $A$, i.e. $A/_I$ has a unit.

   *ii*) $I$ is a maximal ideal, i.e. for each ideal $J \subset A$ which contains $I$, $J$ is either equal to $A$ or to $I$.

   (Note that an ideal which contains a modular ideal is modular itself. Hence there is no need to distinguish between maximal and modular ideals on the one hand and modular ideals which are maximal among the modular ideals on the other hand. So there is no ambiguity in talking of maximal modular ideals.)

   **Proof:** *i*) follows from $A/_I \cong I\!K$.
   Let now $J \subset A$ be an ideal in $A$ containing $I$. If $J \neq I$, it follows:

   $$I\!K = \chi(J) \Longrightarrow A = \chi^{-1}(I\!K) = \chi^{-1}(\chi(J)) = J + I = J.$$

   □

   Thus $\operatorname{kern}(\chi) \subset \operatorname{kern}(\psi)$ yields $\operatorname{kern}(\chi) = \operatorname{kern}(\psi)$, hence, by the next remark, $\chi = \psi$.

8. The following is e.g. stated in [21, p. 588]. Let $\chi, \psi \in \sigma(A)$ be given such that $\operatorname{kern}(\psi) \subset \operatorname{kern}(\chi)$. Then $\chi = \psi$.

   **Proof:** Since $\chi$ vanishes on the kernel of $\psi$, and since $\psi$, as a character, is an algebra-epimorphism, there is $\alpha \in I\!K \setminus \{0\}$ such that $\chi = \alpha\psi$. But multiplicativity implies $\alpha = 1$. □

**Convention:** For the sequel, we make the following convention. Let two sets $X$ and $Y$ be given. We define the canonical projections $pr_y : X^Y \to X, f \mapsto f(y)$. In case $X = I\!K$, the projection $pr_y$ will be denoted by $\delta_y$.

**Examples 2.1.**

1. Obviously: $\sigma(I\!K) = \{id_{I\!K}\}$. More generally:

   $$\sigma(I\!K^n) = \{\delta_k : 1 \leq k \leq n\} \ (n \in I\!N) \ \text{ and } \ \sigma(\omega) = \{\delta_n : n \geq 0\}$$

   (we identify $I\!K^n = I\!K^{\{1,\ldots,n\}}$).

2. For any algebra $A$, and any number $\rho \in I\!K \setminus \{0\}$, and the multiplication $\odot : A \times A \to A, (x,y) \mapsto x \odot y := \rho xy$ we have the representation:

   $$\sigma(A, \odot) = \{\rho\psi : \psi \in \sigma(A)\}.$$

   **Proof:** The inclusion '$\supset$' is obvious. For '$\subset$' only note that for any character $\chi \in \sigma(A, \odot)$ the linear functional $\psi := \frac{1}{\rho}\chi$ is a character on $A$. □

3. We now characterize $\sigma(A[X])$ for an arbitrary algebra $A$. First note that for any family of linear spaces $(E_s)_{s\in T}$ there is the isomorphism

$$\phi : \prod_{s\in T} E_s^* \longrightarrow \left(\bigoplus_{s\in T} E_s\right)^*, (f_s)_{s\in T} \longmapsto \left(\bigoplus_{s\in T} E_s \to \mathbb{K}, (x_s)_{s\in T} \mapsto \sum_{s\in T} f_s(x_s)\right).$$

Now we claim the following representation $\tilde{\sigma}(A[X]) =$
$\left\{(f_k)_{k\geq 0} \in (A^*)^{\mathbb{N}\cup\{0\}} : \forall m,n \geq 0\, \forall x,y \in A :\ f_m(x)\cdot f_n(y) = f_{m+n}(x\cdot y)\right\}$.

**Proof:** To verify the inclusion '$\supset$', one easily computes that $\phi\left((f_k)_{k\geq 0}\right)$ is multiplicative for any sequence $(f_k)_{k\geq 0} \in (A^*)^{\mathbb{N}\cup\{0\}}$ satisfying the above requirement.
To see the inclusion '$\subset$', let $(f_k)_{k\geq 0} \in (A^*)^{\mathbb{N}\cup\{0\}}$ be given, such that there are $m,n \geq 0$ and $x,y \in A$ with $f_m(x)f_n(y) \neq f_{m+n}(xy)$. Then we get

$$= \phi((f_k)_{k\geq 0})(xe_m * ye_n) = f_{m+n}(xy) \neq f_m(x)f_n(y) =$$
$$\phi((f_k)_{k\geq 0})(xe_m)\phi((f_k)_{k\geq 0})(ye_n),$$

hence $(f_k)_{k\geq 0} \in (A^*)^{\mathbb{N}\cup\{0\}} \setminus \tilde{\sigma}(A[X])$. □

Note that, in general, the maps $f_n$ are not multiplicative. Take for instance $A = \mathbb{K}$, which yields $A[X] = (\varphi, *)$. Then, by the above considerations, we obtain $\tilde{\sigma}(\varphi) = \{(x_n)_{n\geq 0} \in \omega : \forall m,n \geq 0 :\ x_m x_n = x_{m+n}\}$. Now $x_0^2 = x_0$ implies $x_0 \in \{0,1\}$. If $x_0 = 0$, then $x_n = 0$ for all $n \geq 0$. Moreover, we obtain inductively $x_n = x_1^n$ for all $n \in \mathbb{N}$. Hence, we have the representation

$$\sigma(\varphi) = \{(z^n)_{n\geq 0} : z \in \mathbb{K}\}.$$

This yields $\sigma(\varphi) \cap (\sigma(\mathbb{K}))^{\mathbb{N}\cup\{0\}} = \{e_0, (1)_{n\geq 0}\}$.

4. As in the above example we get $\tilde{\sigma}\left(A^{\mathbb{N}\cup\{0\}}, *\right) =$
$\left\{(f_k)_{k\geq 0} \in (A^*)^{(\mathbb{N}\cup\{0\})} : \forall m,n \geq 0\, \forall x,y \in A :\ f_m(x)\cdot f_n(y) = f_{m+n}(x\cdot y)\right\}$. This implies

$$\tilde{\sigma}\left(A^{\mathbb{N}\cup\{0\}}, *\right) = \tilde{\sigma}(A) \times \{0\}^{\mathbb{N}}.$$

For the scalar case this yields $\sigma(\omega, *) = \{e_0\}$.

5. Let $A = C \rtimes_S B$ as in definition 1.4. Let $q : A \to B$ denote the corresponding algebra-epimorphism. Then we have the representation

$$\sigma(A) = \sigma(C) \cup (\sigma(B) \circ q).$$

**Proof:** If $\psi \in \sigma(C)$, there is a unique extension of $\psi$ to a character on $A$ and if $\varphi \in \sigma(B)$ then $\varphi \circ q \in \sigma(A)$. Conversely, let $\chi \in \sigma(A)$. If $\chi|_C \neq 0$, then $\chi|_C \in \sigma(C)$ and $\chi$ is its unique extension to a character on $A$; if $\chi|_C = 0$, there is $\psi \in \sigma(B)$ satisfying $\chi = \psi \circ q$.
□

In the sequel, there will be more representations of this type.
It is a well known result from the theory of Banach algebras, that for a Banach algebra $A$

$$\sigma(A) = \sigma_c(A) \text{ or, equivalently, } \tilde{\sigma}(A) = \tilde{\sigma}_c(A)$$

holds. It had been an unsolved problem (the so-called Michael problem) for a long time whether this is also true for ($lmc$) Fréchet algebras (see [28, p. 53]). Now it seems that this problem obtained a positive solution recently which is due to B. Stensones (see [35]).

**Lemma 2.1** *Let $A$ be a Banach algebra and $I \subset A$ a proper modular ideal in $A$. Then $I$ is not dense in $A$.*

**Proof:** Let $u \in A$, such that $u + I$ is a unit in $A/_I$. We claim

$$\{x \in A : \|x - u\| < 1\} \cap I = \emptyset.$$

Let $x \in A$, such that $\|x - u\| < 1$. Then $y := \sum_{k=1}^{\infty} (u-x)^k$ exists in $A$. Now $y - y(u-x) = u - x$ implies $y - yu + yx + x - u = 0$. Thus, $u \in yx + x + I$, because $y - yu \in I$. Hence $x \in I$ would imply $u \in I$, which is a contradiction. □

**Theorem 2.1** *(See e.g. [21, Satz 125.2] or [27, p. 170].) Let $A$ be a Banach algebra, $\chi \in \sigma(A)$, then $\chi$ is continuous.*

**Proof:** $I :=$ kern$(\chi)$ is a proper maximal modular ideal in $A$. As a modular ideal $I$ is not dense in $A$, thus closed, because $I$ is maximal. □

It is easy to see that for a normal Banach sequence space $\lambda$ $\rho_n\delta_n : \lambda \to I\!K, (x_k)_{k\in I\!N} \mapsto \rho_n x_n$ is a character on $\lambda$ for all $n \in I\!N$. As a consequence of the above theorem we obtain that all characters on $\lambda$ can be found among the maps $\rho_n\delta_n$ as far as $\lambda$ contains $\varphi$ as a dense subspace.

**Corollary 2.1** *For every normal Banach sequence space $\lambda$ satisfying (sc) we have the representation*

$$\sigma(\lambda) = \{\rho_n\delta_n : n \in I\!N\}.$$

**Proof:** First observe that $\varphi$ is a subalgabra of $\lambda$ with respect to the multiplication $\varphi \times \varphi \to \varphi, ((x_n)_{n\in I\!N}, (y_n)_{n\in I\!N}) \mapsto (\rho_n x_n y_n)_{n\in I\!N}$.
Let $\chi \in \sigma(\lambda)$ and $j_n : I\!K \to \lambda, x \mapsto xe_n$ denote the canonical injections for all $n \in I\!N$. Then one can find $n \in I\!N$ such that $\chi \circ j_n \neq 0$ and such that $\chi \circ j_m = 0$ for all $m \neq n$.
Indeed, if $\chi \circ j_n = 0$ held for all $n \in I\!N$, we would obtain $\chi|\varphi = 0$. Since $\varphi$ is dense in $\lambda$ and $\chi$ is continuous by theorem 2.1, this would imply $\chi = 0$, which is a contradiction. Thus, there is $n \in I\!N$ such that $\chi \circ j_n \neq 0$.
Let us now assume there was $m \in I\!N \setminus \{n\}$ such that $\chi \circ j_m \neq 0$. Then one could find $x, y \in I\!K$ satisfying $0 \neq \chi(j_n(x))\chi(j_m(y))$. But, on the other hand, we have

$$\chi(j_n(x))\chi(j_m(y)) = \chi(j_n(x)j_m(y)) = 0,$$

which is again a contradiction.
Thus, $(\rho_n\delta_n)|\varphi$ is a character on $\varphi$ the kernel of which is contained in the kernel of $\chi|\varphi$. Now, example 2.1.1 implies $\chi|\varphi = (\rho_n\delta_n)|\varphi$. Continuity of $\chi$ and the requirement $(sc)$ for $\lambda$ yield the conclusion. □

**Corollary 2.2** *For $\lambda = l^1$ we have the representation*

$$\sigma(l^1, *) = \{(z^n)_{n\geq 0} : |z| \leq 1\}.$$

**Proof:** Since $(l^1, *)$ is a Banach algebra, we have, by theorem 2.1 $\sigma(l^1, *) \subset l^\infty$. Now go on as in example 2.1.3. □

**Definition 2.2** *A top algebra A will be called functionally bounded or functionally continuous, respectively, if all the characters on A are bounded or continuous, respectively, where a linear functional is called bounded, if it is bounded on bounded sets.*

The following proposition is an application of a well-known result from the theory of *lc* spaces to the theory of *lmc* algebras. (See also [12].)

**Proposition 2.1** *Let $A = (A, \mathcal{T})$ be a Hausdorff lmc algebra and $B \subset A$ a bounded subset of A. By the 1st remark after definition-remark 1.1*

$$< B > := \left[\bigcup_{n\in I\!N} B^n\right]$$

*is the smallest subalgebra of A containing B. There is a metrizable lmc linear topology $\mathcal{S}$ on $< B >$ which is finer than the relative topology $\mathcal{T} \cap < B >$ and B is bounded in $(< B >, \mathcal{S})$.*

**Proof:** Let $\mathcal{V}$ be a 0nbhd-basis in $A$ consisting of closed, and absolutely $m$-convex sets. For all $n \in I\!N$ we define

$$U_n := \left(\bigcap\left\{U \in \mathcal{V} : U \supset \frac{1}{n}B\right\}\right) \cap < B > .$$

Clearly, each $U_n$ is absolutely $m$-convex and closed in $< B >$. It remains to prove, that $U_n$ is absorbant in $< B >$. Any $x \in < B >$ is of the form

$$x = \sum_{k=1}^{N}\sum_{j=1}^{N_k} \lambda_j^{(k)} b_j^{(k)}$$

for some $N, N_1, ..., N_N \in I\!N$, $\lambda_j^{(k)} \in I\!K$, and some $b_j^{(k)} \in B^k$ for all $k \in \{1, ..., N\}$, and $j \in \{1, ..., N_k\}$. Since $\frac{1}{n}B \subset U_n$, we obtain $\left(\frac{1}{n}\right)^k B^k \subset U_n$ for all $k \in \{1, ..., N\}$. This implies $b_j^{(k)} \in [U_n]$ for all $k \in \{1, ..., N\}$, and $j \in \{1, ..., N_k\}$, thus $x \in [U_n]$. Now, as $U_n$ is absolutely convex, one cand find $\mu > 0$ such that $\mu x \in U_n$. □

As an application we obtain the next theorem, which is the most general precision of the relation between question 1 and question 2 in [28, p. 53]. (See also [12].) It is a slight improvement of the corresponding result for complete *lmc* algebras due to Dixon and Fremlin (see [14]). However, their proof also yields the sligtly more general result stated in theorem 2.2.

**Theorem 2.2** *All lmc Fréchet algebras are functionally continuous iff all Hausdorff locally complete, lmc algebras are functionally bounded.*

**Proof:** The condition above is obviously sufficient. For the converse let $A$ be a Hausdorff, locally complete, *lmc* algebra. Let us assume there is $\chi \in \sigma(A)$, and $B = \overline{\Gamma B} \subset A$ bounded, such that $\chi(B)$ is not bounded in $\mathbb{K}$. Take $\mathcal{S}$ as in proposition 2.1, then the completion $(< B >, \mathcal{S})\tilde{}$ of $(< B >, \mathcal{S})$ is an *lmc* Fréchet algebra. By [29, 5.1.26] the embedding $< B > \hookrightarrow A$ has a continuous, linear, and multiplicative extension $j : (< B >, \mathcal{S})\tilde{} \to A$. $B$ is still bounded in $(< B >, \mathcal{S})\tilde{}$ and $\psi = \chi \circ j \in \sigma(\widetilde{< B >})$ is continuous. Now

$$\psi(B) = \chi(j(B)) = \chi(B)$$

which is an unbounded subset of $\mathbb{K}$. This is a contradiction. □

There is also a positive result for Hausdorff locally complete, *lmc* algebras, as far as they have a fundamental sequence of bounded sets, which is due to S. Dierolf and J. Wengenroth (see [12]).

**Proposition 2.2** *Let $A = (A, \mathcal{T})$ be a Hausdorff locally complete, lmc algebra with a fundamental sequence of bounded sets $(B_n)_{n\in\mathbb{N}}$. Then A is functionally bounded.*

**Proof:** Let $\mathcal{V}$ be a 0nbhd-basis of A, consisting of absolutely $m$-convex, closed sets. We may assume $B_n = \overline{\Gamma B_n} \subset B_{n+1}$ for all $n \in \mathbb{N}$. The inductive limit $(A, \mathcal{R}) := \mathrm{ind}_{n\in\mathbb{N}}([B_n], p_{B_n})$ is an LB-space. Furthermore, we define:

$$A_n := \bigcap_{m\in\mathbb{N}} \left[\bigcap\left\{U \in \mathcal{V} : \frac{1}{m}B_n \subset U\right\}\right]$$

which is a subalgebra of $A$ for all $n \in \mathbb{N}$ and

$$V_{nm} := \left(\bigcap\left\{U \in \mathcal{V} : \frac{1}{m}B_n \subset U\right\}\right) \bigcap A_n$$

for all $n, m \in \mathbb{N}$. Then $\{V_{nm} : m \in \mathbb{N}\}$ is a decreasing 0nbhd-subbasis of an *lmc* Fréchet topolgy $\mathcal{S}_n$ on $A_n$ for all $n \in \mathbb{N}$.
Indeed, $(A_n, \mathcal{S}_n)$ is obviously semimetrizable. It is metrizable because the inclusion $(A_n, \mathcal{S}_n) \hookrightarrow (A, \mathcal{S})$ is continuous. It remains to prove that $(A_n, \mathcal{S}_n)$ is complete. For this purpose it suffices to show that $(A_n, \mathcal{S}_n)$ is locally complete (see [29, 5.1.9]). Now, by [7], an *lcs* $F$ is locally

complete, iff every local Cauchy sequence converges in $F$. A local Cauchy sequence is a sequence which is contained in the linear span of a bounded set and is Cauchy with respect to the corresponding Minkowski functional. Every local Cauchy sequence $(y_k)_{k\in\mathbb{N}}$ in $A_n$ is a local Cauchy sequence in $A$, because $A_n$ is continuously included in $A$, hence $(y_k)_{k\in\mathbb{N}}$ converges to some $y \in A$. Every local Cauchy sequence is a Cauchy sequence, thus for each $m \in \mathbb{N}$ there is $k_m \in \mathbb{N}$ such that $y_k - y_j \in \frac{1}{m}V_{nm}$ for all $k, j \geq k_m$. This implies that for each $m \in \mathbb{N}$ there is $k_m \in \mathbb{N}$ such that

$$y_k - y \in \frac{1}{m}\overline{\left( \bigcap_{\frac{1}{m}B_n \subset U \in \mathcal{V}} U \right)}^{\mathcal{T}} = \frac{1}{m}\left( \bigcap_{\frac{1}{m}B_n \subset U \in \mathcal{V}} U \right)$$

for all $k \geq k_m$. This yields

$$\forall m \in \mathbb{N}\ \exists k_m \in \mathbb{N}\ \forall k \geq k_m :\ y_k - y \in \frac{1}{m}V_{nm},$$

which proves that $(y_n)_{n\in\mathbb{N}} \to y \in A_n$ $(n \to \infty)$ in $(A_n, \mathcal{S}_n)$.
$(A, \mathcal{S}) := \mathrm{ind}_{n\in\mathbb{N}}(A_n, \mathcal{S}_n)$ is an LF-space. (Note that $A_n \supset B_n$ for all $n \in \mathbb{N}$, hence $A \supset \cup A_n \supset \cup B_n = A$.) $\mathcal{S}$ is finer than $\mathcal{T}$, thus $\mathcal{S}$ is finer than $\mathcal{R}$, because $\mathcal{S}$ is bornological and $\mathcal{R}$ is the coarsest bornological topology which is finer than $\mathcal{T}$. Now, by Grothendieck's factorization theorem (see [29, 1.2.20]), we obtain $\mathcal{S} = \mathcal{R}$ and for all $n \in \mathbb{N}$ there is $k_n \geq n$ such that

$$([B_n], p_{B_n}) \overset{\text{cont.}}{\hookrightarrow} (A_n, \mathcal{S}_n) \overset{\text{cont.}}{\hookrightarrow} ([B_{k_n}], p_{B_{k_n}}).$$

Thus, for all $n \in \mathbb{N}$ one can find an absolutely convex set $V_n \in \mathcal{U}_0(A_n, \mathcal{S}_n)$ such that

$$V_n V_n \subset V_n \subset B_{k_n} \cap A_n.$$

$(A_n, p_{V_n})$ is a normed algebra such that

$$([B_n], p_{B_n}) \overset{\text{cont.}}{\hookrightarrow} (A_n, p_{V_n}) \overset{\text{cont.}}{\hookrightarrow} ([B_{k_n}], p_{B_{k_n}}) \overset{\text{cont.}}{\hookrightarrow} (A, \mathcal{T}).$$

For all $n \in \mathbb{N}$ let now $(\tilde{A}_n, p_n)$ be the completion of $(A_n, p_{V_n})$, which is a Banach algebra and $\eta_n : (\tilde{A}_n, p_n) \to (A, \mathcal{T})$ denote the continuous, linear, and multiplicative extension of $(A_n, p_{V_n}) \hookrightarrow (A, \mathcal{T})$. Then $\mathcal{R}$ is the finest *lc* topology, such that $\eta_n : (\tilde{A}_n, p_n) \to A$ is continuous for all $n \in \mathbb{N}$.
Let now $\chi \in \sigma(A)$. Then $\chi \circ \eta_n \in \sigma(\tilde{A}_n, p_n)$, continuous for all $n \in \mathbb{N}$. This implies that $\chi : (A, \mathcal{R}) \to \mathbb{K}$ is continuous, hence $\chi : (A, \mathcal{T}) \to \mathbb{K}$ is bounded. □

Although Michael's problem appears to be solved, a description of the characters on certain algebras still seems desirable. So our next aim is a characterization of $\sigma(A)$ for a number of certain (Fréchet) algebras. Since not all of these are Fréchet algebras, these results can still be regarded as results on automatic continuity (boundedness) of linear and multiplicative functionals.
More precisely: Starting with an algebra $A$ we form a certain algebra $\tilde{A}$ and get a representation of $\sigma(\tilde{A})$ in terms of $\sigma(A)$ which enables us to conclude: $\tilde{A}$ is functionally continuous (bounded), if $A$ is functionally continuous (bounded).

**Definition-Remark 2.1.** *Let $X$ be any topological space. We define*

$$C(X) := \{f : X \longrightarrow \mathbb{K} : f \text{ is continuous}\}$$

*and*

$$C\mathcal{B}(X) := \{f \in C(X) : f(X) \text{ is bounded}\}$$

*The latter provided with $\|\cdot\|_\infty$ is a Banach algebra. We supply $C(X)$ with the compact open topology, which is generated by the family*

$$(p_K : C(X) \longrightarrow [0,\infty), f \longmapsto \sup_{x\in K} f(x))_{K\subset X \text{ compact}}$$

*of submultiplicative seminorms. Thus $C(X)$ becomes an lmc algebra. If $X$ is completely regular, then $C(X)$ is a Fréchet algebra iff, $X$ is a hemicompact k-space (See [17, p. 69]).*

**Remarks:** Let $X$ be a topological space, $A \in \{C(X), C\mathcal{B}(X)\}$, and $\chi \in \sigma(A)$.

1. In case $A = C(X)$, we have $\chi(f) \in f(X)$ for all $f \in A$ and $\chi(f) \in \overline{f(X)}$, if $A = C\mathcal{B}(X)$, respectively, because otherwise $f - \chi(f)\mathbb{1}$ would be invertible. This is a contradiction, because $\chi(f - \chi(f)\mathbb{1}) = 0$, $\mathbb{1}$ denoting the unit $X \to \mathbb{K}, x \mapsto 1$ of $A$.
2. Consequently, we may conclude: $\chi(f) \geq 0$ if $f \geq 0$ and $\chi(|f|) = |\chi(f)|$ for all $f \in A$.

**Proposition 2.3** *(See e.g. [33, p. 38f].) Let $X$ be any topological space. Then $C(X)$ is functionally bounded with respect to the relative product topology induced on $C(X)$ by $\mathbb{K}^X$. (Thus $C(X)$ is also functionally bounded with respect to the compact open topology.)*

**Proof:** Let $\chi \in \sigma(C(X))$, and $(f_n)_{n\in\mathbb{N}} \in C(X)^{\mathbb{N}}$ be a pointwise bounded sequence. We obtain:

$$g_n := 2^{-n}\frac{|f_n - \chi(f_n)\mathbb{1}|}{\mathbb{1} + |f_n - \chi(f_n)\mathbb{1}|} \in C\mathcal{B}(X).$$

$g := \sum_{n=1}^{\infty} g_n \in C\mathcal{B}(X)$ is well defined and the restriction of $\chi$ to $C\mathcal{B}(X)$ is continuous, because $C\mathcal{B}(X)$ is a Banach algebra. The second remark above implies $\chi(g_n) = 0$ for each $n \in \mathbb{N}$. Thus, continuity of $\chi$ on $C\mathcal{B}(X)$ yields $\chi(g) = \sum_{n=1}^{\infty} \chi(g_n) = 0$. Now, the first remark above implies that there is $x \in X$ such that $g(x) = 0$. Therefore $g_n(x) = 0$ for all $n \in \mathbb{N}$. Thus, we obtain $\chi(f_n) = f_n(x)$ for all $n \in \mathbb{N}$. This yields the conclusion:

$$\sup_{n\in\mathbb{N}} |\chi(f_n)| \;=\; \sup_{n\in\mathbb{N}} |f_n(x)| < \infty.$$

□

**Corollary 2.3** *For every set $S$ the algebra $\mathbb{K}^S = C(S, \mathcal{T}_{discr.})$ is functionally bounded, $\mathcal{T}_{discr.}$ denoting the discrete topology on $S$.*

However, the following shows that it is not clear whether $\mathbb{K}^S$ is always functionally continuous. We are now going to characterize those sets $S$ such that $\mathbb{K}^S$ is functionally continuous.

In [29, p. 179] the follwing result is proved:
For a set $S$ t.f.a.e.

*i)* $\mathbb{K}^S$ is not bornological.

*ii)* There is an ultrafilter $\mathcal{F}$ on $S$ satisfying $\bigcap \mathcal{F} = \emptyset$ and $\bigcap_{n\in\mathbb{N}} F_n \neq \emptyset$ for all $(F_n)_{n\in\mathbb{N}} \in \mathcal{F}^{\mathbb{N}}$.

*iii)* There is a measure $\mu$ on $2^S$ with values in $\{0,1\}$ such that $\mu(S) = 1$ and $\mu(\{s\}) = 0$ for all $s \in S$.

The last property of $\mathcal{F}$ in *ii)* is called the countable intersection property. It is easy to see that the countable intersection property is equivalent to:

$$\forall (F_n)_{n\in\mathbb{N}} \in \mathcal{F}^{\mathbb{N}} : \bigcap_{n\in\mathbb{N}} F_n \in \mathcal{F}$$

The measure $\mu$ in *iii)* is called an Ulam measure. A set $S$ is said to satisfy the Mackey-Ulam condition, if no such measure exists; else it is said to be of measurable cardinality.
The proof of '*i)* $\Leftrightarrow$ *iii)*' is due to Mackey. To prove the implication '*ii)* $\Rightarrow$ *i)*', one constructs a bounded linear functional $\chi : \mathbb{K}^S \to \mathbb{K}$ which is not the 0-map but vanishes on $\mathbb{K}^{(S)}$. Going through the proof again, $\chi$ turns out to be a character on $\mathbb{K}^S$. A set $S$ with property *i)*, *ii)*, or *iii)* is said to be of strongly inaccessible cardinality. It is unknown whether such sets do exist. The following is a slightly more general version of what is proved in [29, theorem 6.2.23], '*ii)* $\Rightarrow$ *i)*':

**Proposition 2.4** *If a set $S$ is of strongly inaccessible cardinality, then for every algebra $A$ different from $\{0\}$ which is not of strongly inaccessible cardinality there is a linear and multiplicative map $\chi : A^S \to A$ different from the 0-map which vanishes on $A^{(S)}$. If, in addition, $A$ is a tvs with a fundamental sequence of bounded sets, $\chi$ is also bounded.*

**Proof:** Let $\mathcal{F}$ be an ultrafilter on $S$ as in the remark above. For $x = (x_s)_{s\in S} \in A^S$ we define $\mathcal{B}_x := \{\{x_s : s \in F\} : F \in \mathcal{F}\}$. $\mathcal{B}_x$ is a filterbasis on $A$ and the filter $\mathcal{F}_x$ generated by $\mathcal{B}_x$ is an ultrafilter on $A$. Indeed, let $G \subset A$, not belonging to $\mathcal{F}_x$. Then we have $\{s \in S : x_s \in G\} \notin \mathcal{F}$, hence $F := \{s \in S : x_s \notin G\} \in \mathcal{F}$ and $\{x_s : s \in F\} \subset A \setminus G$ implies $A \setminus G \in \mathcal{F}_x$.
It is easy to see that $\mathcal{F}_x$ has the countable intersection property, hence $\bigcap \mathcal{F}_x \neq \emptyset$, because $A$ is not of strongly inaccessible cardinality. Since $\bigcap \mathcal{F}_x$ is an ultrafilter, this implies

$$\bigcap \mathcal{F}_x = \{\chi(x)\}$$

for some $\chi(x) \in A$.
We claim that $\chi : A^S \to A$ is linear and multiplicative. For all $x \in A^S$ first observe that $\mathcal{F}_x$ converges to $\chi(x)$ in $A$ with its discrete topology. Now, it is easy to see that $\mathcal{F}_{x+y} \to \chi(x) + \chi(y)$

and $\mathcal{F}_{\lambda x} \to \lambda\chi(x)$ and $\mathcal{F}_{xy} \to \chi(x)\cdot\chi(y)$.
Moreover, let $x \in A \setminus \{0\}$. Then, clearly $\chi((x)_{s\in S}) = \{x\}$, hence $\chi \neq$ 0-map. Let now $x = (x_s)_{s\in S} \in A^{(S)}$. Then $E := \{s \in S : x_s \neq 0\}$ is finite. It follows that $S \setminus E \in \mathcal{F}$, because otherwise one would obtain $\bigcap \mathcal{F} \neq \emptyset$. This implies $\chi(x) \in \{x_s : s \in S \setminus E\} = \{0\}$.
Let, finally, $(B_n)_{n\in \mathbb{N}}$ be a fundamental sequence of bounded sets for $A$, and $B \subset A^S$ any bounded set. One can find a family $(B_s)_{s\in S}$ of bounded sets in $A$ such that $B \subset \prod_{s\in S} B_s$. For all $n \in \mathbb{N}$ we define $F_n := \{s \in S : B_s \subset B_n\}$ and get $S = \bigcup_{n\in\mathbb{N}} F_n$. Thus, we can find $n \in \mathbb{N}$ such that $F_n \in \mathcal{F}$. This implies $\chi(x) \in \{x_s : s \in F_n\} \subset B_n$ for all $x = (x_s)_{s\in S} \in B$, hence $\chi$ is bounded. □

**Corollary 2.4** *For every set S, t.f.a.e.:*

*i)* $\sigma(\mathbb{K}^S) = \{(\delta_s : \mathbb{K}^S \to \mathbb{K}, (x_t)_{t\in S} \mapsto x_s) : s \in S\}$.

*ii)* $\mathbb{K}^S$ *is functionally continuous.*

*iii) S is not of strongly inaccessible cardinality.*

**Proof:** The implication ' *i)* ⇒ *ii)*' is obvious. The proof of ' *ii)* ⇒ *i)*' is in complete analogy to the proof of corollary 2.1 and the proof of the equivalence ' *ii)* ⇔ *iii)*' is an immediate consequence of corollary 2.3, proposition 2.4 and the remark after corollary 2.3. □

**Corollary 2.5** *Let S be a set which is not of strongly inaccessible cardinality and* $(A_s)_{s\in S}$ *a family of algebras. Then we have the representation:*

$$\sigma\left(\prod_{s\in S} A_s\right) = \{\psi \circ pr_t : t \in S, \psi \in \sigma(A_t)\}$$

**Proof:** We may assume that each $A_s$ has a unit $e_s$, because $\prod_{s\in S} A_s$ is an ideal in $\prod_{s\in S} (A_s)_e$, hence $\chi \in \sigma\left(\prod_{s\in S} A_s\right)$ has a unique extension to a character $\phi \in \sigma\left(\prod_{s\in S} (A_s)_e\right)$. If $\phi = \psi \circ pr_t$ with $t \in S$ and $\psi \in \sigma((A_t)_e)$ then, clearly, $\chi = (\psi|A_t) \circ pr_t$.
$j : \mathbb{K}^S \to \prod_{s\in S} A_s, (\mu_s)_{s\in S} \mapsto (\mu_s e_s)_{s\in S}$ is an algebra embedding. Let now $\chi \in \sigma\left(\prod_{s\in S} A_s\right)$ then $\chi \circ j \in \sigma(\mathbb{K}^S)$ (note that $\chi \circ j((1)_{s\in S}) = 1$). Hence $\chi \circ j = \delta_t$ for some $t \in S$.
Let $(x_s)_{s\in S} \in \prod_{s\in S} A_s$ such that $x_t = 0$. We set $\mu_s = 1$, if $s \neq t$ and $\mu_t = 0$. Then

$$\chi((x_s)_{s\in S}) = \chi((\mu_s e_s x_s)_{s\in S}) = \chi(j((\mu_s)_{s\in S})(x_s)_{s\in S}) =$$

$$= (\chi \circ j)((\mu_s)_{s\in S})\chi((x_s)_{s\in S}) = 0.$$

Now $pr_t : \prod_{s\in S} A_s \to A_t$ is an algebra-epimorphism the kernel of which is contained in the kernel of $\chi$ which yields the conclusion. □

**Corollary 2.6** *Let $S$ be a set which is not of strongly inaccessible cardinality. If $(A_s)_{s\in S}$ is a family of functionally bounded (continuous) top algebras, then the algebra $\prod_{s\in S} A_s$ is functionally bounded (continuous).*

For sets of strongly inaccessible cardinality we have the following characterization in terms of characters on product algebras:

**Proposition 2.5** *Let a set $S$ be given. $S$ is of strongly inaccessible cardinality, iff for every family of algebras $(A_s)_{s\in S}$ satisfying $\sigma(A_s) \neq \emptyset$ for all $s \in S$ there is a character $\chi \in \sigma\left(\prod_{s\in S} A_s\right)$ which cannot be found among the characters in $\{\psi \circ pr_s : s \in S, \psi \in \sigma(A_s)\}$.*

**Proof:** If there is a character $\chi \in \sigma\left(\mathbb{K}^S\right) \setminus \{\delta_s : s \in S\}$, then, by corollary 2.4, $S$ is of strongly inaccessible cardinality.
For the converse let $(\psi_s)_{s\in S} \in \prod_{s\in S} \sigma(A_s)$. We define

$$\psi := \prod_{s\in S} \psi_s : \prod_{s\in S} A_s \longrightarrow \mathbb{K}^S, (x_s)_{s\in S} \longmapsto (\psi_s(x_s))_{s\in S},$$

which is an algebra-epimorphism. Take $\chi \in \sigma(\mathbb{K}^S)$ as in proposition 2.4 and we define $\varphi := \chi \circ \psi \in \sigma\left(\prod_{s\in S} A_s\right)$. Then, clearly, $\varphi$ vanishes on $\bigoplus_{s\in S} A_s$. Thus, $\chi$ cannot be of the form $\psi \circ pr_s$ for any $s \in S$ and any $\psi \in \sigma(A_s)$. □

**Proposition 2.6** *(See [33, p. 42f].) Let $X$ be a Tychonov space. $X$ is realcompact, iff $\sigma(C(X)) = \{(\delta_x : C(X) \to \mathbb{K}, f \mapsto f(x)) : x \in X\}$.*

**Proof:** The realcompactification of $X$ can be defined by

$$\nu : X \longrightarrow \sigma(C(X)), x \longmapsto \delta_x,$$

which is a homeomorphism onto its range satisfying $\overline{\nu(X)} = \sigma(C(X))$. Now, $X$ is realcompact, iff $\nu$ is surjective. □

**Corollary 2.7** *For the characters on $l^\infty$ we have the representation*

$$\sigma(l^\infty) = \{\delta_z : z \in \beta\mathbb{N}\}$$

*($\beta\mathbb{N}$ denoting the Stone-Čech compactification of the natural numbers).*

**Proof:** It is easy to see that

$$\phi : \mathcal{C}(\beta\mathbb{N}) \longrightarrow l^\infty, f \longmapsto (f(n))_{n\in\mathbb{N}}$$

is an algebra isomorphism. Now, by [15, theorem 3.11.1] and the above proposition, every character on $\mathcal{C}(\beta\mathbb{N})$ is of the form $\delta_z$ for some $z \in \beta\mathbb{N}$. □

In the sequel we are going to describe the characters on $\mathcal{C}(X,A)$, where $\mathcal{C}(X,A)$ denotes the algebra of continuous functions $f : X \to A$ for a topological space $X$ and a *top* algebra $A$ in some important cases. Moreover, we will generalize corollary 2.1 to vector-valued sequence spaces $\lambda(A)$ for *lc* algberas $A$ and normal Banach sequence spaces $\lambda$. Last, but not least, there is a description of $\sigma(\mathcal{H}(\Omega,A))$, $\mathcal{H}(\Omega,A)$ denoting the algebra of holomorphic functions from an open subset $\Omega$ of the complex plane to a locally complete *lc* algebra $A$.
These results are due to S. Dierolf, Schröder, and Wengenroth. See [11].

**Proposition 2.7** *Let X be a realcompact topological space and A a metrizable top algebra. Then*

$$\sigma(\mathcal{C}(X,A)) = \{\psi \circ pr_x : x \in X, \psi \in \sigma(A)\}.$$

**Proof:** We may assume that $A$ has a unit $e$. Indeed $\mathcal{C}(X,A)$ is an ideal in $\mathcal{C}(X,A_e)$. Now, for all $\chi \in \sigma(\mathcal{C}(X,A))$, one can find a unique extension $\varphi \in \sigma(\mathcal{C}(X,A_e))$ of $\chi$. If $\varphi = \psi \circ pr_x$, for some $x \in X$ and some $\psi \in \sigma(A_e)$, then, clearly, $\chi = (\psi|A) \circ pr_x$.
We define

$$j : \mathcal{C}(X) \longrightarrow \mathcal{C}(X,A), f \longmapsto (f \otimes e : X \to A, x \mapsto f(x)e),$$

which is well-defined, linear, and multiplicative. Then $\chi \circ j$ is a character on $\mathcal{C}(X)$ (note that $(\chi \circ j)(1\!\!1) = 1$). Thus, proposition 2.6 implies that we can find $x \in X$ such that $\chi \circ j = \delta_x$. $I :=$ kern$(pr_x) = \{f \in \mathcal{C}(X,A) : f(x) = 0\}$ is an ideal in $\mathcal{C}(X,A)$. Now we claim that $I$ is contained in the kernel of $\chi$.
Let therefore $f \in I$. Since $A$ is metrizable, there is by [22, theorem 2.8.1] an $F$-norm $||\cdot|| : A \to [0,\infty)$ on $A$ generating its linear topology. (Note that an $F$-norm satisfies the triangle inequality and $||\lambda w|| \le ||w||$ for all $w \in A$ and for all $\lambda \in B_{\mathbb{K}}[0,1]$.) We define $g := \sqrt{||f||} \in \mathcal{C}(X)$. Then $g$ has obviously the same zeros as $f$. Moreover, we define a function $h$ on $X$ by

$$h(z) := \begin{cases} ||f(z)||^{-\frac{1}{2}} f(z), & if\ f(z) \neq 0 \\ 0, & if\ f(z) = 0 \end{cases}$$

for all $z \in X$. First observe that $f = j(g)h$.
Now we claim that, in fact, $h \in \mathcal{C}(X,A)$. $h$ is certainly continuous in all points $z \in X$ with $f(z) \neq 0$. Let now $z \in X$ such that $f(z) = 0$, and $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $\frac{2}{n} < \varepsilon$. We get

$$V := \left\{ y \in X : ||f(y)|| \le \frac{1}{n^2} \right\} \in \mathcal{U}_z(X).$$

and for $y \in V$ either $||h(y)|| = 0$ or one can find $k \in \mathbb{N}$, $k \ge n$ such that

$$\frac{1}{(k+1)^2} < ||f(y)|| \le \frac{1}{k^2}.$$

But this yields

$$\lambda := \frac{1}{k+1}\frac{1}{\sqrt{\|f(y)\|}} < \frac{k+1}{k+1} = 1$$

and, by the second remark on $F$-norms above, we obtain

$$\|h(y)\| = \left\| \frac{1}{k+1}\frac{1}{\sqrt{\|f(y)\|}}(k+1)f(y) \right\| \leq \|(k+1)f(y)\|.$$

Now, the triangle inequality yields

$$\|h(y)\| \leq (k+1)\|f(y)\| \leq \frac{k+1}{k^2} = \frac{1+\frac{1}{k}}{k} \leq \frac{2}{k} \leq \frac{2}{n} < \varepsilon,$$

which proves that h is continuous.
Now we conclude

$$\chi(f) = \chi(j(g)h) = (\chi \circ j)(g)\chi(h) = g(x)\chi(h) = 0.$$

Since $pr_x : C(X,A) \to A$ is an epimorphism and $\text{kern}(pr_x) \subset \text{kern}(\chi)$, we can find a character $\psi$ on $A$ satisfying $\chi = \psi \circ pr_x$. □

**Corollary 2.8** *For a realcompact space $X$ and a metrizable top algebra $A$, $C(X,A)$ is functionally bounded (continuous) with respect to the relative product topology induced by $A^X$, if $A$ is functionally bounded (continuous).*

Let $A$ be a realcompact, metrizable *top* algebra and $X$ a Tychonov space. By [15, theorem 3.11.16] every $f \in C(X,A)$ has a unique extension $g \in C(\nu X,A)$. This implies $C(X,A) = C(\nu X,A)$ and we obtain:

**Corollary 2.9** *Let $X$ be a Tychonov space and $A$ a realcompact, metrizable top algebra. Then*

$$\sigma(C(X,A)) = \{\psi \circ pr_x : x \in \nu X, \psi \in \sigma(A)\}.$$

**Corollary 2.10** *For a Tychonov space $X$ and a realcompact, metrizable top algebra $A$, the algebra of continuous functions on $X$ with values in $A$, $C(X,A)$, is functionally bounded (continuous) with respect to the relative product topology induced by $A^X$, if $A$ is functionally bounded (continuous).*

Note that $A$ as a metrizable *tvs* is realcompact, if it is not of strongly inaccessible cardinality (see [15, p. 465]).
In case that in proposition 2.7 $X$ is compact, the requirement $A$ being metrizable can be relaxed.

**Definition 2.3** *An lcs $E$ satisfies the strict Mackey condition (sMc), if for every bounded subset $B \subset E$ one can find an absolutely convex and bounded subset $D \subset E$ containing $B$ such that the Minkowski functional $p_D$ induces the original topology on $B$.*

**Remark:** By [29, theorem 5.1.27 *ii)*] every metrizable *lcs* satisfies $(sMc)$.

**Proposition 2.8** *Let X be a compact topological space and A a Hausdorff lc algebra satisfying (sMc). Then*

$$\sigma(C(X,A)) = \{\psi \circ pr_x : x \in X, \psi \in \sigma(A)\}.$$

**Proof:** Let $\chi \in \sigma(C(X,A))$. As in proposition 2.7 we may assume that $A$ has a unit $e$. Again we define

$$j : C(X) \longrightarrow C(X,A), f \longmapsto (f \otimes e : X \to A, x \mapsto f(x)e)$$

and obtain $\chi \circ j = \delta_x$ for some $x \in X$, because by [15, theorem 3.11.1] every compact space is realcompact. Again we claim

$$\chi(\{f \in C(X,A) : f(x) = 0\}) = \{0\}.$$

Let $f \in C(X,A)$ such that $f(x) = 0$. $f(X)$ is bounded in $A$. So, by $(sMc)$, $f(X)$ is contained in a bounded set $D = \Gamma D \subset A$ such that the Minkowski functional $p_D$ induces the original topology on $f(X)$. Now go on as in the proof of proposition 2.7 only replacing the $F$-norm $\|\cdot\|$ by the Minkowski functional $p_D$. □

Observe that what is really needed in the above proof is a weaker condition than $(sMc)$, namely: Every compact subset $K$ is contained in an absolutely convex, bounded subset $B$ such that $p_B$ induces the original topology on $K$.
As a consequence of proposition 2.8 we obtain:

**Corollary 2.11** *For a compact space X and a Hausdorff lc algebra A satisfying (sMc), $C(X,A)$ is functionally bounded (continuous) with respect to the relative product topology induced by $A^X$, if A is functionally bounded (continuous).*

If $X = \alpha\mathbb{N} = \mathbb{N} \cup \{\infty\}$ is the Alexandroff compactification of $\mathbb{N}$ the requirement for $A$ can be weakened to the Mackey convergence condition.
Observe that $C(\alpha\mathbb{N},A) = c(A)$ is the algebra of convergent sequences in $A$. Indeed

$$\phi : c(A) \longrightarrow C(\alpha\mathbb{N},A),$$

$$(x_n)_{n\in\mathbb{N}} \longmapsto \left( g : \alpha\mathbb{N} \to A, t \mapsto \begin{cases} x_t, & if\ t \in \mathbb{N} \\ \lim\limits_{n\in\mathbb{N}} x_n, & if\ t = \infty \end{cases} \right)$$

is an algebra-isomorphism.

**Definition 2.4** *A tvs E satisfies the Mackey convergence condition $(Mcc)$, if for all $(x_n)_{n\in\mathbb{N}} \in c_0(E)$ there is a sequence $(\mu_n)_{n\in\mathbb{N}} \in c_0 \cap (0,\infty)^{\mathbb{N}}$ such that*

$$(\mu_n^{-1} x_n)_{n\in\mathbb{N}} \in c_0(E).$$

**Remark:** By [29, 5.1.30 *ii)*] $(sMc)$ implies $(Mcc)$.

We are first going to consider the algebra $c_0(A)$ of nullsequences in $A$ where $A$ is a *top* algebra.

**Proposition 2.9** *Let $A$ be a top algebra satisfying (Mcc). Then the characters on $c_0(A)$ admit the representation*

$$\sigma(c_0(A)) = \{\psi \circ pr_n : \psi \in \sigma(A), n \in \mathbb{N}\}.$$

**Proof:** Again we may assume that $A$ has a unit $e$, because $c_0(A)$ is an ideal in $c_0(A_e)$. Form the canonical embedding

$$j : c_0 \longrightarrow c_0(A), (\alpha_n)_{n\in\mathbb{N}} \longmapsto (\alpha_n e)_{n\in\mathbb{N}}.$$

$\chi \circ j$ is a character on $c_0$. Indeed, let us assume $\chi \circ j = 0$ and let any $(y_k)_{k\in\mathbb{N}} \in c_0(A)$ be given. Choose $(\mu_k)_{k\in\mathbb{N}} \in c_0 \cap (0,\infty)^{\mathbb{N}}$ such that $(\mu_k^{-1} y_k)_{k\in\mathbb{N}} \in c_0(A)$. This yields

$$\chi\left((y_k)_{k\in\mathbb{N}}\right) = \chi\left(j((\mu_k)_{k\in\mathbb{N}})\right)\chi\left((\mu_k^{-1} y_k)_{k\in\mathbb{N}}\right) = 0,$$

which is a contradiction.
Thus, by corollary 2.1, one can find $n \in \mathbb{N}$ such that $\chi \circ j = \delta_n$. Let now $(x_k)_{k\in\mathbb{N}} \in c_0(A)$ be given such that $x_n = 0$. One can find $(\alpha_k)_{k\in\mathbb{N}} \in c_0 \cap (0,\infty)^{\mathbb{N}}$ such that $(\alpha_k^{-1} x_k)_{k\in\mathbb{N}} \in c_0(A)$. Now we define

$$\beta_k := \begin{cases} \alpha_k, & k \neq n \\ 0, & k = n \end{cases}$$

for all $k \in \mathbb{N}$. Then $(\beta_k)_{k\in\mathbb{N}} \in c_0$ and we obtain:

$$(x_k)_{k\in\mathbb{N}} = j((\beta_k)_{k\in\mathbb{N}}) \cdot (\alpha_k^{-1} x_k)_{k\in\mathbb{N}}.$$

This yields

$$\chi((x_k)_{k\in\mathbb{N}}) = \chi(j((\beta_k)_{k\in\mathbb{N}})) \cdot \chi((\alpha_k^{-1} x_k)_{k\in\mathbb{N}}) = 0.$$

Now conclude as in the proof of proposition 2.7. □

**Corollary 2.12** *Let $A$ be a top algebra satisfying (Mcc), then*

$$\sigma(C(\alpha\mathbb{N}, A)) = \{\psi \circ pr_x : x \in \alpha\mathbb{N}, \psi \in \sigma(A)\}.$$

**Proof:** First observe that $c_0(A)$ is an ideal in $c(A)$. Let $\chi \in \sigma(c(A))$. Now, if $\chi(c_0(A)) \neq \{0\}$, then there is $n \in \mathbb{N}$ and $\psi \in \sigma(A)$ such that $\chi|c_0(A) = \psi \circ pr_n$. Hence $\chi = \psi \circ pr_n$.
If $\mathrm{kern}(\chi) \supset c_0(A) = \mathrm{kern}(pr_\infty)$, one can find $\psi \in \sigma(A)$ satisfying $\chi = \psi \circ pr_\infty$. □

**Corollary 2.13** *Let $\lambda$ be a normal Banach sequence space containing $\varphi$ as a dense subspace, and $A$ an lc algebra satisfying (Mcc), then*

$$\sigma(\lambda(A)) = \{\psi \circ \rho_n pr_n : n \in \mathbb{N}, \psi \in \sigma(A)\}.$$

**Proof:** Let $\chi \in \sigma(\lambda(A))$. By [16, theorem 3.3] $A^{(\mathbb{N})}$ is dense in $\lambda(A)$, hence

$$j : \lambda(A) \longrightarrow c_0(A), (x_n)_{n\in\mathbb{N}} \longmapsto (\rho_n x_n)_{n\in\mathbb{N}}$$

is a well-defined isomorphism onto an ideal in $c_0(A)$. (See the 3rd remark after definition 1.5.) Now, by proposition 2.9, $\chi \circ j^{-1} = \psi \circ pr_n$ for some $n \in \mathbb{N}$ and some $\psi \in \sigma(A)$, thus $\chi = \psi \circ pr_n \circ j = \psi \circ \rho_n pr_n$. □

**Corollary 2.14** *For every normal Banach sequence space $\lambda$ and every lc algebra A satisfying (Mcc), $\lambda(A)$ is functionally bounded (continuous), if A is functionally bounded (continuous).*

We are now going to describe the characters on $\mathcal{H}(\Omega,A)$, the algebra of vector-valued holomorphic functions, where $\Omega$ is an open subset of $\mathbb{C}$ and $A$ an *lc* $\mathbb{C}$-algebra.

**Definition 2.5** *Let $\Omega$ be an open subset of $\mathbb{C}$ and A an lc $\mathbb{C}$-algebra. $f : \Omega \to A$ is called holomorphic, iff it is weakly holomorphic, i.e. $x' \circ f$ is holomorphic in the usual sense for all $x' \in A'$.*

**Remark:** Let $\Omega$ be given as above and $A$ an *lc* algebra such that the closed absolutely convex hull of any compact subset of $A$ is again compact (the so-called convex-compactness-property (*cc*)). Then, by [19, théorème 1] every weakly holomorphic function is holomorphic in a stronger sense: They have locally uniformly convergent Taylor series. This implies that $\mathcal{H}(\Omega,A)$ is an algebra with respect to pointwise operations.
By [18, II theorem 5.5] the requirement (cc) for $A$ in Grothendiecks's theorem can be weakened to local completeness.

**Proposition 2.10** *Let $\Omega$ be an open subset of $\mathbb{C}$ and A a locally complete lc $\mathbb{C}$-algebra. Then, for the characters on $\mathcal{H}(\Omega,A)$ we have the representation*

$$\sigma(\mathcal{H}(\Omega,A)) = \{\psi \circ pr_z : z \in \Omega, \psi \in \sigma(A)\}.$$

**Proof:** Let $\chi \in \sigma(\mathcal{H}(\Omega,A))$. We may again assume that $A$ has a unit $e$, thus there is the canonical embedding $j : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega,A), f \mapsto f \otimes e$. It is well known that, as a character on $\mathcal{H}(\Omega)$, $\chi \circ j$ satisfies $\chi \circ j = \delta_z$ for some $z \in \Omega$ (see [28, theorem 12.7]). Again we prove $\chi(\mathrm{kern}(pr_z)) = \{0\}$. Let therefore $f \in \mathcal{H}(\Omega,A)$ be given, such that $f(z) = 0$. We define

$$g(\zeta) := \begin{cases} \frac{f(\zeta)}{\zeta - z}, & \text{if } \zeta \neq z, \\ f'(z), & \text{if } \zeta = z \end{cases}$$

for all $\zeta \in \Omega$. Since $f$ has a Taylor expansion $f(\zeta) = \sum_{n=1}^{\infty} \frac{f^{(n)}(z)}{n!}(\zeta - z)^n$ around $z$, it follows that $g$ is again holomorphic. $h : \Omega \to \mathbb{C}, \zeta \mapsto \zeta - z \in \mathcal{H}(\Omega)$ satisfies $f = j(h)g$ and the same reasoning as in proposition 2.7 yields the conclusion. □

**Corollary 2.15** *For an open subset $\Omega$ of the complex plane and every locally complete lc $\mathbb{C}$-algebra $A$, $\mathcal{H}(\Omega, A)$ is functionally bounded (continuous) with respect to the relative product topology induced by $A^{\Omega}$, if $A$ is functionally bounded (continuous).*

We are now concluding this chapter presenting a more general result than corollary 2.13.

**Proposition 2.11** *Let $B$ and $C$ be lc algebras, $B$ satisfying (Mcc). Let $\lambda$ be a normal Banach sequence space containing $\varphi$ as a dense subspace and $f : C \to B$ a linear, multiplicative, and continuous map. Let us denote by*

$$A := \left\{(y_n)_{n\in\mathbb{N}} \in C^{\mathbb{N}} : (f(y_n))_{n\in\mathbb{N}} \in \lambda(B)\right\}$$

*the projective limit of Moscatelli type (see [25, prop. 3.1.1]). Then we have the following representation of the characters on $A$:*

$$\sigma(A) = \{\psi \circ pr_n : n \in \mathbb{N}, \psi \in \sigma(C)\}.$$

**Proof:** W.l.o.g. we may assume that $f$ is surjective. Let $\chi \in \sigma(A)$. By the usual argument it follows that there is $n \in \mathbb{N}$ and $\psi \in \tilde{\sigma}(C)$ such that $\chi|C^{(\mathbb{N})} = \psi \circ pr_n$. Now, it suffices to prove that $\chi|C^{(\mathbb{N})} = 0$ implies $\chi = 0$ for all $\chi \in \tilde{\sigma}(A)$.
Let $\chi \in \tilde{\sigma}(A)$ such that $\chi|C^{(\mathbb{N})} = 0$. We assume $\chi \in \sigma(A)$. As in the proof of corollary 2.13 observe that $A$ is an ideal in $D := \text{proj}\left(C \xrightarrow{f} B, c_0\right)$. Thus $\chi$ has a unique extension to a character on $D$, hence we may assume $\lambda = c_0$.
Moreover, we may assume that $C$ has a unit $e$. Indeed, if $C$ has no unit, form $C_e$. In case $B$ has a unit $e_B$, $f : C_e \to B, y + \lambda e \mapsto f(y) + \lambda e_b$ is a linear, multiplicative, and continuous extension of $f$. In case $B$ has no unit $\tilde{f} : C_e \to B_e, y + \lambda e \mapsto f(y) + \lambda e$ is again a linear, multiplicative, and continuous extension of $f$. $A = \text{proj}\left(C \xrightarrow{f} B, c_0\right)$ is an ideal in $D := \text{proj}\left(C_e \xrightarrow{f} B, c_0\right)$ and the unique extension $\psi$ of $\chi$ to a character on $D$ vanishes on $C_e^{(\mathbb{N})}$, because $C_e^{(\mathbb{N})} \cdot A \subset C^{(\mathbb{N})}$, and $\psi(x) = \chi(ux)$ for all $x \in D$ and some (fixed) $u \in A$ satisfying $\chi(u) = 1$ (see definition of the extension of a character on an ideal in the proof of the 2nd remark after definition 2.1).
$j : c_0 \to A, (\alpha_n)_{n\in\mathbb{N}} \mapsto (\alpha_n e)_{n\in\mathbb{N}}$ is an algebra embedding. This yields $\chi \circ j \in \tilde{\sigma}(c_0)$ and $\chi \circ j|\mathbb{K}^{(\mathbb{N})} = 0$, thus $\chi \circ j = 0$, because $c_0$ is a Banach algebra.
Let now $(y_k)_{k\in\mathbb{N}} \in A$ be given. One can find $(\gamma_k)_{k\in\mathbb{N}} \in c_0 \cap (0,\infty)^{\mathbb{N}}$ such that $(\frac{1}{\gamma_k} f(y_k))_{k\in\mathbb{N}} \in c_0(B)$ and we conclude by

$$(y_k)_{k\in\mathbb{N}} = \left(\frac{1}{\gamma_k} y_k\right)_{k\in\mathbb{N}} \cdot (\gamma_k y_k)_{k\in\mathbb{N}}$$

which yields

$$\chi((y_k)_{k\in\mathbb{N}}) = \chi\left(\left(\frac{1}{\gamma_k} y_k\right)_{k\in\mathbb{N}}\right) \cdot \chi((\gamma_k y_k)_{k\in\mathbb{N}}) = 0,$$

which is a contradiction. □

**Corollary 2.16** *For lc algebras B and C, B satisfying (Mcc), and a linear, multiplicative, and continuous map $f : C \to B$, and a normal Banach sequence space $\lambda$ satisfying (sc), the Moscatelli algebra proj $\left(C \xrightarrow{f} B, \lambda\right)$ is functionally bounded (continuous), if C is functionally bounded (continuous).*

## Chapter 3

## Inductive Topologies on Locally Convex Algebras

We have already hinted that an algebra $A$ endowed with the $lc$ inductive topology with respect to a family of linear and multiplicative maps $(f_t : A_t \to A)_{t\in T}$, where each $A_t$ is an $l(m)c$ algebra, is, in general, not $l(m)c$.

However, one may always consider the finest $l(m)c$ topology on $A$ such that each $f_t$ is continuous, but generally nothing essential about this topology can be proved unless it coincides with the $lc$ inductive topology.

**Example 3.1.** Let $(A_s)_{s\in T}$ be a family of $l(m)c$ algebras, then $(\prod_{s\in E} A_s)_{E\subset S \text{ finite}}$ is an inductive spectrum of $l(m)c$ algebras each provided with componentwise multiplication, which is directed by inclusion. The $lc$ inductive limit is equal to $\bigoplus_{s\in S} A_s$ with its $lc$ direct sum topology.

$$\left\{ \bigoplus_{s\in S} U_s : (U_s)_{s\in S} \in \prod_{s\in S} \mathcal{U}_0(A_s) \right\}$$

is a 0nbh-basis in $\bigoplus_{s\in S} A_s$, hence $\left(\bigoplus_{s\in S} U_s\right)^2 = \bigoplus_{s\in S} U_s^2$ implies that $\bigoplus_{s\in S} A_s$ is $l(m)c$ again.

Note that however 1.4.3 is an example of an $lc$ inductive limit of $lmc$ algebras which has continuous multipliation but fails to be $lmc$, the corresponding linear maps are *not* multiplicative.

In the sequel we will focuss countable inductive limits of $l(m)c$ algebras and aim at conditions such that the $lc$ inductive limit is again an $l(m)c$ algebra.
For the first result which is presented in this section see [12]. Proposition 3.1 is due to M. Akkar, C. Nacir (see [1]) where a result of [30] is applied. There, again, at a key point, a statement of [4] is quoted. A commutative version is stated in [23].

**Lemma 3.1** *Let $(A, \|\cdot\|_A)$ and $(C, \|\cdot\|_C)$ be seminormed algebras and $f : A \to C$ a continuous algebra-homomorphism. Then there is a submultiplicative seminorm $|||\cdot|||$ on C which is equivalent to $\|\cdot\|_C$ such that*

$$|||f(x)||| \leq \|x\|_A$$

*for all $x \in A$.*

**Proof:** Let $B_A$ and $B_C$ denote the closed unit balls on $A$ and on $C$, respectively. $U := f(B_A)$ is bounded and multiplicative, thus one can find $\lambda \geq 1$ such that $U \subset \lambda B_C$. Now we define

$$D := \Gamma\left(\bigcup_{k\in\mathbb{N}} \left(U \cup \frac{1}{\lambda}B_C\right)^k\right)$$

which is the smallest absolutely $m$-convex subset of $C$ containing $U \cup \frac{1}{\lambda}B_C$. We claim $D \subset \lambda B_C$. It suffices to prove $\left(U \cup \frac{1}{\lambda}B_C\right)^k \subset \lambda B_C$ for all $k \in \mathbb{N}$.
Now, $\left(U \cup \frac{1}{\lambda}B_C\right)^k$ is the union of $U^k$, $(\frac{1}{\lambda}B_C)^k$ and of finite products of sets of the form

$$U\frac{1}{\lambda}B_C,\ \frac{1}{\lambda}B_C U,\ U\frac{1}{\lambda}B_C U,\ \frac{1}{\lambda}B_C U\frac{1}{\lambda}B_C$$

all of which being contained in $\lambda B_C$. By $\frac{1}{\lambda}B_C \subset D \subset \lambda B_C$ we obtain that $|||\cdot||| := p_D$ is equivalent to $\|\cdot\|_C$ and $U \subset D$ yields $|||f(\cdot)||| \leq \|\cdot\|_A$. □

**Proposition 3.1** *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of seminormed algebras such that each $A_n$ is continuously embedded in $A_{n+1}$. Then $A := \bigcup_{n\in\mathbb{N}} A_n$ endowed with the finest lc topology such that all the inclusions $A_n \hookrightarrow A$ are continuous (i.e. the lc inductive limit of $((A_n)_{n\in\mathbb{N}}, (A_n \hookrightarrow A_{n+1})_{n\in\mathbb{N}})$) is an lmc algebra.*

**Proof:** As an application of the above lemma we find inductively a sequence $(B_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathcal{U}_0(A_n)$ of bounded, absolutely $m$-convex sets, such that $B_n \subset B_{n+1}$ for all $n \in \mathbb{N}$. Hence

$$\left\{\Gamma\left(\bigcup_{n\in\mathbb{N}} \varepsilon_n B_n\right) : (\varepsilon_n)_{n\in\mathbb{N}} \in (0,1]^{\mathbb{N}}\right\}$$

is a 0nbhd-basis of the *lc* inductive limit consisting of absolutely $m$-convex sets. □

For *lc* algebras satisfying the countable neighbourhood condition, we can prove a more general result, namely:

**Proposition 3.2** *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of lc algebras satisfying (cnc) such that each $A_n$ is continuously included in $A_{n+1}$. Then $A := ind_{n\in\mathbb{N}} A_n$ is an lc algebra.*

**Proof:** Let an absolutely convex 0nbhd $U \in \mathcal{U}_0(A)$ be given. Now, the assertion will be proved within two steps:

1st step. To begin with, we claim the existence of a sequence $(V_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathcal{U}_0(A_n)$ and of sequences $(W_k^{(n)})_{k>n} \in \prod_{k>n} \mathcal{U}_0(A_k)$ such that for all $n \in \mathbb{N}$ and for all $k > n$

$$V_n \cup (V_n W_k^{(n)}) \cup (W_k^{(n)} V_n) \cup W_k^{(n)} \subset U.$$

Indeed, let $n \in \mathbb{N}$ be given. Since the $lc$ topology on $A_n$ is finer than then the relative $lc$ topology induced by $A_k$ for all $k > n$, the restrictions of multiplication on $A_k$ to $A_n \times A_k$ and to $A_k \times A_n$, respectively, are continuous. Finally, $A_k$ is continuously included in the $lc$ inductive limit $A$. Thus, we have

$$\cdot : A_n \times A_k \overset{\text{cont.}}{\to} A_k \overset{\text{cont.}}{\hookrightarrow} A \quad \text{and} \quad \cdot : A_k \times A_n \overset{\text{cont.}}{\to} A_k \overset{\text{cont.}}{\hookrightarrow} A,$$

hence one can find $V^{(k)} \in \mathcal{U}_0(A_n)$ and $W^{(k)} \in \mathcal{U}_0(A_k)$ satisfying

$$(V^{(k)}W^{(k)}) \cup (W^{(k)}V^{(k)}) \subset U.$$

Now, by the countable neighbourhood condition, one can find $(\rho_k)_{k\in\mathbb{N}} \in \prod_{k>n} (0,\infty)$ such that $V_n := \bigcap_{k>n} \rho_k V^{(k)} \cap U \in \mathcal{U}_0(A_n)$. Thus, we obtain:

$$V_n\left(\left(\frac{1}{\rho_k}W^{(k)}\right)\cap U\right) \cup \left(\left(\frac{1}{\rho_k}W^{(k)}\right)\cap U\right)V_n \subset U$$

for all $k > n$.

2nd step. Now we claim the existence of a sequence $(U_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathcal{U}_0(A_n)$ of absolutely convex sets such that:

*i)* $\forall n \in \mathbb{N} : U_n \subset U_{n+1} \subset U$.

*ii)* $\forall m,n \in \mathbb{N} : U_m U_n \subset U$.

*iii)* $\forall n \in \mathbb{N}\, \forall k > n\, \exists W \in \mathcal{U}_0(A_k) : W \cup (U_n W) \cup (W U_n) \subset U$.

We will prove the second assertion inductively: Since $A_1$ is an $lc$ algebra, one can find $U_1 = \Gamma U_1 \in \mathcal{U}_0(A_1)$ such that $U_1 \cup U_1^2 \subset V_1$. Condition *iii)* is guaranteed by the 1st step.
Let us now assume that we have already found $(U_1, ..., U_n) \in \prod_{k=1}^{n} \mathcal{U}_0(A_k)$ satisfying the conditions *i)* – *iii)*. Because of *iii)* we can find $V \in \mathcal{U}_0(A_{n+1})$ such that $V \cup (U_n V) \cup (V U_n) \subset U$. Furthermore, there is $\tilde{V} \in \mathcal{U}_0(A_{n+1})$, $\tilde{V} \subset V_{n+1} \cap V$ such that $\tilde{V}^2 \subset V$. Now we define

$$U_{n+1} := \Gamma(U_n \cup \tilde{V}) \in \mathcal{U}_0(A_{n+1})$$

and we obtain:

*i)* $U_n \subset U_{n+1} \subset U$.

*ii)* $U_{n+1}U_k \cup U_k U_{n+1} \subset U_{n+1}^2 \subset \Gamma(U_n^2 \cup \tilde{V}^2 \cup U_n\tilde{V} \cup \tilde{V}U_n) \subset \Gamma(U \cup V \cup U_n V \cup V U_n) \subset U$ for all $k \le n+1$.

It only remains to prove that condition *iii*) is satisfied. Let therefore $k > n+1$. By the choice of $U_1, ..., U_n$ we can find $W = \Gamma W \in \mathcal{U}_0(A_k)$ such that $(U_n W) \cup (W U_n) \cup W \subset U$ and $(V_{n+1} W) \cup (W V_{n+1}) \subset U$. Now we obtain $U_{n+1} W \subset \Gamma(U_n W \cup \tilde{V} W) \subset \Gamma(U \cup V_{n+1} W) \subset U$ and, similarly, $W U_{n+1} \subset U$. This yields *iii*).

Now $V := \bigcup_{n \in \mathbb{N}} U_n \in \mathcal{U}_0(A)$ is the desired 0nbhd, because

$$VV \subset \Gamma\left(\bigcup_{m,n \in \mathbb{N}} U_m U_n\right) \subset U.$$

□

Note that in the 2nd step of the above proof condition *i*) and *iii*) are only needed for technical reasons.
Seeking to relax the requirement in proposition 3.1 each $A_n$ being a seminormed algebra one must assume, that all $A_n$ are *commutative lmc* algebras with (*cnc*). This result is due to S. Dierolf and Wengenroth (see [12]).

**Proposition 3.3** *Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of commutative, lmc algebras satisfying (cnc) such that for all $n \in \mathbb{N}$ $A_n$ is continuously enbedded in $A_{n+1}$. Then $A := ind_{n \in \mathbb{N}} A_n$ is an lmc algebra.*

**Proof:** Let $U = \Gamma U \in \mathcal{U}_0(A)$. Inductively, we are going to construct an increasing sequence $(U_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \mathcal{U}_0(A_n)$ of absolutely $m$-convex sets each contained in $U$ such that for all $n \in \mathbb{N}$ and for all $k \geq n$ there is an absolutely $m$-convex 0nbhd $V_k^{(n)} \in \mathcal{U}_0(A_k)$ satisfying

$$U_n V_k^{(n)} \subset U \quad \text{and} \quad V_k^{(n)} \subset U.$$

<u>$n = 1$:</u> For all $k \in \mathbb{N}$ we can find an absolutely $m$-convex set $W_k \in \mathcal{U}_0(A_k)$ which is contained in $U$. Choose $(\rho_k)_{k \in \mathbb{N}} \in (1, \infty)^{\mathbb{N}}$ and an absolutely $m$-convex set $U_1 \in \mathcal{U}_0(A_1)$ such that

$$U_1 \subset \left(\bigcap_{k \in \mathbb{N}} \rho_k W_k\right) \cap U.$$

For all $k \in \mathbb{N}$ we set $V_k^{(1)} := \frac{1}{\rho_k} W_k$ and we obtain

$$U_1 V_k^{(1)} \subset W_k W_k \subset W_k \subset U.$$

<u>$n \mapsto n+1$:</u> Let us assume that we have already found $U_1 \subset ... \subset U_n$ and $\left(\left(V_k^{(j)}\right)_{k \geq j}\right)_{j=1}^{n}$ as required above. Choose $(\rho_k)_{k \geq n+1} \in (1, \infty)^{\mathbb{N}}$ and an absolutely $m$-convex set $W \in \mathcal{U}_0(A_{n+1})$ such that $W \subset \left(\bigcap_{k \geq n+1} \rho_k V_k^{(n)}\right) \cap U \cap V_{n+1}^{(n)}$. We define

$$U_{n+1} := \Gamma(U_n \cup W \cup W U_n) \in \mathcal{U}_0(A_{n+1}).$$

Since $A_{n+1}$ is commutative, $U_{n+1}$ is absolutely $m$-convex. Moreover, $WU_n \subset V_{n+1}^{(n)} U_n \subset U$ implies $U_n \subset U_{n+1} \subset U$. Further on, for all $k \geq n+1$ we define

$$V_k^{(n+1)} := \left(\frac{1}{\rho_k} V_k^{(n)}\right) \cap U \ \ (\subset V_k^{(n)})$$

and we obtain:

$$U_{n+1} V_k^{(n+1)} \subset \Gamma\left(U_n V_k^{(n)} \cup V_k^{(n)} V_k^{(n)} \cup U_n V_k^{(n)} V_k^{(n)}\right) \subset U.$$

Now $V := \bigcup_{n \in \mathbb{N}} U_n \in \mathcal{U}_0(A)$ is an absolutely $m$-convex 0nbhd which is contained in $U$. □

Observe that in the above proof the sets $V_k^{(n)}$ are only needed for technical reasons.

In the sequel we are going to investigate inductive algebras of Moscatelli type. Inductive limits of Moscatelli type have their origin in the theory of $lc$ spaces. But the concept is also adaptable (algebraically) to algebras. We are now going to study conditions which yield that the inductive limit of Moscatelli type of $l(m)c$ algebras is again an $l(m)c$ algebra. For this purpose we need some preparation.

**Definition-Remark 3.1.** *Let $l(m)c$ algebras $B$ and $C$ be given such that $C$ is continuously included in $B$. Let, furthermore, $\lambda$ be a normal Banach sequence space. By corollary 1.6 $A_n := \prod_{1 \leq k < n} B \times \lambda((C)_{k \geq n})$ is an $l(m)c$ algebra with respect to the multiplication*

$$\cdot : A_n \times A_n \longrightarrow A_n, ((x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}}) \longmapsto (\rho_k x_k y_k)_{k \in \mathbb{N}}$$

*for all $n \in \mathbb{N}$. $(A_n)_{n \in \mathbb{N}}$ is an inductive system directed by inclusion and $A_n \overset{cont.}{\hookrightarrow} A_{n+1}$ for all $n \in \mathbb{N}$. It is easy to see that*

$$A := ind(C \hookrightarrow B, \lambda) := ind_{n \in \mathbb{N}} A_n \ = \ B^{(\mathbb{N})} + \lambda(C).$$

*Moreover, a 0nbhd-basis of the lc inductive limit is given by*

$$\left\{ \bigoplus_{n \in \mathbb{N}} U_n + \lambda(V) : (U_n)_{n \in \mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}, V = \overline{\Gamma V} \in \mathcal{U}_0(C) \right\}.$$

*$A$ will be called the Moscatelli algebra (with respect to $B$, $C$, and $\lambda$). We are aiming at conditions such that $A$ is an $l(m)c$ algebra.*

**Observation:** Since $\bigoplus_{n \in \mathbb{N}} U_n \cup \lambda(V) \subset \bigoplus_{n \in \mathbb{N}} U_n + \lambda(V) \subset 2\Gamma(\bigoplus_{n \in \mathbb{N}} U_n \cup \lambda(V))$, a 0nbhd-basis of $A$ is also given by

$$\mathcal{V} := \left\{ \Gamma\left( \bigoplus_{n \in \mathbb{N}} U_n \cup \lambda(V) \right) : (U_n)_{n \in \mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}, V = \overline{\Gamma V} \in \mathcal{U}_0(C) \right\}.$$

Thus it follows:

$$A \text{ is an } lc \text{ algebra} \iff$$

$$\forall (U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}, V=\overline{\Gamma V}\in \mathcal{U}_0(C)\ \exists (\tilde U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}, \tilde V=\overline{\Gamma \tilde V}\in \mathcal{U}_0(C):$$

$$\left(\bigoplus_{n\in\mathbb{N}} \tilde U_n \cup \lambda(\tilde V)\right)^2 \subset \bigoplus_{n\in\mathbb{N}} U_n + \lambda(V) \iff$$

$$\forall (U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}, V=\overline{\Gamma V}\in \mathcal{U}_0(C)\ \exists (\tilde U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}, \tilde V=\overline{\Gamma \tilde V}\in \mathcal{U}_0(C):$$

$$\left(\bigoplus_{n\in\mathbb{N}} \tilde U_n\right)\lambda(\tilde V)\ \cup\ \lambda(\tilde V)\left(\bigoplus_{n\in\mathbb{N}} \tilde U_n\right)\ \subset\ \bigoplus_{n\in\mathbb{N}} U_n + \lambda(V),$$

because each $\tilde U_n$ and $\tilde V$ can be assumed to satisfy $\tilde U_n^2 \subset U_n$, and $\tilde V^2 \subset V$.
We are now giving a componentwise characterization such that the inductive algebra of Moscatelli type of $lc$ algebras $B$ and $C$ as it is introduced above is an $lc$ algebra (this generalizes [12, proposition 1]).

**Proposition 3.4** *Let $B$ and $C$ be lc algebras such that $C$ is continuously included in $B$. Let $\lambda$ be a normal Banach sequence space and let $A = ind(C \hookrightarrow B, \lambda)$ denote the corresponding inductive limit of Moscatelli type. $A$ is an lc algebra, iff for all $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ and $V \in \mathcal{U}_0(C)$ there are $(\tilde U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ and $\tilde V \in \mathcal{U}_0(C)$ such that for all $n \in \mathbb{N}$*

$$\tilde U_n \tilde V \cup \tilde V \tilde U_n \subset U_n + \rho_n^{-1} V$$

*(where multiplication is the ordinary multiplication on $B$).*

**Proof:** To see that the above condition is necessary, let $n \in \mathbb{N}$, $x_n \in \tilde U_n$, and $y \in \tilde V$. Since $B_{l^1} \subset B_{\rho\lambda}$, it follows that $\rho_n^{-1} y e_n \in \lambda(\tilde V)$. Hence one can find $u_n \in U_n$ and $z = (z_k)_{k\in\mathbb{N}} \in \lambda(V)$ such that $x_n y = \rho_n x_n \rho_n^{-1} y_n = u_n + z_n$. It remains to prove $z_n \in \rho_n^{-1} V$. But this follows from

$$p_V(\rho_n z_n) = \|p_V(z_n) e_n\|_\lambda \le \|p_V((z_k)_{k\in\mathbb{N}})\|_\lambda \le 1.$$

Accordingly, one proves $\tilde V \tilde U_n \subset U_n + \rho_n^{-1} V$.

To see that the condition is sufficient let $(x_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} \tilde U_n$, $(y_n)_{n\in\mathbb{N}} \in \lambda(\tilde V)$. If $p_{\tilde V}(y_n) \ne 0$ for some $n \in \mathbb{N}$ we obtain:

$$\frac{1}{p_{\tilde V}(y_n)} y_n \in \tilde V.$$

Thus, there are $u_n \in U_n$ and $z_n \in V$ such that $x_n \frac{1}{p_{\tilde V}(y_n)} y_n = u_n + \rho_n^{-1} z_n$. This implies

$$\rho_n x_n y_n = p_{\tilde V}(\rho_n y_n) u_n + p_{\tilde V}(y_n) z_n.$$

Since $p_{\tilde V}(\rho_n y_n) u_n \in U_n$, it remains to prove that $(p_{\tilde V}(y_n) z_n)_{n\in\mathbb{N}} \in \lambda(V)$. But since $p_V(z_n) \le 1$, we obtain:

$$\|(p_V(p_{\tilde V}(y_n) z_n))_{p_{\tilde V}(y_n)\ne 0}, (0)_{p_{\tilde V}(y_n)=0}\|_\lambda \le \|(p_{\tilde V}(y_n))_{n\in\mathbb{N}}\|_\lambda.$$

Let now $n \in \mathbb{N}$ such that $p_{\tilde{V}}(y_n) = 0$. We choose any sequence $(\mu_k)_{k\in\mathbb{N}} \in B_\lambda \cap \prod_{k\in\mathbb{N}} (0, \rho_k^{-1})$ and conclude:

$$\mu_n^{-1} y_n \in \tilde{V} \Longrightarrow \exists u_n \in U_n, z_n \in V : \ x_n \mu_n^{-1} y_n = u_n + \rho_n^{-1} z_n \Longrightarrow$$

$$\rho_n x_n y_n = \rho_n \mu_n u_n + \mu_n z_n,$$

where $U_n = \Gamma U_n$ implies $\rho_n \mu_n u_n \in U_n$, because $\rho_n \mu_n \leq 1$. Finally

$$\|(p_V(\mu_n z_n))_{p_{\tilde{V}}(y_n)=0}, (0)_{p_{\tilde{V}}(y_n)\neq 0}\|_\lambda \leq \|\mu\|_\lambda \leq 1.$$

Altogether, we get

$$(x_n)_{n\in\mathbb{N}} (y_n)_{n\in\mathbb{N}} \in 2\left(\bigoplus_{n\in\mathbb{N}} U_n + \lambda(V)\right).$$

□

As a first application of the above proposition we obtain:

**Proposition 3.5** *Let B and C be lc algebras such that C is continuously included in B and let $\lambda$ be a normal Banach sequence space. If for all $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ there is a sequence $(\mu_n)_{n\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ such that*

$$\bigcap_{n\in\mathbb{N}} \mu_n (U_n \cap C) \in \mathcal{U}_0(C),$$

*then $A = ind(C \hookrightarrow B, \lambda)$ is an lc algebra.*

**Proof:** Let $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$, and $V \in \mathcal{U}_0(C)$. Since $B$ is an *lc* algebra, we can find an absolutely convex set $W_n \in \mathcal{U}_0(B)$ such that $W_n^2 \subset U_n$ for all $n \in \mathbb{N}$. Choose $(\mu_n)_{n\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ such that $\tilde{V} := \bigcap_{n\in\mathbb{N}} \mu_n (W_n \cap C) \in \mathcal{U}_0(C)$. $\tilde{U}_n := \frac{1}{\mu_n} W_n$ yields

$$\tilde{U}_n \tilde{V} \cup \tilde{V} \tilde{U}_n \subset W_n^2 \subset U_n \subset U_n + \frac{1}{\rho_n} V.$$

□

**Corollary 3.1** *Let B, C, $\lambda$, A be given as above such that (cnc) holds either for B or for C. Then A is an lc algebra.*

The proof is immediate.

**Proposition 3.6** *Let an lc algebra B and a subalgebra $C \subset B$ endowed with the relative topology induced by B be given. Moreover, let $\lambda$ be a normal Banach sequence space. If $\overline{C}$ is an ideal in B then $A = ind(C \hookrightarrow B, \lambda)$ is again an lc algebra.*

**Proof:** Let $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$, and $V \in \mathcal{U}_0(C)$ be given. Both

$$\cdot : B \times \overline{C} \to \overline{C}, (x,y) \mapsto xy \quad \text{and} \quad \cdot : \overline{C} \times B \to \overline{C}, (y,x) \mapsto yx$$

are well-defined and continuous, thus one can find $U \in \mathcal{U}_0(B)$ and $\tilde{V} \in \mathcal{U}_0(C)$ such that $U\tilde{V} \cup \tilde{V}U \subset \overline{V}$. $\tilde{U}_n := \rho_n^{-1}U$ satisfies $\tilde{U}_n\tilde{V} \cup \tilde{V}\tilde{U}_n \subset \rho_n^{-1}\overline{V} \subset U_n + \rho_n^{-1}V$. □

**Corollary 3.2** *Let B, C, λ, and A as in the above proposition, C provided with the relative topology induced by B such that either C is dense in B or C is an ideal in B, then the lc inductive limit A is again an lc algebra.*

**Remark:** Let now, in addition $B$ and $C$ be *lmc* algebras. Then $A$ is an *lmc* algebra, iff for all $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ and for all $V = \overline{\Gamma V} \in \mathcal{U}_0(C)$ there are $(\tilde{U}_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ and $\tilde{V} = \overline{\Gamma\tilde{V}} \in \mathcal{U}_0(C)$ such that

$$\bigcup_{k\in\mathbb{N}} \left( \bigoplus_{n\in\mathbb{N}} \tilde{U}_n \cup \lambda(\tilde{V}) \right)^k \subset \bigoplus_{n\in\mathbb{N}} U_n + \lambda(V).$$

Since

$$\bigcup_{k\in\mathbb{N}} \left( \bigoplus_{n\in\mathbb{N}} \tilde{U}_n \cup \lambda(\tilde{V}) \right)^k =$$

$$\bigcup_{k\in\mathbb{N}} \bigoplus_{n\in\mathbb{N}} \tilde{U}_n^k \cup \bigcup_{k\in\mathbb{N}} \lambda(\tilde{V})^k \cup \bigcup_{k\in\mathbb{N}} \left( \left( \bigoplus_{n\in\mathbb{N}} \tilde{U}_n \right) \lambda(\tilde{V}) \right)^k \cup$$

$$\cup \bigcup_{k\in\mathbb{N}} \left( \lambda(\tilde{V}) \left( \bigoplus_{n\in\mathbb{N}} \tilde{U}_n \right) \right)^k \cup \bigcup_{k\in\mathbb{N}} \left( \lambda(\tilde{V}) \left( \bigoplus_{n\in\mathbb{N}} \tilde{U}_n \right) \lambda(\tilde{V}) \right)^k \cup$$

$$\bigcup_{k\in\mathbb{N}} \left( \left( \bigoplus_{n\in\mathbb{N}} \tilde{U}_n \right) \lambda(\tilde{V}) \left( \bigoplus_{n\in\mathbb{N}} \tilde{U}_n \right) \right)^k,$$

and every $\tilde{U}_n$ and $\tilde{V}$ can actually be assumed as absolutely $m$-convex, we get that $A$ is an *lmc* algebra, iff for all $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ and for all $V = \overline{\Gamma V} \in \mathcal{U}_0(C)$ there are $(\tilde{U}_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ and $\tilde{V} = \overline{\Gamma\tilde{V}} \in \mathcal{U}_0(C)$ such that

$$\bigcup_{k\in\mathbb{N}} \left( \left( \bigoplus_{n\in\mathbb{N}} \tilde{U}_n \right) \lambda(\tilde{V}) \right)^k \cup \bigcup_{k\in\mathbb{N}} \left( \lambda(\tilde{V}) \left( \bigoplus_{n\in\mathbb{N}} \tilde{U}_n \right) \right)^k \cup$$

$$\bigcup_{k\in\mathbb{N}} \left( \lambda(\tilde{V}) \left( \bigoplus_{n\in\mathbb{N}} \tilde{U}_n \right) \lambda(\tilde{V}) \right)^k \cup \bigcup_{k\in\mathbb{N}} \left( \left( \bigoplus_{n\in\mathbb{N}} \tilde{U}_n \right) \lambda(\tilde{V}) \left( \bigoplus_{n\in\mathbb{N}} \tilde{U}_n \right) \right)^k \subset$$

$$\subset \bigoplus_{n\in\mathbb{N}} U_n + \lambda(V),$$

However, the search for a 'componentwise' characterization for the *lmc* case as it is given for the *lc* case in proposition 3.4 was unsuccessful. Anyhow, we have:

**Proposition 3.7** *Let B and C be lmc algebras and $\lambda$ a normal Banach sequence space such that C is continuously included in B. Let us denote by A the corresponding inductive limit of Moscatelli type, $ind(C \hookrightarrow B, \lambda)$. If for all $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ one can find $(\tilde{U}_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ and $\tilde{V} = \overline{\Gamma\tilde{V}} \in \mathcal{U}_0(C)$ such that*

$$\tilde{U}_n\tilde{V} \cup \tilde{V}\tilde{U}_n \cup \tilde{V}\tilde{U}_n\tilde{V} \subset U_n,$$

*for all $n \in \mathbb{N}$, then A is an lmc algebra.*

**Proof:** By the above considerations, it suffices to prove

$$\left(\bigoplus_{n\in\mathbb{N}} \tilde{U}_n\right)\lambda(\tilde{V}) \bigcup \lambda(\tilde{V})\left(\bigoplus_{n\in\mathbb{N}} \tilde{U}_n\right) \bigcup \lambda(\tilde{V})\left(\bigoplus_{n\in\mathbb{N}} \tilde{U}_n\right)\lambda(\tilde{V})$$

$$\bigcup \left(\bigoplus_{n\in\mathbb{N}} \tilde{U}_n\right)\lambda(\tilde{V})\left(\bigoplus_{n\in\mathbb{N}} \tilde{U}_n\right) \subset \bigoplus_{n\in\mathbb{N}} U_n,$$

because every $U_n$ can be assumed as absolutely $m$-convex. Moreover, since every $\tilde{U}_n$ can be assumed to be contained in $U_n$, it finally suffices to prove:

$$\left(\bigoplus_{n\in\mathbb{N}} \tilde{U}_n\right)\lambda(\tilde{V}) \bigcup \lambda(\tilde{V})\left(\bigoplus_{n\in\mathbb{N}} \tilde{U}_n\right) \bigcup \lambda(\tilde{V})\left(\bigoplus_{n\in\mathbb{N}} \tilde{U}_n\right)\lambda(\tilde{V}) \subset \bigoplus_{n\in\mathbb{N}} U_n.$$

Let us first concentrate on the inclusion $\left(\bigoplus_{n\in\mathbb{N}} \tilde{U}_n\right)\lambda(\tilde{V}) \subset \bigoplus_{n\in\mathbb{N}} U_n$. Let, therefore, $(x_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} \tilde{U}_n$, $(y_n)_{n\in\mathbb{N}} \in \lambda(\tilde{V})$. Then $p_{\tilde{V}}(\rho_n y_n) \leq \|(p_{\tilde{V}}(y_n))_{n\in\mathbb{N}}\|_\lambda \leq 1$ implies $\rho_n y_n \in \tilde{V}$ $(n \in \mathbb{N})$, thus we obtain: $\rho_n x_n y_n \in U_n$.

The same considerations yield

$$\lambda(\tilde{V})\left(\bigoplus_{n\in\mathbb{N}} \tilde{U}_n\right) \subset \bigoplus_{n\in\mathbb{N}} U_n \quad \text{and} \quad \lambda(\tilde{V})\left(\bigoplus_{n\in\mathbb{N}} \tilde{U}_n\right)\lambda(\tilde{V}) \subset \bigoplus_{n\in\mathbb{N}} U_n.$$

□

**Proposition 3.8** *Let B and C be lmc algebras and $\lambda$ a normal Banach sequence space such that C is continuously included in B. Again we denote the inductive limit of Moscatelli type, $ind(C \hookrightarrow B, \lambda)$, by A. If for all $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ there is a sequence $(\mu_n)_{n\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ such that*

$$\bigcap_{n\in\mathbb{N}} \mu_n(U_n \cap C) \in \mathcal{U}_0(C),$$

*then A is again an lmc algebra.*

**Proof:** Let $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$. We may assume that each $U_n$ is absolutely $m$-convex. We can find a sequence $(\mu_n)_{n\in\mathbb{N}} \in (1,\infty)^{\mathbb{N}}$ and an absolutely $m$-convex set $\tilde{V} \in \mathcal{U}_0(C)$ such that $\tilde{V} \subset \bigcap_{n\in\mathbb{N}} \mu_n(U_n \cap C)$. Now, for every $n \in \mathbb{N}$ $\tilde{U}_n := \left(\frac{1}{\mu_n}\right)^2 U_n$ satisfies:

$$\tilde{U}_n\tilde{V} \cup \tilde{V}\tilde{U}_n \cup \tilde{V}\tilde{U}_n\tilde{V} \subset U_n^2 \cup U_n^2 \cup U_n^3 \subset U_n.$$

□

**Corollary 3.3** *Let B, C, λ, and A be given as above, such that either B or C satisfies (cnc). Then A is an lmc algebra.*

The proof is immediate.

Let again $B$ and $C$ be *lmc* algebras such that C is continuously included in B, and $\lambda$ a normal Banach sequence space, and $A$ denote the corresponding inductive limit of Moscatelli type, $\operatorname{ind}(C \hookrightarrow B,\lambda)$. It is clear that $A$ is an *lmc* algebra, if for all $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$, and $V = \overline{\Gamma V} \in \mathcal{U}_0(C)$ there are $(\tilde{U}_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ and $\tilde{V} = \overline{\Gamma\tilde{V}} \in \mathcal{U}_0(C)$ satisfying $\tilde{V} \subset V$ and $\tilde{U}_n \subset U_n$ for all $n \in \mathbb{N}$ and

$$\left(\bigoplus_{n\in\mathbb{N}} \tilde{U}_n + \lambda(\tilde{V})\right)^2 \subset \bigoplus_{n\in\mathbb{N}} \tilde{U}_n + \lambda(\tilde{V}).$$

Since

$$\left(\bigoplus_{n\in\mathbb{N}} \tilde{U}_n + \lambda(\tilde{V})\right)^2 = \bigoplus_{n\in\mathbb{N}} \tilde{U}_n^2 + \lambda(\tilde{V})^2 + \bigoplus_{n\in\mathbb{N}} \tilde{U}_n\lambda(\tilde{V}) + \lambda(\tilde{V}) \bigoplus_{n\in\mathbb{N}} \tilde{U}_n,$$

it follows that $A$ is an *lmc* algebra, if for all $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$, and $V = \overline{\Gamma V} \in \mathcal{U}_0(C)$ there are $(\tilde{U}_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ and $\tilde{V} = \overline{\Gamma\tilde{V}} \in \mathcal{U}_0(C)$ satisfying $\tilde{V} \subset V$ and $\tilde{U}_n \subset U_n$ for all $n \in \mathbb{N}$ and

$$\bigoplus_{n\in\mathbb{N}} \tilde{U}_n\lambda(\tilde{V}) + \lambda(\tilde{V}) \bigoplus_{n\in\mathbb{N}} \tilde{U}_n \subset \bigoplus_{n\in\mathbb{N}} \tilde{U}_n + \lambda(\tilde{V}).$$

Going through the second part of the proof of proposition 3.4 replacing $U_n$ by $\tilde{U}_n$ and $V$ by $\tilde{V}$ yields that $A$ is *lmc*, if for all $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$, and $V = \overline{\Gamma V} \in \mathcal{U}_0(C)$ there are $(\tilde{U}_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ and $\tilde{V} = \overline{\Gamma\tilde{V}} \in \mathcal{U}_0(C)$ satisfying $\tilde{V} \subset V$ and $\tilde{U}_n \subset U_n$ and

$$\tilde{U}_n\tilde{V} \cup \tilde{V}\tilde{U}_n \subset \tilde{U}_n + \rho_n^{-1}\tilde{V}.$$

for all $n \in \mathbb{N}$.

**Proposition 3.9** *Let an lmc algebra B and a subalgebra C ⊂ B be given such that C is endowed with the relative topology induced by B. Moreover, let λ be a normal Banach sequence space. If C is an ideal in B, then A = ind(C ↪ B, λ) is an lmc algebra.*

**Proof:** Let $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$, $V \in \mathcal{U}_0(C)$. We may assume that all these sets are absolutely $m$-convex. Continuity of $\cdot : B\times C \to C$ and $\cdot : C\times B \to C$ implies that there is an absolutely $m$-convex set $U \in \mathcal{U}_0(B)$ such that $U(U\cap C)\cup(U\cap C)U \subset V$. We define $\tilde{U}_n := \frac{1}{\rho_n}U\cap U_n$ and $\tilde{V} := (U\cap C)\cap V$ and obtain $\tilde{U}_n\tilde{V}\cup\tilde{V}\tilde{U}_n \subset \rho_n^{-1}\tilde{V} \subset \tilde{U}_n + \rho_n^{-1}\tilde{V}$. □

**Proposition 3.10** *Let $B$, $C$, $\lambda$, and $A$ be given as above, such that $C$ carries the relative topology induced by $B$ and such that $C$ is dense in $B$. Then $A$ is an lmc algebra.*

**Proof:** Let $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$, $V \in \mathcal{U}_0(C)$. We may assume that all these sets are absolutely $m$-convex. We define $\tilde{U}_n := \frac{1}{\rho_n}\overline{V}\cap U_n$ and $\tilde{V} := V$ and obtain $\tilde{U}_n\tilde{V}\cup\tilde{V}\tilde{U}_n \subset \rho_n^{-1}\overline{V} \subset \tilde{U}_n+\rho_n^{-1}\tilde{V}$.
□

The next result is a sharper version of proposition 3.9 and of proposition 3.10. It is also an application of propostion 1.9 to the theory of *lc* inductive limits in the category of *lmc* algebras:

**Proposition 3.11** *Let an lmc algebra $B$ and a subalgebra $C \subset B$ be given such that $C$ is endowed with the relative topology induced by $B$. Moreover, let $\lambda$ be a normal Banach sequence space and let us denote $ind(C \hookrightarrow B,\lambda)$ by $A$. If $\overline{C}$ is an ideal in $B$, then $A$ is an lmc algebra.*

**Proof:** By proposition 3.6, $A$ is an *lc* algebra. We define $D := \overline{C}$ and obtain that $D^{(\mathbb{N})}+\lambda(C)$ is an *lmc* algebra by proposition 3.10. Moreover $D^{(\mathbb{N})}+\lambda(C)$ is an ideal in $A$.
Indeed, let $(x_k)_{k\in\mathbb{N}} \in D^{(\mathbb{N})}$, $(v_k)_{k\in\mathbb{N}} \in B^{(\mathbb{N})}$, $(y_k)_{k\in\mathbb{N}}, (z_k)_{k\in\mathbb{N}} \in \lambda(C)$, then

$$((x_k)_{k\in\mathbb{N}} + (y_k)_{k\in\mathbb{N}})\cdot((v_k)_{k\in\mathbb{N}} + (z_k)_{k\in\mathbb{N}}) =$$

$$(\rho_k x_k v_k)_{k\in\mathbb{N}} + (\rho_k x_k z_k)_{k\in\mathbb{N}} + (\rho_k y_k v_k)_{k\in\mathbb{N}} + (\rho_k y_k z_k)_{k\in\mathbb{N}},$$

where $(\rho_k x_k v_k)_{k\in\mathbb{N}} + (\rho_k x_k z_k)_{k\in\mathbb{N}} + (\rho_k y_k v_k)_{k\in\mathbb{N}} \in D^{(\mathbb{N})}$ and $(\rho_k y_k z_k)_{k\in\mathbb{N}} \in \lambda(C)$, hence $\left(D^{(\mathbb{N})}+\lambda(C)\right)A \subset D^{(\mathbb{N})}+\lambda(C)$. Accordingly, we get $A\left(D^{(\mathbb{N})}+\lambda(C)\right) \subset D^{(\mathbb{N})}+\lambda(C)$.
We form $\left(D^{(\mathbb{N})}+\lambda(C)\right)\times_S B^{(\mathbb{N})}$ (according to example 1.5.3) which is an *lmc* algebra by proposition 1.4 and proposition 1.5. We claim that $A$ is a quotient algebra of $D^{(\mathbb{N})}+\lambda(C)\times_S B^{(\mathbb{N})}$. Therefore we must prove that

$$q : D^{(\mathbb{N})}+\lambda(C)\times_S B^{(\mathbb{N})} \longrightarrow A,$$

$$((x_k)_{k\in\mathbb{N}} + (y_k)_{k\in\mathbb{N}}, (v_k)_{k\in\mathbb{N}}) \longmapsto (x_k+v_k)_{k\in\mathbb{N}} + (y_k)_{k\in\mathbb{N}}$$

is linear, multiplicative, continuous and open.
Obviously, $q$ is linear and multiplicative. For continuity of $q$ it suffices to prove that $D^{(\mathbb{N})}+\lambda(C)$ is continuously included in $A$. Since for all $n \in \mathbb{N}$, $\prod_{k<n} B\times\lambda((C)_{k\geq n}) \overset{\text{cont.}}{\hookrightarrow} A$, also $\prod_{k<n} D\times\lambda((C)_{k\geq n}) \overset{\text{cont.}}{\hookrightarrow} A$ holds. Clearly, this yields $D^{(\mathbb{N})}+$

$\lambda(C) \overset{\text{cont.}}{\hookrightarrow} A$. It remains to prove that $q$ is open. Let therefore $U \in \mathcal{U}_0\left(D^{(\mathbb{N})} + \lambda(C)\right)$ and $V \in \mathcal{U}_0\left(B^{(\mathbb{N})}\right)$ be given. Then the following holds:

$$q(U \times V) = U + V \supset (U \cap \lambda(C)) + V \in \mathcal{U}_0(B^{(\mathbb{N})} + \lambda(C)),$$

thus $q$ is also open. □

In [37] Warner gives a rather complicated example of an inductive limit of metrizable *lmc* algebras which is not *lmc*. I do not know whether multiplication is continuous on the inductive limit.

In [12] the authors prove that in case $B = \mathcal{C}(\mathbb{C})$, $C = \mathcal{H}(\mathbb{C})$ (the algebra of entire functions), both $B$ and $C$ endowed with the compact open topology, and $\lambda = l^\infty$, on $A$, which is in fact the strict inductive limit of commutative, *lmc* Fréchet algebras, multiplication is not even continuous. Now we are going to introduce a whole class of such counterexamples.

**Definition-Remark 3.2.** *Let again $B$ and $C$ be lc algebras, such that $C$ is continuously included in $B$. For all $n \in \mathbb{N}$ we form*

$$A_n := B^{n-1} \times \prod_{k \geq n} C$$

*which is an lc algebra with respect to the product topology and componentwise multiplication. $A_n$ is lmc, iff both $B$ and $C$ are lmc.*
*For all $n \in \mathbb{N}$ we have $A_n \overset{\text{cont.}}{\hookrightarrow} A_{n+1}$. We set*

$$A := ind(C \hookrightarrow B) := ind((A_n)_{n \in \mathbb{N}}) = B^{(\mathbb{N})} + C^{\mathbb{N}},$$

*which is a subalgebra of $B^{\mathbb{N}}$, and endow $A$ with the lc inductive limit topology. Then a 0nbhd-basis on $A$ is given by*

$$\left\{ \bigoplus_{n \in \mathbb{N}} U_n + \prod_{k \geq m} C : (U_n)_{n \in \mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}, m \in \mathbb{N} \right\}$$

*or by*

$$\left\{ \Gamma\left( \bigoplus_{n \in \mathbb{N}} U_n \cup \prod_{k \geq m} C \right) : (U_n)_{n \in \mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}, m \in \mathbb{N} \right\},$$

*respectively, where $\prod_{k \geq m} C$ stands in abbreviation for $\{0\}^{m-1} \times \prod_{k \geq m} C$.*

The next result characterizes when $A$ is an $l(m)c$ algebra.

**Proposition 3.12** *Let lc algebras $B$ and $C$ be given, such that $C$ is continuously included in $B$. Let us denote the lc inductive limit, $ind(C \hookrightarrow B)$, by $A$. Then $A$ is an lc algebra, iff $\overline{C}^B$ is an ideal in $B$.*
*Moreover, if $B$ is in fact an lmc algebra, then $A$ is lmc, iff $\overline{C}^B$ is an ideal in $B$, as well.*

**Proof:** To see that the above condition is necessary, we first prove that for all $U \in \mathcal{U}_0(B)$ there is $\tilde{U} \in \mathcal{U}_0(B)$ such that

$$\tilde{U}C \cup C\tilde{U} \subset U + C,$$

if $A$ is an $lc$ algebra. Indeed, one can find $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ and $m \in \mathbb{N}$ such that $\Gamma\left(\bigoplus_{n\in\mathbb{N}} U_n \cup \prod_{k\geq m} C\right)^2 \subset U^{(\mathbb{N})} + C^{\mathbb{N}}$. This clearly implies $U_m C \cup C U_m \subset U + C$.
Now we claim

$$BC \cup CB \subset U + C$$

for all $U \in \mathcal{U}_0(B)$; thus, $BC \cup CB \subset \bigcap_{U\in\mathcal{U}_0(B)} (U + C) = \overline{C}^B$. But this implies

$$B\overline{C}^B \cup \overline{C}^B B \subset \overline{C}^B,$$

since multiplication is continuous on $B$. Let $x \in B$, $y \in C$, and $U \in \mathcal{U}_0(B)$. We can find $\tilde{U} \in \mathcal{U}_0(B)$ such that $\tilde{U}C \cup C\tilde{U} \subset U + C$. Since $\tilde{U}$ is absorbant, there is $\rho > 0$ such that $\rho x \in \tilde{U}$. Now we get: $xy = \rho x \frac{1}{\rho} y \in U + C$ and $yx \in U + C$, respectively.
For the converse let $(U_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ and $m \in \mathbb{N}$. We can find $(\tilde{U}_n)_{n\in\mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ satisfying $\tilde{U}_n^2 \subset U_n$ For all $n \in \mathbb{N}$. It suffices to prove

$$\left(\bigoplus_{n\in\mathbb{N}} \tilde{U}_n \cup \prod_{k\geq m} C\right)^2 \subset \bigoplus_{n\in\mathbb{N}} U_n + \prod_{k\geq m} C;$$

but this is clear, because $\tilde{U}_n C \cup C\tilde{U}_n \subset \overline{C}^B \subset C + U_n$.
In case $B$ is $lmc$, we may assume $\tilde{U}_n = U_n$ and we obtain

$$\frac{1}{2}\left(\bigoplus_{n\in\mathbb{N}} U_n + \prod_{k\geq m} C\right)^2 \subset \bigoplus_{n\in\mathbb{N}} U_n + \prod_{k\geq m} C.$$

□

So the $lc$ inductive limit $A$ is, in general, not an $lc$ algebra. In case $B$ is $lmc$, $A$ is $lmc$, iff multiplication is continuous on $A$.

**Lemma 3.2** *Let an algebra $A$ and a subalgebra $B \subset A$ be given. Then the smallest ideal in $A$ containing $B$, $>B<$, is of the form*

$$\Gamma(AB \cup BA \cup ABA \cup BAB \cup B).$$

*(Hence, in case $A$ is a top algebra, $\overline{>B<}$ is the smallest closed ideal in $A$ containing $B$.)*

**Proof:** We must only prove that $I := \Gamma(AB \cup BA \cup ABA \cup BAB \cup B)$ is an ideal. Since $I$ is absolutely convex, it suffices to verify $\mathbb{K} \cdot I \subset I$ and $AI \cup IA \subset I$. For the former only note that $\mathbb{K}B \subset B$ and $\mathbb{K}A \subset A$ hold. For the latter it suffices to show

$$A(AB \cup BA \cup ABA \cup BAB \cup B) \cup (AB \cup BA \cup ABA \cup BAB \cup B)A$$

$$\subset (AB \cup BA \cup ABA \cup BAB \cup B)\,.$$

But this is obvious. □

For the inductive limit topology in the category of *lc* algebras we obtain the following characterization:

**Proposition 3.13** *Let B and C be lc algebras such that C is continuously included in B. Let us denote by D the closed ideal generated by C in B. Now we claim that the strongest lc algebra topology $\mathcal{T}$ on $A = ind(C \hookrightarrow B)$ such that all the inclusions $\prod_{k=1}^{n-1} B \times \prod_{k \geq n} C \hookrightarrow (A, \mathcal{T})$ are continuous is equal to the relative topology $\mathcal{R}$ induced by the lc inductive limit $B^{(\mathbb{N})} + D^{\mathbb{N}} = ind(D \hookrightarrow B)$.*

**Proof:** As $D$ is an ideal in $B$, the *lc* inductive limit $B^{(\mathbb{N})} + D^{\mathbb{N}}$ is an *lc* algebra, hence $\mathcal{R} \subset \mathcal{T}$. Let now $U = \overline{\Gamma U} \in \mathcal{U}_0(A, \mathcal{T})$ be given. Choose $V \in \mathcal{U}_0(A, \mathcal{T})$ such that

$$V \cup V^2 \cup V^3 \subset U.$$

Since $V$ is in particular a 0nbhd in the *lc* inductive limit topology, we can find a sequence $(V_n)_{n \in \mathbb{N}} \in \mathcal{U}_0(B)^{\mathbb{N}}$ and $m \in \mathbb{N}$ such that $\bigoplus_{n \in \mathbb{N}} V_n + \prod_{k \geq m} C \subset V$. Each $V_n$ is absorbing in $B$, thus,

$$\bigoplus_{k \geq m} (BC \cup CB \cup BCB \cup CBC \cup C) \subset U.$$

Since $U$ is absolutely convex and, in particular, closed with respect to the *lc* inductive limit topology on $B^{(\mathbb{N})} + C^{\mathbb{N}}$, we obtain that

$$U \supset \overline{\bigoplus_{k \geq m} \overline{\Gamma(BC \cup CB \cup BCB \cup CBC \cup C)}^{B}} \supset \prod_{k \geq m} D,$$

which proves that $U$ is a 0nbhd with respect to the relative topology induced by the *lc* inductive limit $B^{(\mathbb{N})} + D^{\mathbb{N}}$. □

**Proposition 3.14** *Let B, C, D, and A be given as above such that, in addition, B is lmc. Then the lmc inductive topology $\mathcal{T}$ on A is equal to the relative lc inductive topopolgy $\mathcal{R}$ induced by $B^{(\mathbb{N})} + D^{\mathbb{N}}$ on $B^{(\mathbb{N})} + D^{\mathbb{N}}$.*

**Proof:** The proof of the inclusion '$\mathcal{R} \subset \mathcal{T}$' is in complete analogy to the proof of the above proposition.
For the inclusion '$\mathcal{R} \supset \mathcal{T}$' let $U = \overline{\Gamma U}^{\mathcal{T}} \in \mathcal{U}_0(B^{(\mathbb{N})} + C^{\mathbb{N}}, \mathcal{T})$ be absolutely $m$-convex and go on as in the above proof. □

**Corollary 3.4** *Let B, C, and A be given as in proposition 3.13 or in proposition 3.14, respectively, such that, in addition, C is dense in B. Then the l(m)c inductive algebra topology on A is equal to the relative product topology induced on A by* $B^{\mathbf{N}}$.

**Proof:** The proof is an immediate consequence of proposition 3.13, 3.14, respectively. □

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