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COMBINATORIAL PROPERTIES OF CONVEX CONES IN $\mathbf{R}^n$

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Abstract. *In this paper we obtain sets of conditions under which the convex hull of a family of convex cones is an acute cone. Some intersectional results for families of convex sets there are also given. Finally, using two combinatorial results concerning families of convex cones, a lower bound for the Ramsey numbers $R_{2n}(m,3n)$ is established.*

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1 Introduction

The convex cones play an important role in optimization and in theory of inequalities . At the same time to each convex set one can associate some convex cones such as the generated convex cone, the polar cone, the dual cone, the recession cone (see [1], [9]) which make possible to deduce many results about convex sets.

All the cones considered in our paper will have 0 (the origin of the space $\mathbf{R}^n$) as vertex. The origin itself may or may not belong to the cone.

A convex cone $C \subset \mathbf{R}^n$ will be said to be:

- *pointed* if $0 \in C$,
- *acute* if $C \cap (-C) = \{0\}$.

In this work we present some results with combinatorial character for families of convex cones in $\mathbf{R}^n$. Thus in the next section we obtain sufficient conditions in order that the convex hull of a family of convex cones should be an acute convex cone. In Section 3 we give some intersection properties for families of convex cones. In the last section, using two results concerning convex cones, we establish a lower bound for the Ramsey numbers $R_{2n}(m,3n)$.

2 Acute convex cones

It is well known that a pointed convex cone C in $\mathbf{R}^n$ (more generally in a vector space) induces a preordering relation $\leq_C$ on $\mathbf{R}^n$ as follows: for $x,y \in \mathbf{R}^n$ we denote $x \leq_C y$ whenever $y-x \in C$. This relation becomes an ordering if and only if the cone C is acute. This being so, the following problem arises naturally: establish certain condition under which the convex hull of a family of convex cone is an acute cone. In this sense we give a first result:

Theorem 1 *Let C be a family of convex cones in $\mathbf{R}^n$ such that for any $n+1$ membered subfamily $\mathcal{B}$ of C,* conv $(\cup \mathcal{B})$ *is an acute convex cone. Then* conv $(\cup C)$ *is an acute convex cone.*

Proof. To prove that the convex cone conv($\cup\, \mathcal{C}$) is acute it is sufficient to show that the origin is an extremal point of this set, that is $0 \notin$conv($\cup\, \mathcal{C} \setminus \{0\}$). Supposing the contrary, by Carathéodory's theorem there exists

$$x_i \in \cup\, \mathcal{C} \setminus \{0\},\ \alpha_i \geq 0\ (1 \leq i \leq n+1),\ \text{with} \sum_{i=1}^{n+1} \alpha_i = 1$$

such that $0 = \alpha_1 x_1 + \ldots + \alpha_{n+1} x_{n+1}$.
Moreover, the points x_i may be supposed from distinct cones C_i. Indeed if two of the points with positive coefficients, for instance x_1 and x_2, belong to the same cone C, then the sum $\alpha_1 x_1 + \alpha_2 x_2$ can be replaced by αx, where $\alpha = \alpha_1 + \alpha_2$ and $x = \frac{\alpha_1}{\alpha} x_1 + \frac{\alpha_2}{\alpha} x_2 \in C$. If $x_i \in C_i \setminus \{0\}\ (1 \leq i \leq n+1)$, then $0 \in$conv $\left(\bigcup_{i=1}^{n+1} C_i \setminus \{0\}\right)$, hence conv $\left(\bigcup_{i=1}^{n+1} C_i\right)$ is not an acute cone. This contradicts the hypothesis and completes the proof. □

Carathéodory's theorem is the best possible, in the sense that if the number $n+1$ is reduced the theorem is no longer true for all subsets of $\mathbf{R}^n$. However, the theorem can be sharpened when the attention is restricted to special classes of subsets of $\mathbf{R}^n$.

Lemma 2 *Suppose that A is a nonempty set in $\mathbf{R}^n$ which satisfies one of the following conditions:*
(i) A has at most n connected components.
(ii) A is a union of a family of convex sets, all meeting a fixed hyperplane.
Then each point of conv A *is a convex combination of n or fewer points of A.*

The variant (i) of the previous lemma was obtained by Fenchel [5] for compact sets and by Bunt [3] without the compactness condition, while the variant (ii) is due to Tverberg [11].
The proof of the following result is similar to that of Theorem 1, using Lemma 2 instead of Carathéodory's theorem.

Theorem 3 *Let $\mathcal{C}$ be a family of convex cones in $\mathbf{R}^n$ which satisfies one of the following conditions:*
(i) $\cup\, \mathcal{C} \setminus \{0\}$ has at most n connected components.
(ii) There exists a hyperplane H such that $H \cap (C \setminus \{0\}) \neq \emptyset$ for each cone $C \in \mathcal{C}$.
If for any n membered subfamily $\mathcal{B}$ of $\mathcal{C}$ conv ($\cup\, \mathcal{B}$) *is an acute convex cone, then* conv ($\cup\, \mathcal{C}$) *is an acute convex cone.*

Theorem 4 *Let $\mathcal{C} = \{C_1, \ldots, C_m\}$ $(m \geq n)$ be a family of convex cones in $\mathbf{R}^n$ satisfying the following conditions:*
(i) For each $i \in \{1, 2, \ldots, m\}$ card $\mathcal{C}_i \geq \frac{m-n}{n+1}$,
where $\mathcal{C}_i = \{C \in \mathcal{C} \setminus \{C_i\} : C \cap C_i \setminus \{0\} \neq \emptyset\}$.
(ii) For each n membered subfamily $\mathcal{B}$ of $\mathcal{C}$ conv ($\cup\, \mathcal{B}$) *is an acute convex cone.*
Then conv ($\cup\, \mathcal{C}$) *is an acute convex cone.*

Proof. According to Theorem 3 it is sufficient to prove that $\cup\, \mathcal{C} \setminus \{0\}$ has at most n connected components. The *intersection graph* of $\mathcal{C} \setminus \{0\}$ [4], denoted by $G(\mathcal{C} \setminus \{0\})$ is the graph whose vertices are in one-to-one correspondence with the members of $\mathcal{C}$ and in which two vertices are joined by an edge only when the corresponding cones have a common point different from 0. It is almost obvious that the connected components of the set $\cup\, \mathcal{C}\setminus\{0\}$ are in one-to-one correspondence with the connected components of the graph $G(\mathcal{C}\setminus\{0\})$. Suppose $G(\mathcal{C}\setminus\{0\})$ has p connected components. If x is a vertex of the component G_i $(1 \le i \le p)$ then G_i contains all the vertex incident to x, hence

$$\text{card } G_i \ge \frac{m-n}{n+1} + 1 = \frac{m+1}{n+1}.$$

We have

$$m = \sum_{i=1}^{p} \text{card } G_i \ge p\frac{m+1}{n+1}$$

whence

$$p \le \frac{m}{m+1}(n+1) < n+1$$

and the proof is complete. □

3 Intersection properties of convex cones

A hyperplane H in $\mathbf{R}^n$ is said to be *homogenous* if it contains the origin of the space. The closure of a set A in $\mathbf{R}^n$ will be denoted by $\bar{A}$.

Lemma 5 *If C is a convex cone for which there exists a homogenous hyperplane H such that $C \cap H = \{0\}$ then C is an acute cone. Conversely, if C is an acute convex cone which either is closed or has the property that $C\setminus\{0\}$ is a relatively open set, then there exists a homogenous hyperplane H such that $C \cap H = \{0\}$.*

Proof. If H is a hyperplane satisfying $C \cap H = \{0\}$, then the set $C\setminus\{0\}$ is contained in one of the open half-spaces bounded by H and $-C\setminus\{0\}$ lies in the opposite open half-space. It follows that $C \cap (-C)\setminus\{0\} = \emptyset$.

In order to prove the converse we analyse in turn each of the two variants.

Case I. Suppose that C is a closed acute convex cone. Denote by $B = \{x \in \mathbf{R}^n : \|x\| < 1\}$. It is clear that the set $A = C \cap \left(\bar{B} - \frac{1}{2}B\right)$ is compact, hence conv$A$ is compact too. Let us show that $0 \notin$conv$A$. In contrary case there exist m points $x_i \in C\setminus\{0\}$ and m positive numbers α_i with $\sum_{i=1}^{m} \alpha_i = 1$ such that $0 = \sum_{i=1}^{m} \alpha_i x_i$. Therefore

$$-x_1 = \sum_{i=2}^{m} \frac{\alpha_i}{\alpha_1} x_i \in C$$

since C is convex. By $x_1 \in C$ and $-x_1 \in C$ it follows that C is not an acute cone; a contradiction.

The set B being absorbent, the cone generated by conv$A$ (the smallest pointed convex cone containing conv$A$) coincides with C. By a classical strongly separation theorem, applied to the disjoint compact convex sets conv$A$ and $\{0\}$, there exists a linear functional $f : \mathbf{R}^n \to \mathbf{R}$ such that

$$f(x) > f(0) = 0 \quad \text{foreach } x \in \text{ conv}\, A.$$

Hence $f(x) > 0$ for each $x \in C \setminus \{0\}$ and the hyperplane $H = \{x \in \mathbf{R}^n : f(x) = 0\}$ has the required property.

Case II. Suppose that C is an acute convex cone such that $C \setminus \{0\}$ is relatively open set. The relatively open convex set $C \setminus \{0\}$ and the affine set $\{0\}$ being disjoint, there is a hyperplane H containing the origin and disjoint from $C \setminus \{0\}$ (see [9, p. 96]) and the proof is complete. □

Remark 1 The second part of the above lemma is a variant more nuancé of the following known result (see [2, p. 224], or [9, p. 101]): If C is a convex cone in $\mathbf{R}^n$ other than $\mathbf{R}^n$ itself, then there exists a homogenous hyperplane H such that C is contained in one of the closed halfspaces bounded by H.

The following lemma is an immediate consequence of Corollary 9.1.3 in [9].

Lemma 6 *If $\mathcal{C}$ is a finite family of closed convex cones in $\mathbf{R}^n$ such that* conv $(\cup\, \mathcal{C})$ *is an acute cone, then* conv $(\cup\, \mathcal{C})$ *is closed.*

Theorem 7 *Let $\mathcal{C}$ be a family of closed convex cones in $\mathbf{R}^n\,(n \geq 2)$ such that for each n membered subfamily $\mathcal{B}$ of $\mathcal{C}$,* conv $(\cup\, \mathcal{B})$ *is an acute cone and $\cap\, \mathcal{B} \setminus \{0\} \neq \emptyset$. Then $\cap\, \mathcal{C} \setminus \{0\} \neq \emptyset$.*

Proof. Since every two cones have a common point different from 0, the set $\cup\, \mathcal{C} \setminus \{0\}$ is connected and by Theorem 3 (i), conv$(\cup\, \mathcal{C})$ is an acute cone.

Let us suppose for the beginning the family of cones $\mathcal{C}$ finite. In this case, by Lemma 6, the set conv$(\cup\, \mathcal{C})$ is closed. From Lemma 5 we infer the existence of a homogenous hyperplane H_0 disjoint from conv$(\cup\, \mathcal{C}) \setminus \{0\}$. Let H be a translate of H_0 which intersects the set conv$(\cup\, \mathcal{C}) \setminus \{0\}$. For each $C_i \in \mathcal{C}$ denote by $A_i = H \cap C_i$ and let $\mathcal{A}$ be the family of all sets A_i. Then each C_i coincides with the cone generated by A_i and conv$(\cup\, \mathcal{C})$ is the cone generated by $\cup\, \mathcal{A}$. Since any n members of the family $\mathcal{C}$ have a common half-line, it follows that each n members of the family $\mathcal{A}$ have a common point. By Helly's theorem there is at least an $x \in \cap\, \mathcal{A}$, whence $\{\alpha x : \alpha > 0\} \subset \cap\, \mathcal{C}$.

Pass to the proof of the theorem in the case when the family $\mathcal{C}$ is infinite. Let $S = \{x \in \mathbf{R}^n : \|x\| = 1\}$ and $\mathcal{B} = \{C \cap S : C \in \mathcal{C}\}$. Obviously the members of the family $\mathcal{B}$ are compact sets and according to the first part of the proof $\mathcal{B}$ has the finite intersection property. It follows $\cap\, \mathcal{B} \neq \emptyset$, hence $\cap\, \mathcal{C} \setminus \{0\} \neq \emptyset$. □

Remark 2 As the next example shows, the hypothesis of Theorem 7 cannot be weakened. Consider the family of planar cones $\mathcal{C} = \{C_1, C_2, C_3\}$, where $C_1 = \{(x,y) \in \mathbf{R}^2 : x \geq 0,\ y \geq 0\}$, $C_2 = \{(x,y) \in \mathbf{R}^2 : x \leq 0,\ y \geq 0\}$ and $C_3 = \{(x,y) \in \mathbf{R}^2 : y = 0\}$. The intersection of any two cones contains at least one half-line, but conv$(C_1 \cup C_2)$ is not an acute cone and the conclusion of the theorem is not true.

The assertion of Theorem 7 holds in the situation presented below.

Theorem 8 *Let $\mathcal{C}$ be a finite family of pointed convex cones in $\mathbf{R}^n$ $(n \geq 2)$ such that for each*

$C \in \mathcal{C}$, $C \setminus \{0\}$ is open. If for each n membered subfamily $\mathcal{B}$ of $\mathcal{C}$, conv $(\cup \mathcal{B})$ *is an acute cone and* $\cap \mathcal{B} \setminus \{0\} \neq \emptyset$, *then* $\cap \mathcal{C} \setminus \{0\} \neq \emptyset$.

Proof. By Theorem 3 (i) conv $(\cup \mathcal{C})$ is an acute cone. Since the convex hull of an open set is open [13, p. 122] Lemma 5 is again applicable for the cone conv $(\cup \mathcal{C})$ and further on the proof is similar to the proof of Theorem 7. □

Sandgren [10] and Valentine [13] employed the duality theory of convex cones in order to obtain some Helly's type theorems. The same technique will be used further on, obtaining new combinatorial results for families of cones. Recall that if C is a nonempty convex cone in $\mathbf{R}^n$ the set

$$C^0 = \{x \in \mathbf{R}^n : \langle x, y \rangle \leq 0, \ \forall\, y \in C\}$$

is called the *polar* of C. Observe that C^0 is a pointed closed cone. The proofs of the following lemmas are elementary and they can be found, in close variants, for instance in [1], [9], [10].

Lemma 9 *If $\mathcal{C}$ is a family of closed convex cones in $\mathbf{R}^n$ then* $(\cap \mathcal{C})^0 = \overline{\text{conv}\,(\cup \mathcal{C})}$.

Lemma 10 *If $\{C_i : i \in I\}$ is a finite family of closed convex cones in $\mathbf{R}^n$ then the following assertions are equivalent:*
(i) $\cap\{C_i : i \in I\} = \{0\}$.
(ii) conv $\left(\cup \left\{C_i^0 : i \in I\right\}\right) = \mathbf{R}^n$.

Lemma 11 *If C is a closed convex cones in $\mathbf{R}^n$, then C (respectively C^0) is acute if and only if C^0 (respectively C) is n - dimensional.*

The following theorems concerning families of convex cones will be obtained by means of the associated polar cones and of the previous lemmas.

Theorem 12 *Let $\mathcal{C} = \{C_1, ..., C_m\}$ $(m \geq n)$ be a family of convex cones in $\mathbf{R}^n$ satisfying the following conditions:*
(a) *For each $i \in \{1, 2, ..., m\}$* card $\mathcal{C}_i \geq \frac{m-n}{n+1}$, where

$$\mathcal{C}_i = \{C \in \mathcal{C} \setminus \{C_i\} : \text{conv } (C \cup C_i) \neq \mathbf{R}^n\}.$$

(b) *The intersection of any n members of $\mathcal{C}$ is n- dimensional.*
Then $\cap\, \mathcal{C}$ is n - dimensional.

Proof. We divide the proof into two parts. For the beginning suppose that all cones $C_i \in \mathcal{C}$ are closed. Consider the family of the polar cones $\mathcal{C}^0 = \left\{C^0 : C \in \mathcal{C}\right\}$ and for each cone $C_i \in \mathcal{C}$ denote by

$$\mathcal{C}_i^0 = \left\{C^0 \in \mathcal{C}^0 \setminus \{C_i^0\} : C^0 \cap C_i^0 \neq \{0\}\right\}.$$

By Lemma 10 the condition (a) is equivalent with card $\mathcal{C}_i^0 \geq \frac{m-n}{n+1}$. From (b), taking into account Lemma 11, it follows that for any n membered subfamily $\mathcal{B}^0$ of $\mathcal{C}^0$ the set $\overline{\text{conv}\,(\cup\, \mathcal{B}^0)}$ is an acute convex cone and thus, by Lemma 6, conv$(\cup\, \mathcal{B})$ is closed.

Therefore the family of cones $\mathcal{C}^0$ fulfils the conditions of Theorem 4. Applying this theorem we obtain that $\operatorname{conv}(\cup \mathcal{C}^0)$ is an acute convex cone. Moreover, by Lemma 6 this cone is closed. Using once again Lemmas 9 and 11 it follows that $\cap\, \mathcal{C}$ is n - dimensional.

Pass now to the general case, removing the closedness condition imposed to the cone C_i. If $\operatorname{conv}(C \cup C_i) \neq \mathbf{R}^n$, taking into account Remark 1, there exists a closed homogenous half-space S which contains $\operatorname{conv}(C \cup C_i)$. Then $\operatorname{conv}(\bar{C} \cup \bar{C}_i)$ will be also contained in S, hence $\operatorname{conv}(\bar{C} \cup \bar{C}_i) \neq \mathbf{R}^n$. Thus the family $\bar{\mathcal{C}} = \{\bar{C}_1, ..., \bar{C}_m\}$ satisfies the conditions (a), (b). From the first part of the proof $\cap\, \bar{\mathcal{C}}$ is n - dimensional, hence being a convex set it has nonempty interior. So there exits an open ball B such that $\bar{B} \subset \cap\, \bar{\mathcal{C}}$. Since, for a convex set A in a topological vector space $\operatorname{int} A = \operatorname{int} \bar{A}$ [13, p.123], for each $C \in \mathcal{C}$ we have $B \subset \operatorname{int} C \subset C$. It follows $B \subset \cap\, \mathcal{C}$, whence $\cap\, \mathcal{C}$ is n - dimensional. □

Theorem 13 *Let $\mathcal{C}$ be a family of convex cones in $\mathbf{R}^n$ $(n \geq 2)$, such that for each n membered subfamily $\mathcal{B}$ of $\mathcal{C}$, $\cap\, \mathcal{B}$ is an n - dimensional cone and* $\operatorname{conv}(\cup\, \mathcal{B}) \neq \mathbf{R}^n$. *Then* $\operatorname{conv}(\cup\, \mathcal{C}) \neq \mathbf{R}^n$.

Proof. As in the previous proof let $\mathcal{C}^0 = \{C^0 : C \in \mathcal{C}\}$. It is easy to check that the hypothesis of the theorem can be reformulated as follows: for each n membered subfamily $\mathcal{B}^0$ of $\mathcal{C}^0$, $\operatorname{conv}(\cup\, \mathcal{B}^0)$ is an acute cone and $\cap\, \mathcal{B} \setminus \{0\} \neq \emptyset$. By Theorem 7, $\cap\, \mathcal{C}^0 \setminus \{0\} \neq \emptyset$. If $b \in \cap\, \mathcal{C}^0 \setminus \{0\}$ it follows that

$$C \subset \{x \in \mathbf{R}^n : \langle x, b \rangle \leq 0\} \quad \text{foreach } C \in \mathcal{C},$$

whence

$$\operatorname{conv}(\cup\, \mathcal{C}) \subset \{x \in \mathbf{R}^n : \langle x, b \rangle \leq 0\}$$

and the proof is complete. □

4 A lower bound for the Ramsey numbers $R_{2n}(m, 3n)$

For a nonempty set M and a positive integer $k \leq \operatorname{card} M$, we denote by $\mathcal{P}_k(M)$ the family of all subsets with k elements of M. We recall that, for positive integer, k, m_1, m_2 the Ramsey number $R_k(m_1, m_2)$ is the minimal integer r with the following property:

if M is a set with r elements and $(\mathcal{B}_1, \mathcal{B}_2)$ is an ordered partition of the family $\mathcal{P}_k(M)$, then for an index $i \in \{1, 2\}$ there is a subset M_i of M with cardinality m_i such that all the subsets with k elements of M_i are contained in $\mathcal{B}_i$.

The problem of determining the Ramsey number is far from being solved. The literature contains some partial solutions to this problem and different lower or upper bounds for the Ramsey numbers (see [6]). In this section we establish a lower bound for the Ramsey numbers $R_{2n}(m, 3n)$, $m \geq 2n$.

The two following lemmas will be helpful. The first is a classical result (see [7] for a historical account). The second is a result of Katchalski [8].

Lemma 14 *Let $\mathcal{C}$ be a finite family of pointed convex cones in $\mathbf{R}^n$ with* card $\mathcal{C} \geq 2n$. *If $\cap\, \mathcal{B} \neq \{0\}$ for any n membered subfamily $\mathcal{B}$ of $\mathcal{C}$, then $\cap\, \mathcal{C} \neq \{0\}$.*

Lemma 15 *Let* C *be a family of* m *pointed convex cones in* $\mathbf{R}^n$ $(m > n)$ *such that:*
(*i*) $\cap\, C = \{0\}$,
(*ii*) $\cap\, \mathcal{B} \neq \{0\}$ *for any* n *membered subfamily* $\mathcal{B}$ *of* C.
Then there exists a $n + \left[\frac{m-n}{2}\right]$ *membered subfamily* $\mathcal{K}$ *of* C *such that* $\cap\, \mathcal{K} \neq \{0\}$. *Moreover, for any positive integers* $n < m$, $n + \left[\frac{m-n}{2}\right]$ *in the above statement cannot be replaced by* $n + \left[\frac{m-n}{2}\right] + 1$.

Theorem 16 *If* m *and* n *are positive integers such that* $m \geq 2n$, *then* $R_{2n}(m, 3n) \geq 2m - n$.

Proof. By the last part of Lemma 15 there exists a family C of $2m - n - 1$ convex cones such that
(1) $\cap\, \mathcal{B} \neq \{0\}$ for any n membered subfamily $\mathcal{B}$ of C;
(2) $\cap\, \mathcal{K} = \{0\}$ for any m membered subfamily $\mathcal{K}$ of C.
We partition the set $\mathcal{P}_{2n}(C)$ of $2n$ membered subfamilies of C into two classes $(\mathfrak{B}_1, \mathfrak{B}_2)$ such that $\mathfrak{B}_1$ constitutes the class of all $2n$ membered subfamilies $\mathcal{B}$ for which $\cap\, \mathcal{B} \neq \{0\}$. Supposing by way of contradiction that $R_{2n}(m, 3n) \leq 2m - n - 1$, then by the definition of the Ramsey numbers, it would follow that there exists a subfamily $\mathcal{K}$ of C having one of the following properties:

(*a*) $\mathcal{K}$ has m members and any $2n$ membered subfamily of $\mathcal{K}$ belongs to $\mathfrak{B}_1$. By Lemma 14 we get $\cap\, \mathcal{K} \neq \{0\}$ which contradicts (2).

(*b*) $\mathcal{K}$ has $3n$ members and any $2n$ membered subfamily of $\mathcal{B}$ of $\mathcal{K}$ belongs to $\mathfrak{B}_2$ i.e. $\cap\, \mathcal{B} = \{0\}$. On the other side, in view of (1), Lemma 15 implies that $\mathcal{K}$ contains a $2n$ membered subfamily $\mathcal{B}$ such that $\cap\, \mathcal{B} \neq \{0\}$. Thus we arrived again to a contradiction and the proof is complete. □

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