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The projective theory of ruled surfaces

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Abstract. The aim of this paper is to get some results about ruled surfaces which configure a projective theory of scrolls and ruled surfaces. Our ideas follow the viewpoint of Corrado Segre, but we employ the contemporaneous language of locally free sheaves. The results complete the exposition given by R. Hartshorne and they have not appeared before in the contemporaneous literature.

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Introduction

Through this paper, a *geometrically ruled surface*, or simply a *ruled surface*, will be a $\mathbf{P}^1$-bundle over a smooth curve X of genus g. It will be denoted by $\pi : S = \mathbf{P}(\mathcal{E}_0) \longrightarrow X$ and we will follow the notation and terminology of R. Hartshorne's book [8], V, section 2. We will suppose that $\mathcal{E}_0$ is a normalized sheaf and X_0 is the section of minimum self-intersection that corresponds to the surjection $\mathcal{E}_0 \longrightarrow \mathcal{O}_X(\mathfrak{e}) \longrightarrow 0$, $\bigwedge^2 \mathcal{E} \cong \mathcal{O}_X(\mathfrak{e})$. Which are the linear equivalence classes $D \sim mX_0 + \mathfrak{b}f$, $\mathfrak{b} \in \mathrm{Pic}(X)$, that correspond to very ample divisors?. When $g = 1$ and $m = 1$, a characterization is known ([8], V, ex.2.12), but the classification of elliptic scrolls obtained by Corrado Segre in [17] does not follow directly from this. A scroll is the birational image of a ruled surface $\pi : S = \mathbf{P}(\mathcal{E}_0) \longrightarrow X$ by an unisecant complete linear system.

The philosophy of this work is to develop a theory of ruled surfaces that allows their projective classification, by using the modern language of $\mathbf{P}^n$-bundles and rescuing the classical viewpoint introduced by C. Segre in [18]. This Segre's paper was reviewed with criticism by F. Severi in [19], but only some of the results of this work were reformulated nowadays. The study of directrix curves with minimum self-intersection and the formalization of the concept of ruled

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surface of general type was made by F. Ghione in [6]. The calculus of the genus of a curve on a ruled surface appeared in Ghione-Sacchiero [7]. The Hilbert scheme of the nonspecial ruled surfaces was studied by the second author in [1] and [16], where the property of maximal rank was proved. The theorem of C.Segre which says that $e \geq -g$ in a ruled surface $\pi : \mathbf{P}(\mathcal{E}_0) \longrightarrow X$ was proved by M. Nagata in [15], by H. Lange in [10], and it was generalized to higher rank by H. Lange in [11]. Finally, M. Maruyama studied ruled surfaces by using elementary transforms in [13]. He applies the classification theorem of Nagata (all geometrically ruled surface $\pi : \mathbf{P}(\mathcal{E}_0) \longrightarrow X$ is obtained from $X \times \mathbf{P}^1$ by applying a finite number of elementary transformations, [21], V,§1) to study the moduli of ruled surfaces of genus $g \leq 3$. Anyway, the results of this paper complete the exposition about ruled surfaces given in [8] and they have not appeared before in the contemporaneous literature.

The paper is organized in the following way:

1: Ruled surfaces and scrolls.

2: Unisecant linear systems on a ruled surface.

3: Decomposable ruled surfaces.

4: Elementary transformation of a ruled surface.

5: Speciality of a scroll.

6: Segre Theorems.

In section 1, we introduce the basic facts about ruled surfaces and we relate them to the scroll. The classical authors define a scroll as a surface $R \subset \mathbf{P}^N$ such that there exists a line contained in R that passes through the generic point (see [20], 204). We show that any scroll is the birational image of a geometrically ruled surface $S = \mathbf{P}(\mathcal{E}_0)$ by an unisecant linear system. In a modern way, this is the equivalence between morphisms $\phi : X \longrightarrow G(1, N)$, where X is a smooth curve, and surjections $\mathcal{O}_G^{N+1} \longrightarrow \mathcal{E}$, where $\mathcal{E} = \phi^* U$ is the locally free sheaf of rank 2 obtained from the universal bundle U.

In section 2, we characterize when a complete linear system defined by an unisecant divisor $H \sim X_0 + \mathfrak{b} f$ in S is base-point-free. The most important result is Theorem 20 which describes the points where the regular map $\phi_H : S \longrightarrow \mathbf{P}^N$ is not a local isomorphism. Equivalently, this characterizes the singular locus of the scroll $R = \phi_H(S)$, $\phi_H^{-1}(\text{sing}(R)) = \{x \in S / x \text{ is a base point of } |H - Pf|, P \in X\}$, and when $|H|$ is very ample.

In section 3, we consider a decomposable ruled surface $\mathcal{E}_0 \cong \mathcal{O}_X \oplus \mathcal{O}_X(\mathfrak{e})$. There exist two disjoint sections X_0 and X_1, which correspond to the surjections $\mathcal{E}_0 \longrightarrow \mathcal{O}_X(\mathfrak{e}) \longrightarrow 0$ and $\mathcal{E}_0 \longrightarrow \mathcal{O}_X \longrightarrow 0$. We prove some results that localize the base points of a unisecant complete linear system $H \sim X_0 + \mathfrak{b} f$ over X_0 or X_1. We study the existence of sections in $|H|$ and we give a sufficient condition for ϕ_H to be an isomorphism in points out of X_0 or X_1. The main result of this section is Theorem 34, where we describe the support of the singular locus of the regular map $\phi_H : S \longrightarrow R \subset \mathbf{P}^N$. We finish this section studying the base-point-free and very ample m-secant complete linear systems.

In section 4, we make a classical study of the elementary transformation of a ruled surface. We describe some elementary properties and we show that the elementary transform corresponds to the projection of a scroll from a nonsingular point.

The study of how the divisor $\mathfrak{e}$ is transformed by the elementary transformation at a point x in the minimum self-intersection section allow us to give an easy demonstration of the result of C.Segre (Corollary 48): any indecomposable scroll is obtained from a decomposable one by applying a finite number of elementary transformations. We use that $e = -\partial(\mathfrak{e}) \leq 2g - 2$ in a decomposable ruled surface.

The main result of this section is Theorem 50, where we identify the elementary transforms of a decomposable ruled surface at a point x according to its position.

In section 5, we introduce the special ruled surfaces. Then we use the elementary transformation to give a geometrical meaning, according to Riemann-Roch, of the speciality of a scroll. In this way, we pose the problem of the existence of scrolls with speciality 1 over a smooth curve of genus $g \geq 1$ and such that any special scroll is obtained by projection from them. This problem is solved in [5].

Finally, in section 6, we rescue the results of Segre in [18] about special ruled surfaces. We conserve the spirit of Segre's methods, although we write them in modern way. In fact, Segre proved that a special ruled surface of genus g and degree $d \geq 4g-2$ always has a special directrix curve, but the condition over the degree is not necessary: any special ruled surface has a special directrix curve (see [5]).

Most of the results that appear in this paper generalize to higher rank and will be studied in a forthcoming paper.

1 Ruled surfaces and scrolls

1 Definition. A geometrically ruled surface, or simply ruled surface, is a surface S, together with a surjective morphism $\pi : S \longrightarrow X$ to a smooth curve X, such that the fibre S_x is isomorphic to $\mathbf{P}^1$ for every point $x \in C$, and such that π admits a section (i.e., a morphism $i : X \longrightarrow S$ such that $\pi \circ i = id_X$).

2 Proposition. *If $\pi : S \longrightarrow X$ is a ruled surface, then there exist a locally free sheaf $\mathcal{E}$ of rank 2 on X such that $S \cong \mathbf{P}(\mathcal{E})$ over X. Conversely, every such $\mathbf{P}(\mathcal{E})$ is a ruled surface over X. If $\mathcal{E}$ and $\mathcal{E}'$ are two locally free sheaves of rank*

2 on X, then $\mathbf{P}(\mathcal{E})$ and $\mathbf{P}(\mathcal{E}')$ are isomorphic as ruled surfaces over X if and only if there is an invertible sheaf $\mathcal{L}$ on X such that $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$.

PROOF. See [8], V, 2.2. QED

If $\pi : S \longrightarrow X$ is a ruled surface, we can choose $S \cong \mathbf{P}(\mathcal{E}_0)$ where $\mathcal{E}_0$ is a locally free sheaf of rank 2 on X with the property $H^0(\mathcal{E}_0) \neq 0$ but for all invertible sheaves $\mathcal{L}$ on X with $\deg(\mathcal{L}) < 0$, we have $H^0(\mathcal{E}_0 \otimes \mathcal{L}) = 0$. In this case we say $\mathcal{E}_0$ is *normalized*. The sheaf $\mathcal{E}_0$ is not determined uniquely, but it is determined $e = -\deg(\mathcal{E})$.

Let $\mathfrak{e}$ be the divisor on X corresponding to the invertible sheaf $\bigwedge^2 \mathcal{E}_0$, then $e = -\deg(\mathfrak{e})$. Moreover, there is a section $i : X \longrightarrow S$ with image X_0, such that $\mathcal{O}_S(X_0) \cong \mathcal{O}_S(1)$.

3 Proposition. *Under the above assumptions:*

$$\mathrm{Pic}(S) \cong \mathbf{Z} \oplus \pi^* \mathrm{Pic}(X)$$

where $\mathbf{Z}$ is generated by X_0. Also

$$Num(S) \cong \mathbf{Z} \oplus \mathbf{Z}$$

generated by X_0 and f, and satisfying $X_0.f = 1$, $f^2 = 0$.

PROOF. See [8], V, 2.3. QED

Thus, if $\mathfrak{b} \in Div(X)$, we denote the divisor $\pi^*\mathfrak{b}$ on S by $\mathfrak{b}f$. Therefore, any element of $\mathrm{Pic}(S)$ can be written $nX_0 + \mathfrak{b}f$ with $n \in \mathbf{Z}$ and $\mathfrak{b} \in \mathrm{Pic}(X)$. Any element of $Num(X)$ can be written $nX_0 + bf$ with $n, b \in \mathbf{Z}$. A linear system $|nX_0 + \mathfrak{b}f|$ will be called *n-secant* because it meets each generator at n points.

4 Proposition. *Let $\mathcal{E}$ be a locally free sheaf of rank 2 on the curve X, and let S be the ruled surface $\mathbf{P}(\mathcal{E})$. Let $\mathcal{O}_S(1)$ be the invertible sheaf $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. Then there is a one-to-one correspondence between sections $i : X \longrightarrow S$ and surjections $\mathcal{E} \longrightarrow \mathcal{L} \longrightarrow 0$, where $\mathcal{L}$ is an invertible sheaf on X, given by $i^*\mathcal{O}_S(1)$.*

Furthermore, if D is a section of S, corresponding to the surjection $\mathcal{E} \rightarrow \mathcal{L} \rightarrow 0$, and $\mathcal{L} = \mathcal{O}_X(\mathfrak{a})$ for any divisor $\mathfrak{a}$ on X, then $\deg(\mathfrak{a}) = X_0.D$, and $D \sim X_0 + (\mathfrak{a} - \mathfrak{e})f$.

PROOF. See [8], V, 2.6 and 2.9. QED

If $\mathcal{E}_0$ is a normalized sheaf and X_0 the corresponding section of the ruled surface $\pi : S \longrightarrow X$, we have that:

$$\pi_*\mathcal{O}_S(X_0) \cong \mathcal{E}_0$$

Moreover, if $H \sim X_0 + \mathfrak{b}f$, by the projection formula:

$$\pi_*\mathcal{O}_S(H) \cong \pi_*(\mathcal{O}_S(X_0) \otimes \pi^*\mathfrak{b}) \cong \mathcal{E}_0 \otimes \mathcal{O}_X(\mathfrak{b})$$

Since $R^i\pi_*\mathcal{O}_S(H) = 0$ for any $i > 0$, we have that $H^i(\mathcal{O}_S(H)) = H^i(\mathcal{E}_0 \otimes \mathcal{O}_X(\mathfrak{b}))$.

From this and from the definition of normalized sheaf, we see that the curve X_0 is the minimum self-intersection curve on S and $X_0^2 = -e$.

The image of a ruled surface by the map defined by an unisecant base-point-free linear system is a surface containing a one-dimensional family of lines.

5 Definition. A scroll $R \subset \mathbf{P}^N$ is an algebraic surface such that it has a line passing through the generic point. The lines of the scroll are called generators.

Let $R \subset \mathbf{P}^N$ be a scroll. Let $\overline{H}$ be a generic hyperplane section of R. $\overline{H}$ is smooth away from the singular locus of R. Thus there is an open set $U \subset \overline{H}$, such that there is a unique generator passing through any point.

Let $G(1,N)$ be the Grassmaniann parameterizing the lines of $\mathbf{P}^N$. We have a map:

$$U \longrightarrow G(1,N)$$

which applies each point of U over the unique generator passing through it.

The map extends uniquely to the nonsingular model X of $\overline{H}$:

$$\eta : X \longrightarrow G(1,N)$$

If X is a curve of genus g, we say that R has genus g, that is, we define the genus of R as the geometric genus of the generic hyperplane section.

6 Definition. Let $V \subset \mathbf{P}^N$ be a projective variety in $\mathbf{P}^N$. We say that V is linearly normal, when there is not any variety $V' \subset \mathbf{P}^{N'}$, with $N' > N$ and $\deg(V) = \deg(V')$ such that V' projects over V.

7 Proposition. *A linearly normal scroll R is the image of a unique ruled surface S by the birational map defined by a base-point-free unisecant complete linear system $|H|$.*

PROOF. Let $R \subset P^N$. Consider the corresponding map $\eta : X \longrightarrow G(1,N)$. We build the following incidence variety:

$$G(1,N)\times\mathbf{P}^N \leftarrow X\times\mathbf{P}^N \supset \mathcal{J}_X := \{(P,x)/x \in l_{\eta(P)}\} \begin{array}{l} \overset{q}{\nearrow}\ \mathbf{P}^N \\ \underset{p}{\searrow}\ X \to G(1,N) \end{array}$$

$\mathcal{J}_X$ and X are smooth varieties and the map $p : \mathcal{J}_X \longrightarrow X$ has fibre $\mathbf{P}^1$ and surjective differential. Then, applying Enriqües–Noether Theorem (see [2], II), there exists an open set $U' \subset X$ verifying $p^{-1}(U') \simeq U' \times \mathbf{P}^1$. Since X is a smooth curve, we deduce that $p : \mathcal{J}_X \longrightarrow X$ has a section and it is a geometrically ruled surface.

The image of the projection q is exactly the scroll R on $\mathbf{P}^N$. The generic fibre of q is a point. Consider the invertible sheaf $\mathcal{L} \cong q^*\mathcal{O}_{P^N}(1)$. Their global sections correspond to the complete linear system $|H|$,where $H := q^*\overline{H}$. It is an unisecant linear system, because it meets the generic generator at a unique point. The map q is determined by a linear subsystem $\delta \subset |H|$, so $|H|$ is base-point-free.

But R is linearly normal, so $H^0(\mathcal{O}_{\mathcal{J}_X}(H)) \cong H^0(\mathcal{O}_{P^N}(1))$. From this q is determined by the complete linear system.

Note that the construction does not depend of the election of the hyperplane section, because any two hyperplane sections are birational equivalent. In fact, the ruled surface J_X is unique:

If we suppose R defined by the birational map determined by a base-point-free unisecant linear system $|H'|$ over the ruled surface $\pi : \mathbf{P}(\mathcal{E}) \longrightarrow X$:

$$\phi_{H'} : \mathbf{P}(\mathcal{E}) \longrightarrow R \subset \mathbf{P}^N$$

we can define a birational map $\eta' : X \longrightarrow G(1, N)$ which applies a point $P \in X$ on the line $\phi_{H'}(Pf)$ on $\mathbf{P}^N$. The maps η and η' are equal up to automorphism of X and then the incidence variety $\mathcal{J}_X$ is isomorphic to $\mathbf{P}(\mathcal{E})$. QED

8 Definition. Let $R \subset \mathbf{P}^N$ be a linearly normal scroll, let S be a ruled surface and let $|H|$ be a base-point-free unisecant linear system defining a birational map $\phi_H : S \longrightarrow \mathbf{P}^N$. If $\phi_H(S) = R$, then we say that S and H are the ruled surface and the linear system associated to R.

9 Definition. A directrix curve of a scroll is a curve meeting each generator at a unique point.

The directrix curves of a scroll R correspond to the sections of the associated ruled surface S. Suppose that R is the image of S by the map defined by the linear system $|X_0 + \mathfrak{b}f|$. We will denote the image of a section D of S by $\overline{D} \subset R$. The curve has degree $\deg(\overline{D}) = D.X_0 + \deg(\mathfrak{b})$. The degree of the scroll R is $(X_0 + \mathfrak{b}f)^2 = X_0^2 + 2\deg(\mathfrak{b})$.

The minimum self-intersection curve X_0 of S corresponds to the minimum degree directrix curve of R. If we take two sections $D_1 \sim X_0 + \mathfrak{a}_1 f$ and $D_2 \sim X_0 + \mathfrak{a}_2 f$ on S, they have non negative intersection. Thus:

$$\begin{aligned} \deg(\overline{D_1}) + \deg(\overline{D_2}) &= 2X_0^2 + 2\deg(\mathfrak{b}) + \deg(\mathfrak{a}_1) + \deg(\mathfrak{a}_2) = \\ &= X_0^2 + 2\deg(\mathfrak{b}) + D_1.D_2 \geq \deg(R) \end{aligned}$$

We see that the sum of the degree of two directrix curves of R is greater than or equal to the degree of R.

We have seen that the study of the scrolls is equivalent to the study of geometrically ruled surfaces and their unisecant linear systems, but it is equivalent to the study of locally free sheaves of rank 2 over the base curve X too. In the next section we begin the study of the unisecant linear systems on a ruled surface and in this way we treat the study of the scrolls.

2 Unisecant linear systems on a ruled surface.

Let $\pi : S \longrightarrow X$ be a geometrically ruled surface. An unisecant complete linear system $|H| = |X_0 + \mathfrak{b}f|$ on S defines a rational map $\phi_H : S \longrightarrow \mathbf{P}^N$. The map ϕ_H is regular out of base points of $|H|$ and it is an isomorphism onto its image when $|H|$ is very ample. In this section we will study general conditions for an unisecant linear system to be base-point-free, to have irreducible elements and to define an isomorphism.

10 Lemma. *Let $\mathfrak{b}$ be a nonspecial divisor on X. Then, if $i \geq 0$:*

$$h^i(\mathcal{O}_S(X_0 + \mathfrak{b}f)) = h^i(\mathcal{O}_X(\mathfrak{b})) + h^i(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e}))$$

PROOF. Because S is a surface, it is sufficient to prove it for $i \leq 2$. Let us consider the exact sequence of X_0 on S:

$$0 \longrightarrow \mathcal{O}_S(-X_0) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{X_0} \longrightarrow 0$$

Tensoring with $\mathcal{O}_S(X_0 + \mathfrak{b}f)$, we get the cohomology sequence

$$\begin{array}{l} 0 \longrightarrow H^0(\mathcal{O}_S(\mathfrak{b}f)) \longrightarrow H^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) \longrightarrow H^0(\mathcal{O}_{X_0}(X_0 + \mathfrak{b}f)) \longrightarrow \\ \longrightarrow H^1(\mathcal{O}_S(\mathfrak{b}f)) \longrightarrow H^1(\mathcal{O}_S(X_0 + \mathfrak{b}f)) \longrightarrow H^1(\mathcal{O}_{X_0}(X_0 + \mathfrak{b}f)) \longrightarrow \\ \longrightarrow H^2(\mathcal{O}_S(\mathfrak{b}f)) \longrightarrow H^2(\mathcal{O}_S(X_0 + \mathfrak{b}f)) \longrightarrow H^2(\mathcal{O}_{X_0}(X_0 + \mathfrak{b}f)) \longrightarrow \end{array}$$

We have $h^i(\mathcal{O}_S(\mathfrak{b}f)) = h^i(\mathcal{O}_X(\mathfrak{b}))$ and $h^i(\mathcal{O}_{X_0}(X_0 + \mathfrak{b}f)) = h^i(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e}))$. But $h^2(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e})) = h^2(\mathcal{O}_X(\mathfrak{b})) = 0$. Since $\mathfrak{b}$ is nonspecial, $h^1(\mathcal{O}_S(\mathfrak{b}f)) = 0$ and the lemma follows. QED

11 Remark. *Note that we have seen that the following inequality always holds:*

$$h^i(\mathcal{O}_S(X_0 + \mathfrak{b}f)) \leq h^i(\mathcal{O}_X(\mathfrak{b})) + h^i(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e})).$$

Furthermore, if we consider the linear system $|mX_0 + \mathfrak{b}f|$ with $m \geq 0$, for each $i > 0$ we have the exact sequence:

$$H^i(\mathcal{O}_S((m-1)X_0 + \mathfrak{b}f)) \longrightarrow H^i(\mathcal{O}_S(mX_0 + \mathfrak{b}f)) \longrightarrow H^i(\mathcal{O}_X(\mathfrak{b} + m\mathfrak{e}))$$

From this, we deduce that $h^i(\mathcal{O}_S((mX_0 + \mathfrak{b}f))) \leq h^i(\mathcal{O}_X(\mathfrak{b} + m\mathfrak{e})) + h^i(\mathcal{O}_S((m-1)X_0 + \mathfrak{b}f))$. We continue in this fashion obtaining:

$$h^i(\mathcal{O}_S(mX_0 + \mathfrak{b}f)) \leq \sum_{k=0}^{m} h^i(\mathcal{O}_X(\mathfrak{b} + k\mathfrak{e}))$$

QED

12 Proposition. *Let S be a geometrically ruled surface and let $\mathfrak{b}$ be a divisor on X. Let $|H| = |X_0 + \mathfrak{b}f|$ be a complete linear system on S. Let P be a point in X. Then:*

(1) $|H|$ is base-point-free on the generator Pf if and only if $h^0(\mathcal{O}_S(H-Pf)) = h^0(\mathcal{O}_S(H)) - 2$.

(2) $|H|$ has a unique base point on the generator Pf if and only if $h^0(\mathcal{O}_S(H - Pf)) = h^0(\mathcal{O}_S(H)) - 1$.

(3) $|H|$ has Pf as a fixed component if and only if $h^0(\mathcal{O}_S(H - Pf)) = h^0(\mathcal{O}_S(H))$.

PROOF. Let us consider the trace of the linear system $|H|$ on the generator Pf:

$$0 \longrightarrow H^0(\mathcal{O}_S(H - Pf)) \longrightarrow H^0(\mathcal{O}_S(H)) \xrightarrow{\alpha} H^0(\mathcal{O}_{Pf}(H))$$

H meets each generator at a point, so $H^i(\mathcal{O}_{Pf}(H)) \cong H^i(\mathcal{O}_{\mathbf{P}^1}(1))$. Therefore $h^0(\mathcal{O}_{Pf}(H)) = 2$ and:

(1) If $\dim(\mathrm{Im}(\alpha)) = 2$, then the linear system $|H|$ traces on Pf the complete linear system of points of $\mathbf{P}^1$. Since this is base-point-free, $|H|$ is base-point-free on Pf.

(2) If $\dim(\mathrm{Im}(\alpha)) = 1$, then the linear system $|H|$ traces on Pf a unique point, so $|H|$ has a unique base point on the generator Pf.

(3) If $\dim(\mathrm{Im}(\alpha)) = 0$, then the generator Pf is a fixed component of the linear system $|H|$.

From the exact sequence we obtain $\dim(\mathrm{Im}(\alpha)) = h^0(\mathcal{O}_S(H)) - h^0(\mathcal{O}_S(H - Pf))$, which completes the proof. QED

13 Corollary. *Let S be a geometrically ruled surface and $|H|$ an unisecant complete linear system on S. $|H|$is base-point-free if and only if for all $P \in X$, $h^0(\mathcal{O}_S(H - Pf)) = h^0(\mathcal{O}_S(H)) - 2$.*

14 Proposition. *Let $\mathfrak{b}$ be a divisor on X. If P is a base point of $|\mathfrak{b} + \mathfrak{e}|$, then $Pf \cap X_0$ is a base point of the complete linear system $|X_0 + \mathfrak{b}f|$.*

PROOF. Let us study the trace of the linear system $|X_0 + \mathfrak{b}f|$ on X_0:

$$0 \to H^0(\mathcal{O}_S(\mathfrak{b}f)) \longrightarrow H^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) \longrightarrow H^0(\mathcal{O}_{X_0}(X_0 + \mathfrak{b}f)) \simeq H^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e}))$$

By hypothesis, P is a base point of $|\mathfrak{b} + \mathfrak{e}|$, so all divisors of $|X_0 + \mathfrak{b}f|$ trace on X_0 a divisor which contains P. We conclude that $Pf \cap X_0$ is a base point of $|X_0 + \mathfrak{b}f|$. QED

15 Lemma. *Let* $\mathfrak{b}$ *be a nonspecial divisor on* X. *Then:*

(1) If P *is not a base point of* $\mathfrak{b}$ *and* $\mathfrak{b}+\mathfrak{e}$, *then the linear system* $|X_0+\mathfrak{b}f|$ *has no base points on the generator* Pf.

(2) If P *is a base point of* $\mathfrak{b}+\mathfrak{e}$ *but not of* $\mathfrak{b}$, *then the linear system* $|X_0+\mathfrak{b}f|$ *has a unique base point on the generator* Pf. *This point is* $X_0 \cap Pf$.

(3) If P *is a base point of* $\mathfrak{b}$ *but not of* $\mathfrak{b}+\mathfrak{e}$, *then the linear system* $|X_0+\mathfrak{b}f|$ *has at most a base point on the generator* Pf.

(4) If P *is a base point of* $\mathfrak{b}$ *and* $\mathfrak{b}+\mathfrak{e}$, *then the linear system* $|X_0+\mathfrak{b}f|$ *has at least a base point on the generator* Pf.

PROOF. By Proposition 12, it is sufficient to compute $h^0(\mathcal{O}_S(X_0+\mathfrak{b}f))$ and $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f))$. Since $\mathfrak{b}$ is nonspecial, $h^0(\mathcal{O}_S(X_0+\mathfrak{b}f)) = h^0(\mathcal{O}_X(\mathfrak{b})) + h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))$. We consider two cases:

(1) If P is not a base point of $\mathfrak{b}$, $\mathfrak{b}-P$ is nonspecial because $\mathfrak{b}$ is nonspecial. Therefore, $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) = h^0(\mathcal{O}_X(\mathfrak{b}-P))+h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}-P)) = h^0(\mathcal{O}_X(\mathfrak{b}))-1+h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}-P))$. Then, if P is not a base point of $\mathfrak{b}+\mathfrak{e}$, $h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}-P)) = h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))-1$ and the linear system is base-point-free on Pf. If P is a base point of $\mathfrak{b}+\mathfrak{e}$, then $h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}-P)) = h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))$ and the linear system has a unique base point on Pf (by Proposition 14, it is at $Pf \cap X_0$).

(2) If P is base point of $\mathfrak{b}$, then $\mathfrak{b}-P$ is special. Then $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) \le h^0(\mathcal{O}_X(\mathfrak{b}-P))+h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}-P)) = h^0(\mathcal{O}_X(\mathfrak{b}))+h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}-P))$. If P is not a base point of $\mathfrak{b}+\mathfrak{e}$, $h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}-P)) = h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))-1$ and the linear system has at most a base point on Pf. If P is a base point of $\mathfrak{b}+\mathfrak{e}$, by Proposition 14, the linear system has at least a base point at $X_0 \cap Pf$.

QED

16 Theorem. *Let* S *be a geometrically ruled surface and let* $\mathfrak{b}$ *be a divisor on* X. *There exists a section* $D \sim X_0+\mathfrak{b}f$ *if and only if one of the following conditions holds:*

(1) $h^0(\mathcal{O}_S(X_0+\mathfrak{b}f)) = 1$ *and* $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) = 0$ *for all* $P \in X$.

(2) $h^0(\mathcal{O}_S(X_0+\mathfrak{b}f)) > 1$, $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) < h^0(\mathcal{O}_S(X_0+\mathfrak{b}f))$ *for all* $P \in X$ *and* $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) = h^0(\mathcal{O}_S(X_0+\mathfrak{b}f))-2$ *for the generic point* $P \in X$.

PROOF. We first note that reducible elements of $|X_0 + \mathfrak{b}f|$ contain at least one generator, so they are in linear subsystems $|X_0 + (\mathfrak{b} - P)f|$.

Let us suppose $h^0(\mathcal{O}_S(X_0+\mathfrak{b}f)) = 1$. If $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) = 1$ for some $P \in X$, then the unique effective divisor of $|X_0 + \mathfrak{b}f|$ contains Pf so it is not irreducible. Conversely, if $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P)f)) = 0$ for all $P \in X$, then the unique effective divisor of the linear system does not contain any generator, so it is irreducible.

Let us now suppose $h^0(\mathcal{O}_S(X_0+\mathfrak{b}f)) > 1$. If $h^0(X_0+(\mathfrak{b}-P)f) = h^0(\mathcal{O}_S(X_0+\mathfrak{b}f))$ for some $P \in X$. Then (by Lemma 12) Pf is a fixed component of the linear system, so there are not irreducible elements in $|X_0 + \mathfrak{b}f|$.

If $h^0(X_0+(\mathfrak{b}-P)f) = h^0(\mathcal{O}_S(X_0+\mathfrak{b}f))-1$ for all $P \in X$, then (Proposition 12) the linear system $|X_0+\mathfrak{b}f|$ has a unique base point on each generator. Hence there exists a fixed unisecant curve in the linear system and, as $h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) > 1$ the generic element is not irreducible.

Conversely, if the codimension of linear subsystems $|X_0 + (\mathfrak{b} - P)f|$ is 2 for the generic point and 1 for the remaining ones, then the reducible elements do not satisfy the linear system, so the generic element is irreducible. QED

The curve X_0 is unique on its class of linear equivalence, except when the ruled surface is $X \times \mathbf{P}^1$.

17 Lemma. *Let $S = \mathbf{P}(\mathcal{E}_0)\longrightarrow X$ be a ruled surface. Then $h^0(\mathcal{O}_S(X_0)) = 2$ when $\mathbf{P}(\mathcal{E}_0) \cong \mathbf{P}^1 \times X$ and $h^0(\mathcal{O}_S(X_0)) = 1$ in other case.*

PROOF. Since X_0 is the minimum self-intersection curve, it corresponds to the normalized sheaf $\mathcal{E}_0$. Then we have that $h^0(\mathcal{O}_S(X_0)) = h^0(\mathcal{E}_0) > 0$ and $h^0(\mathcal{O}_S(X_0 - Pf)) = h^0(\mathcal{E}_0 \otimes \mathcal{O}_X(-P)) = 0$, for any point $P \in X$. From this, $h^0(\mathcal{O}_S(X_0)) \leq h^0(\mathcal{O}_S(X_0 - Pf)) + 2 = 2$.

If $\mathbf{P}(\mathcal{E}_0) \cong \mathbf{P}^1 \times X$ then $\mathcal{E}_0 \cong \mathcal{O}_X \oplus \mathcal{O}_X$ and $h^0(\mathcal{O}_{P(\mathcal{E}_0)}(X_0)) = 2$.

Suppose that $h^0(\mathcal{O}_S(X_0)) = 2$. Then $|X_0|$ is a pencil of unisecant irreducible curves, because $h^0(\mathcal{O}_S(X_0 - Pf)) = 0$. If $X_0', X_0'' \in |X_0|$, $X_0'.X_0'' = -e$ must be positive, so $e \leq 0$. Suppose that $e < 0$. Then the curves X_0' and X_0'' have at least a common point. Because $|X_0|$ has dimension 1, the linear system has a base point. By Proposition 12, there is a point $P \in X$ such that $h^0(\mathcal{O}_S(X_0 - Pf)) > h^0(\mathcal{O}_S(X_0)) - 2 = 0$, but this is false. Therefore $e = 0$ and $\mathbf{P}(\mathcal{E}_0)$ has a pencil of disjoint unisecant curves. We have an isomorphism:

$$|X_0| \times X \xrightarrow{\cong} \mathbf{P}(\mathcal{E}_0)$$

that is, $\mathbf{P}(\mathcal{E}_0) \cong \mathbf{P}^1 \times X$. QED

18 Corollary. *Let $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ be effective divisors on X. If they have no common base points and $\mathfrak{b}$ is nonspecial, then there exists a section $D \sim X_0+\mathfrak{b}f$. Furthermore if $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are base-point-free, then the complete linear system $|X_0 + \mathfrak{b}f|$ is base-point-free.*

PROOF. Because $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ are effective divisors, a generic point P is not a base point of both of them. By Propositions 12 and 15, $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) = h^0(\mathcal{O}_S(X_0+\mathfrak{b}f))-2$.

If P is a base point of $\mathfrak{b}$ or $\mathfrak{b}+\mathfrak{e}$ (by hypothesis P is not a common base point), by applying Proposition 15, we obtain that $|X_0+\mathfrak{b}f|$ has at most a base point on the generator Pf and $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) \leq h^0(\mathcal{O}_S(X_0+\mathfrak{b}f))-1$.

Now, by applying Theorem 16, the first part of the statement follows.

According to Lemma 13, we see that if $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ are base-point-free, then the linear system $|X_0+\mathfrak{b}f|$ is base-point-free. QED

19 Remark. *A unisecant complete base-point-free linear system $|H|$ determines a morphism $\phi_H : S \longrightarrow \mathbf{P}^N$ that gives us a scroll $R=\phi_H(S)$ in $\mathbf{P}^N$.*

The map is injective if it separates points, that is, given $x, y \in X$ with $x \neq y$, there is an element $D \in |H|$, such that $x \in D$, but $y \notin D$.

Furthermore, the differential is injective at $x \in S$ when it separates tangent vectors, that is, given $t \in T_x(S)$, there is $D \in |H|$ such that $x \in D$, but $t \notin T_x(D)$. QED

20 Theorem. *Let S be a geometrically ruled surface and let $|H| = |X_0+\mathfrak{b}f|$ be a base-point-free complete linear system on S. Let $\phi_H : S \longrightarrow \mathbf{P}^N$ be the regular map which $|H|$ defines. Let $K=\{x \in S/x$ is a base point of $|H-Pf|$, for some $P \in X\}$. Then the map ϕ_H is an isomorphism exactly in the open set $S \setminus K$.*

PROOF. Let us first see that ϕ_H is injective in $S\setminus K$ and ${\phi_H}^{-1}(\phi_H(S\setminus K)) = S\setminus K$. Given $x \in S/K$ and $y \in S$ with $x \neq y$, they must be separated by elements of $|H|$:

(1) Suppose x and y lie in the same generator Pf. As we saw in Proposition 12, when $|H|$ is base-point-free, it traces the complete linear system of points of $\mathbf{P}^1$ on each generator Pf. Since this is very ample, it separates x from y, so we can find a divisor D in $|H|$ which meets Pf at x, but not at y.

(2) Suppose x and y lie in different generators, $x \in Pf$, $y \in Qf$. Since $x \notin K$, x is not a base point of the linear system $|H-Qf|$. Moreover, the elements of $|H-Qf|$ correspond to the elements of $|H|$ which contain Qf, so we can find a divisor on $|H|$ which contains Q (and $y \in Q$) but not x.

We now check that the differential map $d\phi_H$ is an isomorphism at points $x \in S \setminus K$. In order to get this we will see that $|H|$ separates tangent directions at x, this is, if $t \in T_x(S)$, then there must be an element D in $|H|$ satisfying $x \in D$ but $t \notin T_x(D)$.

(1) Suppose $x \in Pf$ and $t \in T_x(Pf)$. Because $|H|$ is base-point-free it traces a very ample system on Pf, so there is an element D in $|H|$ which meets Pf transversally at x and $T_x(D) \neq T_x(Pf)$.

(2) Suppose $x \in Pf$ and $t \notin T_x(Pf)$. As $x \notin K$, x is not a base point of $|H-Pf|$. Then there exists a divisor D' in $|H-Pf|$, which does not contain x. Taking $D = D' + Pf$, we obtain an element of $|H|$ which contains x and its tangent direction is Pf, so $T_x(D) = T_x(Pf)$ and $t \notin T_x(D)$.

We have seen that ϕ_H is an isomorphism in $S \setminus K$. In fact, we can see that it is not an isomorphism at points of K.

Let $x \in K$ be a point in Pf. Since $x \in K$, x is a base point of the linear system $|H - Qf|$ for some $Q \in X$.

(1) If $Q \neq P$, all elements of $|H|$ which contain Qf pass through x, so the image of x by ϕ_H lies at a point of $\phi_H(Qf)$. Thus, there exists $y \in Qf$ with $\phi_H(y) = \phi_H(x)$ and ϕ_H is not bijective in K.

(2) Let $Q = P$. Let $C_1 \in |H|$ be a curve which meets Pf transversally at x. It exists because $|H|$ is base-point-free, so $h^0(\mathcal{O}_S(H-Pf)) < h^0(\mathcal{O}_S(H-x))$. Let $t_1 \in T_x(S)$ be the tangent vector to C_1 at x. Suppose that there is other curve $C_2 \in |H|$ which meets Pf transversally at x. Let $t_2 \in T_x(S)$ be its tangent vector at x. Suppose $\langle t_1 \rangle \neq \langle t_2 \rangle$. We can define both curves by local equations u_1 and u_2. Taking $u = \lambda_1 u_1 + \lambda_2 u_2$ we define a curve C on the linear system $|H|$. The tangent vector to C at x is $t = \lambda_1 t_1 + \lambda_2 t_2$. By a suite election of λ_1 and λ_2 we can suppose $t \neq 0$ (so C nonsingular at x) and $\langle t \rangle = T_x(Pf)$, because $\langle t_1 \rangle \neq \langle t_2 \rangle$ and $\langle t_1, t_2 \rangle = T_x(S)$.

On the other hand, we know that an unisecant irreducible curve can not be tangent to a generator. Then the curve C is on the linear system $|H - Pf|$ and it can be written as $C = Pf + C'$. By hypothesis, x is a base point of $|H - Pf|$, so $x \in C'$. From this, x is a singular point of C and we get a contradiction. Note, that we had supposed that there were two curves on $|H|$ which passed through x with different tangent directions. Then, we deduce that all nonsingular curves at x of $|H|$ have a unique tangent direction $\langle t_1 \rangle$ at x.

Finally, let us see that $d\phi_H$ is not an isomorphism. In other case, given the tangent vector $t_1 \in T_x(S)$, there must be a curve $D \in |H|$ with $t_1 \notin T_x(D)$. But, if D is nonsingular at x, then $T_x(D) = \langle t_1 \rangle$. If D is singular at x, then $T_x(D) = T_x(S)$ and $t_1 \in T_x(D)$.

QED

21 Remark. *This theorem yields information about the singular locus of a scroll in* $\mathbf{P}^N$. *Let* $R \subset \mathbf{P}^N$ *be a linearly normal scroll given by the ruled surface* $\mathbf{P}(\mathcal{E})$ *and the unisecant complete linear system* $|H|$ *on* $\mathbf{P}(\mathcal{E})$.

As $\mathbf{P}(\mathcal{E})$ *is smooth, if* ϕ_H *is an isomorphism in an open set* $U \subset \mathbf{P}(\mathcal{E})$, *then* R *is smooth at points of the image* $\phi_H(U)$. *The singular locus of* R *will be supported at points of* $R \setminus \phi_H(U)$.

Let us apply Proposition 12. Since $|X_0 + \mathfrak{b}f|$ *is base-point-free, we have that* $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P)f)) = h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 2$. *Furthermore, the linear system* $|X_0 + (\mathfrak{b} - P)f|$ *has a base point on* Qf *when* $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P - Q)f)) = h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P)f)) - 1$ *and it has* Qf *as a fixed component when* $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P - Q)f)) = h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P)f)$. *From this there can appear the following singularities:*

(1) If $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P - Q)f)) = h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 3$ *with* $P \neq Q$, *then the generators* $\phi_H(Pf)$ *and* $\phi_H(Qf)$ *meet at a unique point.*

(2) If $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - 2P)f)) = h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 3$, *then the generator* $\phi_H(Pf)$ *meets its infinitely near generator at a unique point. It is called torsal generator.*

(3) If $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - P - Q)f)) = h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 2$ *with* $P \neq Q$, *then the generators* $\phi_H(Pf)$ *and* $\phi_H(Qf)$ *coincide and we have a singular generator.*

(4) If $h^0(\mathcal{O}_S(X_0 + (\mathfrak{b} - 2P)f)) = h^0(\mathcal{O}_S(X_0 + \mathfrak{b}f)) - 2$, *then the generator* $\phi_H(Pf)$ *coincides with its infinitely near generator and it is again a singular generator.*

QED

22 Corollary. *Let* S *be a geometrically ruled surface and let* $|H| = |X_0 + \mathfrak{b}f|$ *be a complete linear system on* S. $|H|$ *is very ample if and only if* $h^0(\mathcal{O}_S(H - (P + Q)f)) - h^0(\mathcal{O}_S(H)) - 4$ *for any* $P, Q \in X$.

PROOF. $|H|$ is very ample if it is base-point-free and the morphism $\phi_H : S \longrightarrow \mathbf{P}^N$ is an isomorphism.

Let us suppose $|H|$ is very ample. Since it is base-point-free and according to Corollary 13, we deduce that $h^0(\mathcal{O}_S(H - Pf)) = h^0(\mathcal{O}_S(H)) - 2$ for any $P \in X$. By the above theorem, as ϕ_H is an isomorphism at any point, $|H - Pf|$ is always base-point-free and $h^0(\mathcal{O}_S(H - (P + Q)f)) = h^0(\mathcal{O}_S(H - Pf)) - 2$ for any $Q \in X$. It follows that:

$$h^0(\mathcal{O}_S(H - (P + Q)f)) = h^0(\mathcal{O}_S(H)) - 4.$$

Conversely, let us suppose $h^0(\mathcal{O}_S(H-(P+Q)f)) = h^0(\mathcal{O}_S(H)) - 4$ for any $P, Q \in X$. If $|H|$ were not base-point-free there would be a point $P \in X$ which satisfies $h^0(\mathcal{O}_S(H-Pf)) \geq h^0(\mathcal{O}_S(H)) - 1$ so $h^0(\mathcal{O}_S(H-(P+Q)f)) \geq h^0(\mathcal{O}_S(H)) - 3$, which contradicts the hypothesis. If ϕ_H were not an isomorphism at a point x, by the above theorem, x would be a base point of $|H - Pf|$ and there would be a $Q \in X$ satisfying $h^0(\mathcal{O}_S(H-(P+Q)f)) \geq h^0(\mathcal{O}_S(H-Pf)) - 1$, so $h^0(\mathcal{O}_S(H-(P+Q)f)) \geq h^0(\mathcal{O}_S(H)) - 3$, which contradicts the hypothesis again. QED

23 Proposition. *Let D be a section of a ruled surface S and let $|H| = |D + \mathfrak{b}f|$ be a base-point-free complete linear system. Let $\phi_H : S \longrightarrow \mathbf{P}^N$ the regular map defined by $|D + \mathfrak{b}f|$. If $\mathfrak{b}$ is very ample, then ϕ_H is an isomorphism out of D.*

PROOF. Applying Theorem 20, we see that it is sufficient to check that $|D + (\mathfrak{b} - P)f|$ is base-point-free out of D.

Let $x \in S \setminus D$ be a point in the generator Qf. Let $P \in X$. Since $\mathfrak{b}$ is very ample, $\mathfrak{b} - P$ is base-point-free and we can take a divisor $\mathfrak{b}' \sim \mathfrak{b} - P$ which does not contain Q. Then, $D + \mathfrak{b}'f \sim D + (\mathfrak{b} - P)f$ does not contain x and this is not a base point of $|D + (\mathfrak{b} - P)f|$. QED

24 Proposition. *If $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are very ample divisors on X and $\mathfrak{b}$ is nonspecial, then the complete linear system $|H| = |X_0 + \mathfrak{b}f|$ is very ample.*

PROOF. Because $\mathfrak{b}$ is nonspecial and very ample, given $P \in X$, $\mathfrak{b} - P$ is base-point-free and nonspecial.

Applying Lemma 10 we see that

$$h^0(\mathcal{O}_S(H-(P+Q)f)) = h^0(\mathcal{O}_X(\mathfrak{b} - P - Q)) + h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e} - P - Q))$$

for any $P, Q \in X$.

Since $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are very ample, $h^0(\mathcal{O}_X(\mathfrak{b} - P - Q)) = h^0(\mathcal{O}_X(\mathfrak{b})) - 2$ and $h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e} - P - Q)) = h^0(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e})) - 2$; we obtain $h^0(\mathcal{O}_S(H-(P+Q)f)) = h^0(\mathcal{O}_S(H)) - 4$ and by Corollary 22, $|H|$ is very ample. QED

3 Decomposable ruled surfaces.

25 Definition. Let $S \longrightarrow X$ be a geometrically ruled surface over a nonsingular curve X of genus g. The ruled surface is called *decomposable* if $\mathcal{E}_0$ is a direct sum of two invertible sheaves.

The invariant e on a decomposable geometrically ruled surface is positive:

26 Theorem. *Let S be a ruled surface over the curve X of genus g, determined by a normalized locally free sheaf $\mathcal{E}_0$.*

(1) *If* $\mathcal{E}_0$ *is decomposable then* $\mathcal{E}_0 \cong \mathcal{O}_C \oplus \mathcal{L}$ *for some* $\mathcal{L}$ *with* $\deg(\mathcal{L}) \leq 0$. *Therefore,* $e \geq 0$. *All values of* $e \geq 0$ *are possible.*

(2) *If* $\mathcal{E}_0$ *is indecomposable, then* $-g \leq e \leq 2g-2$.

PROOF. See [8], V, 2.12. and [15]. QED

27 Remark. *Geometrically, a decomposable ruled surface has two disjoint unisecant curves. These unisecant curves are given by surjections* $\mathcal{E}_0 \cong \mathcal{O}_X \oplus \mathcal{O}_X(\mathfrak{e}) \longrightarrow \mathcal{O}_X(\mathfrak{e}) \longrightarrow 0$ *and* $\mathcal{E}_0 \cong \mathcal{O}_X \oplus \mathcal{O}_X(\mathfrak{e}) \longrightarrow \mathcal{O}_X \longrightarrow 0$. *We denote them by* X_0 *and* X_1. *According to ([8], V,2.9), we know* $X_1 \sim X_0 - \mathfrak{e}f$.

Since $\mathcal{E}_0$ *is decomposable the equality* $h^i(\mathcal{O}_S(X_0 + \mathfrak{b}f)) = h^i(\mathcal{O}_X(\mathfrak{b})) + h^i(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))$ *holds always, because* $H^i(\mathcal{O}_S(X_0+\mathfrak{b})) \cong H^i(\mathcal{E}_0 \otimes \mathcal{O}_X(\mathfrak{b}))$ *and* $\mathcal{E}_0 \cong \mathcal{O}_X \oplus \mathcal{O}_X(\mathfrak{e})$. QED

28 Proposition. *Let* S *be a decomposable geometrically ruled surface. Let* $|H| = |X_0 + \mathfrak{b}f|$ *be a complete linear system. Then,* $x \in Pf$ *is a base point of* $|H|$ *if and only if it satisfies some of the following conditions:*

(1) P *is a base point of* $\mathfrak{b}$ *and* $x = Pf \cap X_1$.

(2) P *is a base point of* $\mathfrak{b}+\mathfrak{e}$ *and* $x = Pf \cap X_0$.

(3) P *is a common base point of* $\mathfrak{b}$ *and* $\mathfrak{b}+\mathfrak{e}$. *In this case* Pf *is fixed component of* $|H|$.

Moreover, $|H|$ *is base-point-free if and only if* $\mathfrak{b}$ *and* $\mathfrak{b}+\mathfrak{e}$ *are base-point-free.*

PROOF. Let us examine the trace of the linear system $|X_0 + \mathfrak{b}f|$ on X_0:

$$H^0(\mathcal{O}_S(\mathfrak{b}f)) \hookrightarrow H^0(\mathcal{O}_S(X_0+\mathfrak{b}f)) \xrightarrow{\alpha} H^0(\mathcal{O}_{X_0}(X_0+\mathfrak{b}f)) \cong H^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))$$

According to the above remark, we know that $h^0(\mathcal{O}_S(X_0+\mathfrak{b}f)) = h^0(\mathcal{O}_X(\mathfrak{b})) + h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))$, so the map α is a surjection and $|H|$ traces the complete linear system $|\mathfrak{b}+\mathfrak{e}|$ on X_0. Thus, if P is a base point of $\mathfrak{b}+\mathfrak{e}$, then any divisor of $|H|$ meets X_0 at $X_0 \cap Pf$ and conversely.

The same reasoning applies to the trace of $|H|$ on X_1. Since $H^i(\mathcal{O}_{X_1}(X_0 + \mathfrak{b}f)) \cong H^i(\mathcal{O}_X(\mathfrak{b}))$, we can see that P is a base point of $\mathfrak{b}$ if and only if any divisor of $|H|$ meets X_1 at $X_1 \cap Pf$.

Finally, by Remark 27, we conclude

$$\begin{aligned} &h^0(\mathcal{O}_S(H)) - h^0(\mathcal{O}_S(H-Pf)) = \\ &= (h^0(\mathcal{O}_X(\mathfrak{b})) - h^0(\mathcal{O}_X(\mathfrak{b}-P))) + (h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e})) - h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}-P))). \end{aligned}$$

By Proposition 14, we see:

(1) $|H|$ is base-point-free if and only if $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ are base-point-free.

(2) $|H|$ has a unique base point on Pf if and only if P is a base point of $\mathfrak{b}$ or $\mathfrak{b}+\mathfrak{e}$, but not both.

(3) $|H|$ has Pf as a fixed component if and only if P is a common base point of $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$.

QED

29 Remark. *The above proof shows us that a complete linear system $|X_0 + \mathfrak{b}f|$ traces the complete linear systems $|\mathfrak{b}+\mathfrak{e}|$ and $|\mathfrak{b}|$ on curves X_0 and X_1. Hence, when the linear system $|X_0+\mathfrak{b}f|$ is base-point-free and it defines a regular map on S, X_0 and X_1 apply on linearly normal curves given by the linear systems $|\mathfrak{b}+\mathfrak{e}|$ and $|\mathfrak{b}|$ on X.* QED

30 Theorem. *Let S be a decomposable ruled surface. The generic element of the complete linear system $|X_0+\mathfrak{b}f|$ is irreducible if and only if $\mathfrak{b}\sim 0$, $\mathfrak{b}\sim -\mathfrak{e}$ or $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ are effective without common base points.*

Proof. Let us first suppose there exists an irreducible element $D\sim X_0+\mathfrak{b}f$. If $D\sim X_0$, then $\mathfrak{b}\sim 0$ and if $D\sim X_1$, then $\mathfrak{b}\sim -\mathfrak{e}$. In other case, D meets X_0 and X_1 at effective divisors, so $\pi_*(D\cap X_0)\sim \mathfrak{b}+\mathfrak{e}$ and $\pi_*(D\cap X_1)\sim\mathfrak{b}$ must be effective. Furthermore, if $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ had a common base point P, then, by Proposition 28, Pf would be a fixed component of the linear system and this would not have irreducible elements.

Conversely, if $\mathfrak{b}\sim 0$ or $\mathfrak{b}\sim -\mathfrak{e}$, then $X_0+\mathfrak{b}f\sim X_0$ or $X_0+\mathfrak{b}f\sim X_1$ and the generic element is irreducible.

If $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ are effective divisors without common base points, the generic point P is not a base point of $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$, because they are effective. Thus, by Remark 27, $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) = h^0(\mathcal{O}_S(X_0+\mathfrak{b}f))-2$. A finite number of points P can be base points of $\mathfrak{b}$ or $\mathfrak{b}+\mathfrak{e}$ (but not both), so, in this case, $h^0(\mathcal{O}_S(X_0+(\mathfrak{b}-P)f)) = h^0(\mathcal{O}_S(X_0+\mathfrak{b}f))-1$. Applying Proposition 18 the theorem follows. QED

31 Corollary. *If $\mathbf{P}(\mathcal{E})$ is a decomposable ruled surface it holds $X_0^2=-e$, $X_1^2=e$ and for any other unisecant curve D not linearly equivalent to these, $D^2\geq e+2$. In particular:*

(1) If $D\equiv X_0$ then $D\sim X_0$ when $e>0$ and $D\sim X_0$ or $D\sim X_1$ when $e=0$. Moreover, if $D\sim X_0$ and $\mathfrak{e}\not\sim 0$, then $D=X_0$.

(2) If $D\equiv X_1$ then $D\sim X_1$ when $e>0$ and $D\sim X_0$ or $D\sim X_1$ when $e=0$. Moreover, if $D\sim X_1$, $e=0$ and $\mathfrak{e}\not\sim 0$, then $D=X_1$.

PROOF. We know $X_0^2 = -e$ and $X_1^2 = e$. By the above proposition if $D \sim X_0 + \mathfrak{b}f$ is an irreducible curve no linearly equivalent to the first, then $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ must be effective divisors, so $\deg(\mathfrak{b}) \geq e$ and $\deg(\mathfrak{b}) \geq 0$. Since $\mathfrak{b}+\mathfrak{e}$ is effective, if $\deg(\mathfrak{b}) - e$, then $\mathfrak{b} \sim -\mathfrak{e}$ and $D \sim X_1$. From this, necessarily $\deg(\mathfrak{b}) \geq e+1$ and $D^2 = 2\deg(b) - e \geq e+2$. QED

32 Proposition. *Let S be a decomposable ruled surface. The complete linear system $|X_1| = |X_0 - \mathfrak{e}f|$ satisfies following conditions:*

(1) The set of reducible elements of $|X_1|$ is exactly $\{X_0 + \mathfrak{b}f/\mathfrak{b} \sim -\mathfrak{e}\}$.

(2) If P is not a base point of $-\mathfrak{e}$, then there exists an irreducible curve of $|X_1|$ passing through any point of Pf not in X_0.

(3) If P is a base point of $-\mathfrak{e}$, all irreducible curves of $|X_1|$ pass through a unique base point on Pf.

PROOF. (1) Let $D + \mathfrak{b}f$ be a reducible element of $|X_1|$. Since $D + \mathfrak{b}f \sim X_0 - \mathfrak{e}f$, we have $\deg(\mathfrak{b}) < \deg(-\mathfrak{e})$ and $D \sim X_0 + (-\mathfrak{b} - \mathfrak{e})f$. From this, D is an irreducible curve of self-intersection strictly smaller than X_1. By the above corollary, D must be X_0 and $\mathfrak{b} \sim -\mathfrak{e}$.

(2) According to Proposition 28, we know that if P is not a base point of $-\mathfrak{e}$, the linear system $|X_1|$ has not base points on Pf. Hence, it traces the complete linear system of points of $\mathbf{P}^1$ on the generator. For each point x of Pf there passes an effective divisor of $|X_1|$ not containing the generator. But, as we see at 1, if $x \notin X_0$, the divisor must be irreducible.

(3) According to Proposition 28, if P is a base point of $-\mathfrak{e}$, the linear system $|X_1|$ has a base point on the generator Pf, so all irreducible elements of the linear system pass through it.

QED

33 Theorem. *Let S be a decomposable ruled surface and let $|H| = |X_0 + \mathfrak{b}f|$ be a complete linear system on S. Then:*

(1) If $\mathfrak{b}$ is very ample and $\mathfrak{b}+\mathfrak{e}$ is base-point-free, then $|H|$ defines an isomorphism in $S \setminus X_0$.

(2) If $\mathfrak{b}+\mathfrak{e}$ is very ample and $\mathfrak{b}$ is base-point-free, then $|H|$ defines an isomorphism in $S \setminus X_1$.

(3) $|H|$ is very ample if and only if $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ are very ample.

PROOF. By Proposition 28, if $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ are base-point-free the linear system $|X_0+\mathfrak{b}f|$ is base-point-free.

Since $X_0+\mathfrak{b}f \sim X_1+(\mathfrak{b}+\mathfrak{e})f$, we can apply Proposition 23. Taking $D=X_0$ or $D=X_1$, we obtain the assertions 1 and 2.

The third equivalence is consequence of Corollary 22: $|H|$ is very ample if and only if $h^0(\mathcal{O}_S(H-(P+Q)f)) = h^0(\mathcal{O}_S(H))-4$. Now, it is sufficient to remark that in a decomposable ruled surface it holds $h^0(\mathcal{O}_S(H)) = h^0(\mathcal{O}_X(\mathfrak{b}))+h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))$ and $h^0(\mathcal{O}_S(H-(P+Q)f)) = h^0(\mathcal{O}_X(\mathfrak{b}-P-Q))+h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}-P-Q))$. QED

34 Theorem. *Let S be a decomposable ruled surface and let $|H|=|X_0+\mathfrak{b}f|$ be a base-point-free linear system. Let $\phi_H : S \longrightarrow R \subset \mathbf{P}^N$ be the map defined by $|H|$. Then:*

(1) $N = h^0(\mathcal{O}_X(\mathfrak{b}))+h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))-1$ and $\deg(S)=2\deg(\mathfrak{b})-e$.

(2) $\phi_H(X_0)$ and $\phi_H(X_1)$ are linearly normal curves given by the maps $\phi_{\mathfrak{b}+\mathfrak{e}} : X \longrightarrow \phi_H(X_0)$ and $\phi_{\mathfrak{b}} : X \longrightarrow \phi_H(X_1)$. Moreover, they lie in complementary disjoint spaces of $\mathbf{P}^N$.

(3) The singular locus of R is supported at most in $\phi_H(X_0)$, $\phi_H(X_1)$ and the set $K=\{\phi_H(Pf)/|\mathfrak{b}-P|$ and $|\mathfrak{b}+\mathfrak{e}-P|$ have a common base point$\}$. If $K=S$ the map ϕ_H is not birational. If $K\neq S$ the map ϕ_H is birational and singularities of R are exactly:

a. Singular unisecant curves $\phi_H(X_0)$ or $\phi_H(X_1)$ if the regular maps $\phi_{\mathfrak{b}+\mathfrak{e}} : X \longrightarrow \phi_H(X_0)$ or $\phi_{\mathfrak{b}} : X \longrightarrow \phi_H(X_1)$ are not birational.

b. Isolated singularities on $\phi_H(X_0)$ or $\phi_H(X_1)$ when $\mathfrak{b}+\mathfrak{e}$ or $\mathfrak{b}$ are not very ample but they define birational maps. They correspond to the generators Pf and Qf meeting at a point on $\phi_H(X_i)$. If $P=Q$, the generator Pf is a torsal generator.

c. Double generators when $\mathfrak{b}+\mathfrak{e}-P$ and $\mathfrak{b}-P$ have a common base point Q. Then $\phi_H(Pf)=\phi_H(Qf)$. If $P=Q$, the generator $\phi_H(Qf)$ coincides with its infinitely near generator.

PROOF. The linear system $|H|$ is base-point-free, so it defines a regular map $\phi_H : S \longrightarrow \mathbf{P}^N$. The hyperplane sections of the scroll R correspond to divisors of the linear system $|H|$. If we denote $N_0 = h^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))-1$ and $N_1 = h^0(\mathcal{O}_X(\mathfrak{b}))-1$, then $N = h^0(\mathcal{O}_S(X_0+\mathfrak{b}f))-1 = N_0+N_1+1$ and $\deg(R)=\deg(H)=(X_0+\mathfrak{b}f)^2 = 2\deg(\mathfrak{b})-e$.

At Remark 29 we saw that curves $\phi_H(X_0)$ and $\phi_H(X_1)$ are linearly normal and they are defined by maps $\phi_{\mathfrak{b}+\mathfrak{e}} : X \longrightarrow \phi_H(X_0) \subset \mathbf{P}(H^0(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))^\vee)$ and

$\phi_{\mathfrak{b}} : X \longrightarrow \phi_H(X_1) \subset \mathbf{P}(H^0(\mathcal{O}_X(\mathfrak{b}))^\vee)$. Thus $\phi_H(X_0)$ lies in $\mathbf{P}^{N_0}$ and $\phi_H(X_1)$ lies in $\mathbf{P}^{N_1}$. Since $h^0(\mathcal{O}_S(H - X_0 - X_1)) = 0$, there are not hyperplane sections containing both curves. Hence these lie in complementary disjoint spaces.

Finally, let us study the singular locus of R. Applying Theorem 20, we know that ϕ_H is an isomorphism out of base points of linear systems $|H-Pf|$, $P \in X$.

As we saw in Proposition 28, in a decomposable ruled surface base points of linear system lie in X_0 or X_1, except when there is a base generator. In this case, $\mathfrak{b} - P$ and $\mathfrak{b} + \mathfrak{e} - P$ have a common base point.

It follows that the singular locus of R is supported at most in $\phi_H(X_0)$, $\phi_H(X_1)$ and $K = \{\phi_H(Pf)/|\mathfrak{b} - P|$ and $|\mathfrak{b} + \mathfrak{e} - P|$ have a common base point$\}$. If $K = S$, the map ϕ_H is not an isomorphism at any point so it is not birational. On the contrary, if $K \neq S$ we can see which are exactly the singularities of R. We will reason on the curve $\phi_H(X_0)$, but similar arguments apply to the curve $\phi_H(X_1)$.

If the morphism $\phi_{\mathfrak{b}+\mathfrak{e}} : X \longrightarrow \phi_H(X_0)$ is not birational, then it is a $k:1$ map and we have an unisecant singular curve on the scroll.

If the morphism $X \longrightarrow \phi_H(X_0)$ is birational, the map given by $\mathfrak{b} + \mathfrak{e}$ is 1:1 in an open set, but isolated singularities can appear. This happens when the divisor $\mathfrak{b} + \mathfrak{e} - P$ has a base point Q for some $P \in X$. Then the linear system $|H - Pf|$ have a base point at $X_0 \cap Qf$:

- If Q is not a base point of $\mathfrak{b} - P$, the linear system $|H - Pf|$ has no more base points in Qf and then the unique singular point in $\phi_H(Pf)$ lies at $\phi_H(X_0) \cap \phi_H(Qf) \cap \phi_H(Pf)$. The generators $\phi_H(Qf)$ and $\phi_H(Pf)$ meet at a point. Moreover, if $Q = P$, the generator $\phi_H(Pf)$ meets its infinitely near generator at a unique point and it is a torsal generator.

- If Q is a base point of $\mathfrak{b} - P$, the linear system $|H - Pf|$ has Qf as a fixed component. Then, both generators coincide in the image, so $\phi_H(Pf) = \phi_H(Qf)$ is a singular generator. If $P = Q$ the generator $\phi_H(Qf)$ coincides with its infinitely near generator. QED

We will finish this section by studying conditions for m-secant divisors to be very ample on a decomposable ruled surface.

We begin with a technical result on computing the dimension of a m-secant linear system on a decomposable ruled surface. It is known that:

$$h^i(\mathcal{O}_S(X_0 + \mathfrak{b}f)) = h^i(\mathcal{O}_X(\mathfrak{b})) + h^i(\mathcal{O}_X(\mathfrak{b} + \mathfrak{e})).$$

Let us see the following generalization:

35 Lemma. *Let $|mX_0 + \mathfrak{b}f|$ be a m-secant linear system on a decomposable* *ruled surface S. Then,*

$$h^i(\mathcal{O}_S(mX_0+\mathfrak{b}f))=\sum_{k=0}^{m}h^i(\mathcal{O}_X(\mathfrak{b}+k\mathfrak{e})),\quad i\geq 0$$

PROOF. We note that because S is a surface, then $h^i(\mathcal{O}_S(H))=0$ when $i>2$. The proof is by induction on m:

It is clear that $h^i(\mathcal{O}_S(\mathfrak{b}f))=h^i(\mathcal{O}_X(\mathfrak{b}))$ and in particular $h^2(\mathcal{O}_S(\mathfrak{b}f))=0$.

Assuming the formula holds for $m-1$, we will prove it for m. Let $|H|=|mX_0+\mathfrak{b}f|$ be a m-secant system. Consider the exact sequence:

$$0\longrightarrow\mathcal{O}_S(H-X_1)\longrightarrow\mathcal{O}_S(H)\longrightarrow\mathcal{O}_{X_1}(H)\longrightarrow 0$$

Since $X_1\sim X_0-\mathfrak{e}f$ and introducing cohomology, we have:

$$\begin{array}{ccccccc}0&\to&H^0(\mathcal{O}_S(H-X_1))&\to&H^0(\mathcal{O}_S(H))&\overset{\alpha_0}{\to}&H^0(\mathcal{O}_X(\mathfrak{b}))&\to\\&\to&H^1(\mathcal{O}_S(H-X_1))&\to&H^1(\mathcal{O}_S(H))&\overset{\alpha_1}{\to}&H^1(\mathcal{O}_X(\mathfrak{b}))&\to\\&\to&H^2(\mathcal{O}_S(H-X_1))&\to&H^2(\mathcal{O}_S(H))&\to&0\end{array}$$

where $H-X_1\sim(m-1)X_0+(\mathfrak{b}+\mathfrak{e})f$. The map α_0 is a surjection because given $\mathfrak{b}'\sim\mathfrak{b}$ we have $\mathfrak{b}'=\alpha(mX_0+\mathfrak{b}'f)$, where $mX_0+\mathfrak{b}'f\sim mX_0+\mathfrak{b}f$. α_1 is a surjection too, because $h^2(\mathcal{O}_S((m-1)X_0+(\mathfrak{b}+\mathfrak{e})f))=0$ by induction hypothesis. We conclude that:

$$\begin{aligned}h^i(\mathcal{O}_S(mX_0+\mathfrak{b}f))&=h^i(\mathcal{O}_X(\mathfrak{b}))+\sum_{k=0}^{m-1}h^i(\mathcal{O}_X(\mathfrak{b}+(k+1)\mathfrak{e}))=\\&=\sum_{k=0}^{m}h^i(\mathcal{O}_X(\mathfrak{b}+k\mathfrak{e}))\end{aligned}$$

QED

We will now restrict our attention to study the trace of a m-secant linear system $|mX_0+\mathfrak{b}f|$ on a generator Pf. Consider the exact sequence:

$$0\longrightarrow\mathcal{O}_S(mX_0+(\mathfrak{b}-P)f)\longrightarrow\mathcal{O}_S(mX_0+\mathfrak{b}f)\longrightarrow\mathcal{O}_{Pf}(mX_0+\mathfrak{b}f)\longrightarrow 0$$

Introducing cohomology:

$$0\longrightarrow H^0(\mathcal{O}_S(mX_0+(\mathfrak{b}-P)f))\longrightarrow H^0(\mathcal{O}_S(mX_0+\mathfrak{b}f))\overset{\alpha}{\longrightarrow}H^0(\mathcal{O}_{\mathbf{P}^1}(m))$$

We see that $|mX_0+\mathfrak{b}f|$ traces a linear subsystem of the complete linear system of divisors of degree m on $\mathbf{P}^1$ on the generator Pf.

Let us introduce homogeneous coordinates $[x_0:x_1]$ on $\mathbf{P}^1$, where point $[0:1]$ is $X_0\cap Pf$ and point $[0:1]$ is $X_1\cap Pf$.