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Generalized Topp-Leone-G power series class of distributions: properties and applications

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This note is concerned with the construction, development and applications of a new class of distributions referred to as the Generalized Topp-Leone-G Power Series class of distributions. More importantly, this new generalized class of distributions can be expressed as an infinite linear combination of exponentiated-G distributions, which allows us to develop and obtain the important statistical and mathematical properties. Monte Carlo simulations are conducted to established the consistency of the estimation process. Applications in several areas are presented to illustrate the importance and usefulness of this new class of distributions.

keywords: Generalized Topp-Leone-G, power series distribution, maximum likelihood estimation.

1 Introduction

In the context of survival analysis, modeling lifetime data has become popular. Studies in survival analysis involve examining the lifetimes of both biological and mechanical systems. To model these types of data, several distributions have been introduced in recent years. The fundamental idea behind these distributions is that the lifetime of a system with N components and a continuous random variable, X_i, which denotes the

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lifespan of the i^{th} component, can be represented by a non-negative random variable Z=min$(X_1, X_2, \ldots, X_N)$ if the components are in a series or Z=max$(X_1, X_2, \ldots, X_N)$ if the components are parallel. The power series class of distributions was proposed and studied by Noack (1950). This class of distributions includes binomial, geometric, logarithmic and Poisson distributions as special cases. These distributions may not be useful, however, when a random variable has a high probability of taking the value zero. As a result, a truncated distribution is more appropriate in such situations. These distributions are described in more detail in Johnson et al. (2005) in the context of univariate discrete distributions. Authors have developed new distributions using the power series class of distributions in recent years. Many of the new distributions have been constructed from a mixture of well-known distributions and power series distributions. Some of the power series distributions include the Weibull power series class of distributions by Morais and Barreto-Souza (2011), the generalized exponential power series class of distributions by Mahmoudi and Jafari (2012), the Kumaraswamy power series class of distributions by Bidram and Nekoukhou (2013), the exponentiated generalized power series class of distributions by Oluyede et al. (2020), the Topp-Leone-G power series class of distributions by Makubate et al. (2021), and the odd Weibull Topp–Leone–G power series class of distributions by Oluyede et al. (2021).

The Generalized Topp-Leone-G Power Series (GTL-GPS) class of distributions is motivated by three factors, which can be applied in a variety of real-world situations:

(1) The GTL-GPS class of distributions can arise in many industrial applications and biological organisms from the stochastic representation Z=min$(X_1, X_2, \ldots, X_N)$.

(2) It is possible to use the GTL-GPS distribution class to model appropriately the time to the first failure of a system of identical components.

(3) There are some interesting behaviors observed when looking at the GTL-GPS class of distributions, such as bathtub failure rates, upside bathtub failure rates and decreasing-decreasing-increasing failure rates, all of which are more likely to be encountered in practice.

The Topp-Leone generated family of distributions was introduced by Al-Shomrani et al. (2016). The cumulative distribution function (cdf) and probability density function (pdf) of the Generalized Topp-Leone-G (GTL-G) family of distributions is given by

$$F_{GTL-G}(x; b, \beta, \xi) = 1 - \left[1 - (1 - \bar{G}^2(x; \xi))^b\right]^\beta \tag{1}$$

and

$$\begin{aligned} f_{GTL-G}(x; b, \beta, \xi) &= 2b\beta \left[1 - (1 - \bar{G}^2(x; \xi))^b\right]^{\beta-1} (1 - \bar{G}^2(x; \xi))^{b-1} \\ &\times \bar{G}(x; \xi) g(x; \xi), \end{aligned} \tag{2}$$

respectively, for $x > 0$, $b, \beta > 0$ and parameter vector ξ, where $\bar{G}(x; \xi) = 1 - G(x; \xi)$.

Suppose N is a discrete random variable following a power series distribution and assumed to be truncated at zero with the probability mass function (pmf) given by

$$P(N=n)=\frac{a_n\theta^n}{C(\theta)}, \quad n=1,2,..., \tag{3}$$

where $C(\theta)=\sum_{n=1}^{\infty}a_n\theta^n$ is finite, $\theta>0$ and $\{a_n\}_{n\geq 1}$ a sequence of positive real numbers. The power series family of distributions includes Poisson, binomial, geometric and logarithmic distributions.

2 The Model

In this section, we present the generalized Topp-Leone-G power series distribution and its sub-models. Let X be a random variable such that X_i, for $i=1,2,...,N$ denote the time to failure of a device due to the i^{th} defect with the assumption that the X_i's are independent and identically distributed (iid) GTL-G random variables and $X_{(1)}$ =min$(X_1,X_2,....X_N)$, with the distribution of N given by equation (3). The GTL-GPS class of distributions is defined by the marginal cdf, say $F_\theta(x)$, and is given by

$$F_\theta(x) \quad = \quad 1-\frac{C\left(\theta\left[1-(1-\bar{G}^2(x;\xi))^b\right]^\beta\right)}{C(\theta)}, \tag{4}$$

with the corresponding pdf and hazard rate function (hrf) given as

$$\begin{aligned} f_\theta(x) &= 2b\beta\theta\left[1-(1-\bar{G}^2(x;\xi))^b\right]^{\beta-1}(1-\bar{G}^2(x;\xi))^{b-1}\bar{G}(x;\xi)g(x;\xi) \\ &\times \frac{C'\left(\theta\left[1-(1-\bar{G}^2(x;\xi))^b\right]^\beta\right)}{C(\theta)} \end{aligned} \tag{5}$$

and

$$\begin{aligned} h_\theta(x) &= 2b\beta\theta\left[1-(1-\bar{G}^2(x;\xi))^b\right]^{\beta-1}(1-\bar{G}^2(x;\xi))^{b-1}\bar{G}(x;\xi)g(x;\xi) \\ &\times \frac{C'\left(\theta\left[1-(1-\bar{G}^2(x;\xi))^b\right]^\beta\right)}{C\left(\theta\left[1-(1-\bar{G}^2(x;\xi))^b\right]^\beta\right)}, \end{aligned} \tag{6}$$

respectively, for $b,\beta,\theta>0$ and parameter vector ξ. Sub-classes of the GTL-GPS class of distributions are given in Table 1.

3 Some Model Properties

In this section, we present some statistical properties of the proposed model. The statistical properties considered include the quantile function, linear representation of the density function, moments, generating function, probability weighted moments, distribution of order statistics and Rényi entropy of the GTL-GPS class of distributions.

Table 1: Sub-classes of the GTL-GPS class of distributions

Distribution	a_n	$C(\theta)$	cdf
GTL-G Poisson	$(n!)^{-1}$	$e^{\theta}-1$	$1-\frac{e^{\left(\theta\left(\left[1-(1-\bar{G}^2(x;\xi))^b\right]^{\beta}\right)\right)}-1}{e^{\theta}-1}$
GTL-G Geometric	1	$\theta(1-\theta)^{-1}$	$1-\frac{(1-\theta)\left(\left[1-(1-\bar{G}^2(x;\xi))^b\right]^{\beta}\right)}{\left(1-\theta\left(\left[1-(1-\bar{G}^2(x;\xi))^b\right]^{\beta}\right)\right)}$
GTL-G Logarithmic	n^{-1}	$-\log(1-\theta)$	$1-\frac{\log\left(1-\theta\left(\left[1-(1-\bar{G}^2(x;\xi))^b\right]^{\beta}\right)\right)}{\log(1-\theta)}$
GTL-G Binomial	$\binom{m}{n}$	$(1+\theta)^m-1$	$1-\frac{\left(1+\theta\left(\left[1-(1-\bar{G}^2(x;\xi))^b\right]^{\beta}\right)\right)^m-1}{(1+\theta)^m-1}$

3.1 Quantile Function

The quantile function of the GTL-GPS class of distributions can be computed by inverting the non-linear equation $F_\theta(x) = u$, $0 \leqslant u \leqslant 1$, such that

$$1-\frac{C\left(\theta\left[1-(1-\bar{G}^2(x;\xi))^b\right]^{\beta}\right)}{C(\theta)}=u.$$

By following the detailed derivations in the Appendix section, the quantile function of GTL-GPS class of distributions is given by

$$Q_\theta(u)=G^{-1}\left[1-\left(1-\left(1-\left[\frac{C^{-1}\left(C(\theta)(1-u)\right)}{\theta}\right]^{\frac{1}{\beta}}\right)^{\frac{1}{b}}\right)^{\frac{1}{2}}\right], \tag{7}$$

where G^{-1} and C^{-1} represent the inverse functions of G and C, respectively.

3.2 Linear Representation of the Density Function

In this subsection, we present a series expansion of the density function. The GTL-GPS density can be expressed as an infinite linear combination of the exponentiated-G (Exp-G) densities. *See Appendix section for derivations.* The series representation of the GTL-GPS pdf is given by

$$\begin{aligned}
f_\theta(x) &= 2b\beta\theta\sum_{n=1}^{\infty}\frac{na_n\theta^n}{C(\theta)}\sum_{i,j,k=0}^{\infty}(-1)^{i+j+k}\binom{\beta n-1}{i}\binom{b(i+1)-1}{i}\binom{2j+1}{k} \\
&\times \frac{(k+1)}{(k+1)}g(x;\xi)G^k(x;\xi) \\
&= \sum_{k=0}^{\infty}D_{k+1}g^*_{k+1}(x;\xi),
\end{aligned}$$

where

$$D_{k+1} = 2b\beta\theta \sum_{n=1}^{\infty} \frac{n a_n \theta^n}{C(\theta)} \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j+k}}{(k+1)} \binom{\beta n-1}{i} \binom{b(i+1)-1}{i} \binom{2j+1}{k} \tag{8}$$

and $g^*_{k+1}(x;\xi) = (k+1)g(x;\xi)G^k(x;\xi)$ is the Exp-G density with power parameter $(k+1)$. Hence, the pdf of the GTL-GPS class of distributions can be expressed as an infinite linear combination of Exp-G densities. The series expansion of the density function as well as other functions are meant to obtain the mathematical and statistical properties of the GTL-GPS class of distributions from those of the well known Exp-G distribution. Computations of the moments, skewness and kurtosis and other measures are done via numerical software packages such as R, SAS or MATLAB for specified baseline cdf G and function $C(\theta)$.

3.3 Moments and Generating Function

The r^{th} moment for the GTL-GPS class of distributions is given by

$$E(X^r) = \int_0^{\infty} x^r f_\theta(x)dx = \sum_{k=0}^{\infty} D_{k+1} E(Y^r_{k+1}),$$

where $E(Y^r_{k+1})$ is the r^{th} moment of Y_{k+1} which follows an Exp-G distribution with power parameter $k+1$ and D_{k+1} is given in equation (8). The moment generating function for the GTL-GPS class of distributions is given as

$$M_X(t) = E\left(e^{tX}\right) = \sum_{k=0}^{\infty} D_{k+1} E\left(e^{tY_{k+1}}\right),$$

where $E\left(e^{tY_{k+1}}\right)$ is the moment generating function of the Exp-G family of distributions with power parameter $(k+1)$ and D_{k+1} is given in equation (8).

3.4 Probability Weighted Moments (PWMs)

In this subsection, we present the probability weighted moments of the GTL-GPS class of distributions. The PWMs for a random variable X from the GTL-GPS class of distributions is given by

$$\eta_{a,r} = E(X^a[F(X)]^r) = \int_0^{\infty} x^a f_\theta(x)[F_\theta(x)]^r dx.$$

After following the detailed derivations in the Appendix section, we can write:

$$
\begin{aligned}
f_\theta(x)[F_\theta(x)]^r &= 2b\beta\theta \sum_{n,z=1}^{\infty} \sum_{s,i,j,k=0}^{\infty} \frac{n a_n \theta^n d_{z,s}}{(C(\theta))^{s+1}} (-1)^{s+i+j+k} \binom{r}{s} \binom{\beta(z+n)-1}{i} \\
&\times \binom{b(i+1)-1}{i} \binom{2j+1}{k} \frac{(k+1)}{(k+1)} g(x;\xi) G^k(x;\xi) \\
&= \sum_{k=0}^{\infty} W_{k+1} g^*_{k+1}(x;\xi),
\end{aligned}
$$

where

$$
\begin{aligned}
W_{k+1} &= 2b\beta\theta \sum_{n,z=1}^{\infty} \sum_{s,i,j=0}^{\infty} \frac{n a_n \theta^n d_{z,s}}{(C(\theta))^{s+1}} \frac{(-1)^{s+i+j+k}}{(k+1)} \binom{r}{s} \binom{\beta(z+n)-1}{i} \\
&\times \binom{b(i+1)-1}{i} \binom{2j+1}{k}
\end{aligned}
$$

and $g^*_{k+1}(x;\xi) = (k+1) g(x;\xi) G^k(x;\xi)$ is the Exp-G density with power parameter (k+1). Consequently, the PWMs of the GTL-GPS class of distributions by interchanging the integration and summation signs is given by

$$
\eta_{a,r} = \sum_{k=0}^{\infty} W_{k+1} \int_0^{\infty} x^a g^*_{k+1}(x;\xi) dx. \tag{9}
$$

3.5 Distribution of Order Statistics

The pdf of the i^{th} order statistics for the GTL-GPS class of distributions is presented in this subsection. Let $X_1, X_2, ...X_n$ be a random sample of size n from the GTL-GPS class of distributions, then the pdf of the i^{th} order statistic is given by

$$
f_{i:n}(x) = \frac{n!}{(i-1)!(n-1)!} \sum_{r=0}^{n-i} (-1)^r \binom{n-i}{r} f_\theta(x)[F_\theta(x)]^{i+r-1}.
$$

Based on the detailed derivation given in the Appendix section, we can obtain the pdf of the i^{th} order statistics from GTL-GPS class of distributions as follows:

$$
f_{i:n}(x) = \frac{n!}{(i-1)!(n-1)!} \sum_{r=0}^{n-i} \sum_{k=0}^{\infty} (-1)^r \binom{n-i}{r} V_{k+1} g^*_{k+1}(x;\xi), \tag{10}
$$

where

$$
\begin{aligned}
V_{k+1} &= 2b\beta\theta \sum_{n,z=1}^{\infty} \sum_{s,i,j=0}^{\infty} \frac{n a_n \theta^n d_{z,s}}{(C(\theta))^{s+1}} \frac{(-1)^{s+i+j+k}}{(k+1)} \binom{i+r-1}{s} \binom{\beta(z+n)-1}{i} \\
&\times \binom{b(i+1)-1}{i} \binom{2j+1}{k}
\end{aligned}
$$

and $g^*_{k+1}(x;\xi) = (k+1) g(x;\xi) G^k(x;\xi)$ is the Exp-G density with power parameter (k+1).

4 Rényi Entropy

In this subsection, we present Rényi entropy of the GTL-GPS class of distributions. Rényi entropy (Rényi et al., 1961) of a random variable X following GTL-GPS class of distributions is defined by

$$I_R(\nu) = (1-\nu)^{-1}\log\left[\int_0^\infty f_\theta^\nu(x)dx\right] \text{ for } \nu > 0 \ and \ \nu \neq 1,$$

where $f_\theta(x)$ is given in equation (5). By following the detailed derivations in the Appendix section, the Rényi entropy of the GTL-GPS class of distributions is given by

$$\begin{aligned} I_R(\nu) &= (1-\nu)^{-1}\log\left[\sum_{n=1}^{\infty}\sum_{i,j,k=0}^{\infty}\frac{nd_{\nu,n}(2b\beta\theta)^\nu\theta^{n-1}}{(C(\theta))^\nu}(-1)^{i+j+k}\binom{\beta\nu(n-1)}{i}\binom{b(i+\nu)-1}{j}\right.\\ &\times \left.\binom{2j+\nu}{k}\frac{1}{\left(\frac{k}{\nu}+1\right)^\nu}\int_0^\infty\left(\left(\frac{k}{\nu}+1\right)g(x;\xi)G^{\frac{k}{\nu}}(x;\xi)\right)^\nu dx\right]\\ &= (1-\nu)^{-1}\log\left[\sum_{k=0}^{\infty}R_k^*\exp(1-\nu)I_{REG}\right], \end{aligned}$$

where $I_{REG} = \int_0^\infty\left(\left(\frac{k}{\nu}+1\right)g(x;\xi)G^{\frac{k}{\nu}}(x;\xi)\right)^\nu dx$ is Rényi entropy of Exp-G densities with power parameter $\frac{k}{\nu}+1$ and

$$\begin{aligned} R_k^* &= \sum_{n=1}^{\infty}\sum_{i,j=0}^{\infty}\frac{nd_{\nu,n}(2b\beta\theta)^\nu\theta^{n-1}}{(C(\theta))^\nu}(-1)^{i+j+k}\binom{\beta\nu(n-1)}{i}\binom{b(i+\nu)-1}{j}\\ &\times \binom{2j+\nu}{k}\frac{1}{\left(\frac{k}{\nu}+1\right)^\nu}. \end{aligned}$$

5 Maximum Likelihood Estimation

Let $X \sim GTL-GPS(b,\beta,\theta,\xi)$ and $\Delta = (b,\beta,\theta,\xi)^T$ be an unknown parameter vector, then the log-likelihood function (ℓ_n) of a random sample of size n from the $GTL-GPS(b,\beta,\theta,\xi)$ class of distributions is given by

$$\begin{aligned} \ell_n(\Delta) &= n\log(2b\beta\theta) + (\beta-1)\sum_{i=1}^{n}\log\left[1-(1-\bar{G}^2(x_i;\xi))^b\right]\\ &+ (b-1)\sum_{i=1}^{n}\log[1-\bar{G}^2(x_i;\xi)] + \sum_{i=1}^{n}\log[\bar{G}(x_i;\xi)] + \sum_{i=1}^{n}\log[g(x_i;\xi)]\\ &+ \sum_{i=1}^{n}\log\left[C'\left(\theta\left[1-(1-\bar{G}^2(x_i;\xi))^b\right]^\beta\right)\right] - \sum_{i=1}^{n}\log[C(\theta)]. \end{aligned}$$

The maximum likelihood estimates (mle's) of the parameters can be obtained by solving a system of non-linear equations $\left(\frac{\partial \ell_n}{\partial b}, \frac{\partial \ell_n}{\partial \beta}, \frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \xi_k}\right)^T = 0$ by numerical methods. Elements of the score vector $\Delta = (b, \beta, \theta, \xi)$ are presented in the appendix.

6 Some Special Cases of GTL-GPS Class of Distributions

In this section, by specifying the baseline cdf $G(x;\xi)$ and pdf $g(x;\xi)$ in equations (4) and (5), some special cases of GTL-GPS class of distributions are introduced.

6.1 Generalized Topp-Leone-Log-logistic Power Series (GTL-LLoGPS) Class of Distributions

If the baseline distribution is the log-logistic distribution with cdf and pdf given by $G(x;c) = 1-(1+x^c)^{-1}$ and $g(x;c) = cx^{c-1}(1+x^c)^{-2}$, respectively, for $c, x > 0$, then the pdf and hrf of the GTL-LLoGPS class of distributions are defined as

$$\begin{aligned} f_\theta(x) &= 2bc\beta\theta x^{c-1}(1+x^c)^{-3}\left(1-(1+x^c)^{-2}\right)^{b-1}\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^{\beta-1} \\ &\times \frac{C'\left(\theta\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^\beta\right)}{C(\theta)}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} h_\theta(x) &= 2bc\beta\theta x^{c-1}(1+x^c)^{-3}\left(1-(1+x^c)^{-2}\right)^{b-1}\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^{\beta-1} \\ &\times \frac{C'\left(\theta\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^\beta\right)}{C\left(\theta\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^\beta\right)}, \end{aligned} \tag{12}$$

respectively, for $b > 0, \beta > 0, c > 0, \theta > 0$ and $x > 0$.

Generalized Topp-Leone-Log-logistic Poisson (GTL-LLoGP) Distribution

The pdf and hrf of the GTL-LLoGP class of distribution are defined as

$$\begin{aligned} f_\theta(x) &= 2bc\beta\theta x^{c-1}(1+x^c)^{-3}\left(1-(1+x^c)^{-2}\right)^{b-1}\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^{\beta-1} \\ &\times \frac{\exp\left(\theta\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^\beta\right)}{\exp(\theta)-1}, \end{aligned}$$

and

$$h_\theta(x) = 2bc\beta\theta x^{c-1}(1+x^c)^{-3}\left(1-(1+x^c)^{-2}\right)^{b-1}\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^{\beta-1} \times \frac{\exp\left(\theta\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^\beta\right)}{\exp\left(\theta\left[1-\left(1-(1+x^c)^{-2}\right)^b\right]^\beta\right)-1},$$

respectively, for $b > 0, \beta > 0, c > 0, \theta > 0$ and $x > 0$.

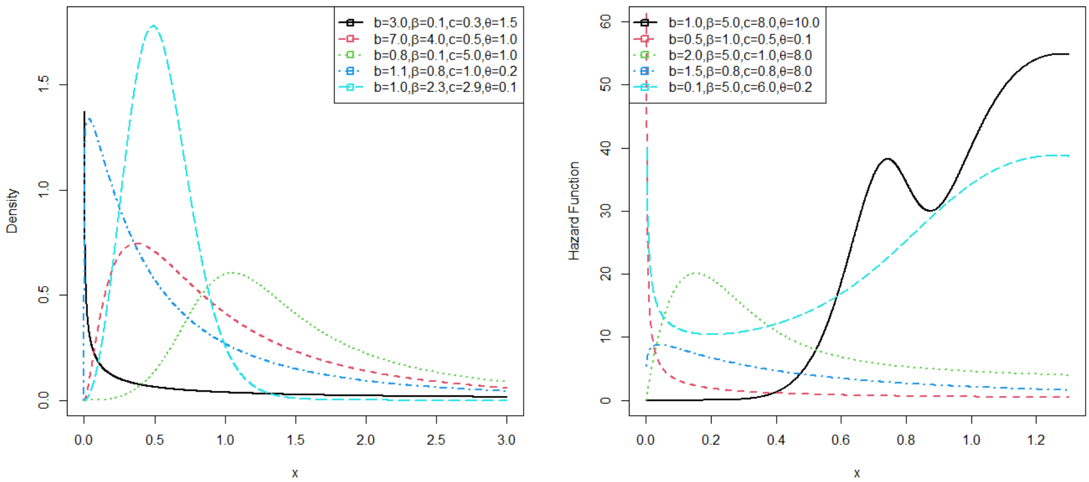
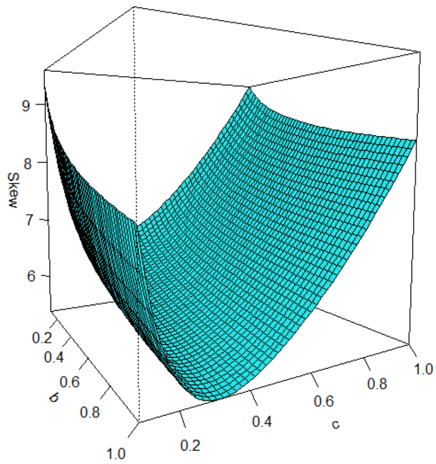


Figure 1: Plots of the pdf and hrf for the GTL-LLoGP distribution

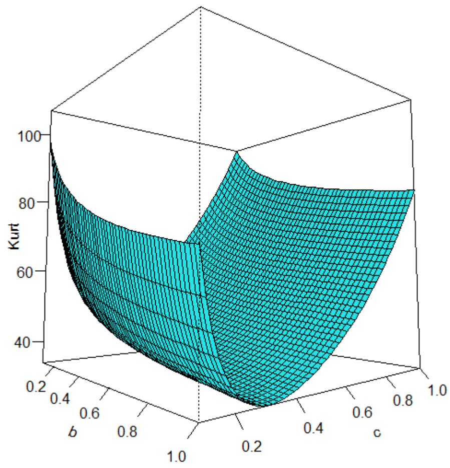
6.2 Generalized Topp-Leone-Weibull Power Series (GTL-WPS) Class of Distributions

If the baseline distribution is the Weibull distribution with cdf and pdf given by $G(x;c) = 1 - e^{-x^\lambda}$ and $g(x;c) = \lambda x^{\lambda-1}e^{-x^\lambda}$, respectively, for $\lambda, x > 0$, then the pdf and hrf of the GTL-WPS class of distributions are defined as

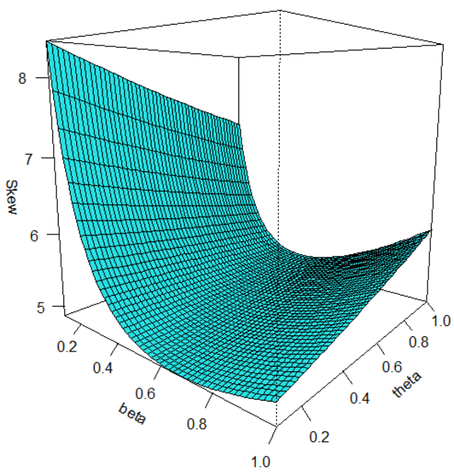
$$f_\theta(x) = 2b\beta\theta\lambda x^{\lambda-1}e^{-2x^\lambda}\left(1-e^{-2x^\lambda}\right)^{b-1}\left[1-\left(1-e^{-2x^\lambda}\right)^b\right]^{\beta-1} \times \frac{C'\left(\theta\left[1-\left(1-e^{-2x^\lambda}\right)^b\right]^\beta\right)}{C(\theta)}, \tag{13}$$



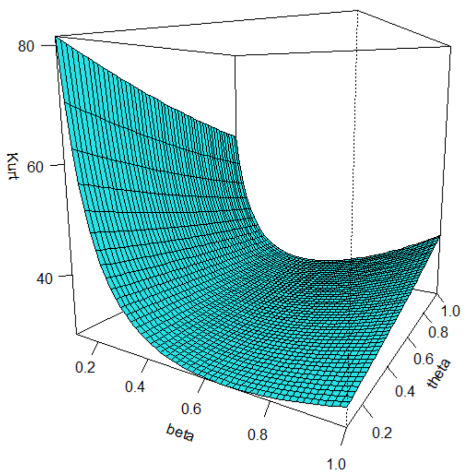
GTL-LLoGP$(b, 0.1, c, 1.5)$



GTL-LLoGP$(b, 0.1, c, 1.5)$



GTL-LLoGP$(4, \beta, 0.1, \theta)$



GTL-LLoGP$(4, \beta, 0.1, \theta)$

Figure 2: 3D plots of the skewness and kurtosis of the GTL-LLoGP distribution for some selected parameter values

and

$$\begin{aligned} h_\theta(x) &= 2b\beta\theta\lambda x^{\lambda-1} e^{-2x^\lambda} \left(1 - e^{-2x^\lambda}\right)^{b-1} \left[1 - \left(1 - e^{-2x^\lambda}\right)^b\right]^{\beta-1} \\ &\times \frac{C'\left(\theta\left[1 - \left(1 - e^{-2x^\lambda}\right)^b\right]^\beta\right)}{C\left(\theta\left[1 - (1 - e^{-2x^\lambda})^b\right]^\beta\right)}, \end{aligned} \tag{14}$$

respectively, for $b > 0, \beta > 0, \lambda > 0, \theta > 0$ and $x > 0$.

Generalized Topp-Leone-Weibull Poisson (GTL-WP) Distribution

The pdf and hrf of the GTL-WP class of distributions are defined as

$$\begin{aligned} f_\theta(x) &= 2b\beta\theta\lambda x^{\lambda-1}e^{-2x^\lambda}\left(1-e^{-2x^\lambda}\right)^{b-1}\left[1-\left(1-e^{-2x^\lambda}\right)^b\right]^{\beta-1} \\ &\times \frac{\exp\left(\theta\left[1-\left(1-e^{-2x^\lambda}\right)^b\right]^\beta\right)}{\exp(\theta)-1}, \end{aligned}$$

and

$$\begin{aligned} h_\theta(x) &= 2b\beta\theta\lambda x^{\lambda-1}e^{-2x^\lambda}\left(1-e^{-2x^\lambda}\right)^{b-1}\left[1-\left(1-e^{-2x^\lambda}\right)^b\right]^{\beta-1} \\ &\times \frac{\exp\left(\theta\left[1-\left(1-e^{-2x^\lambda}\right)^b\right]^\beta\right)}{\exp\left(\theta\left[1-\left(1-e^{-2x^\lambda}\right)^b\right]^\beta\right)-1}, \end{aligned}$$

respecivey, for $b > 0, \beta > 0, \lambda > 0, \theta > 0$ and $x > 0$.

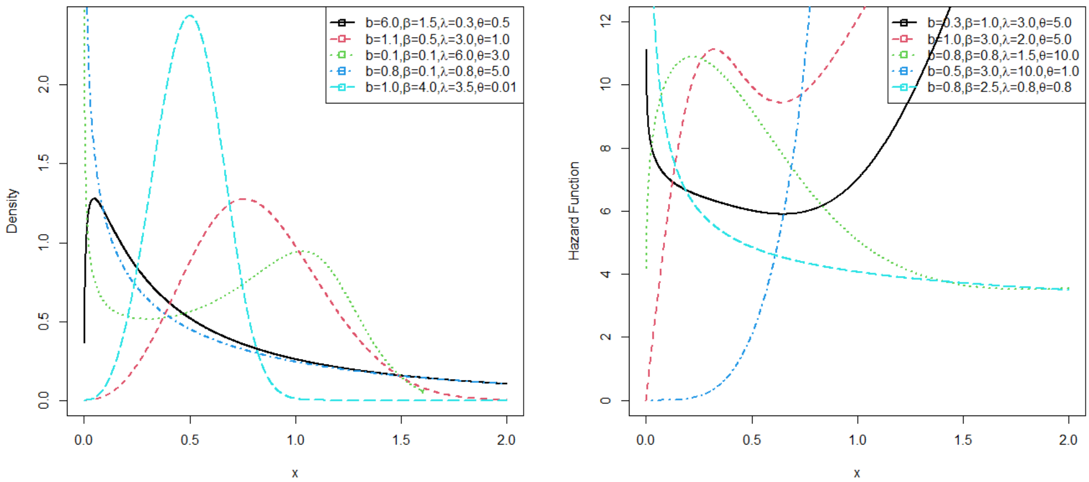
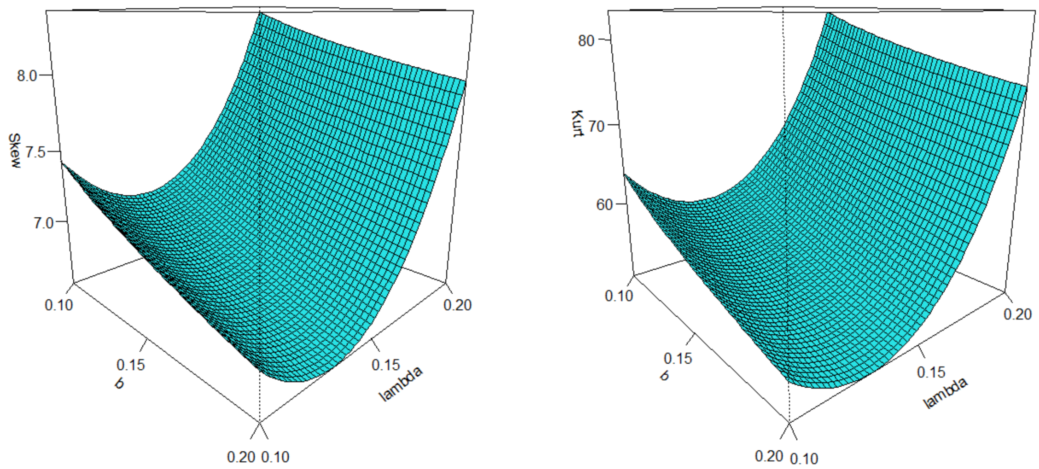
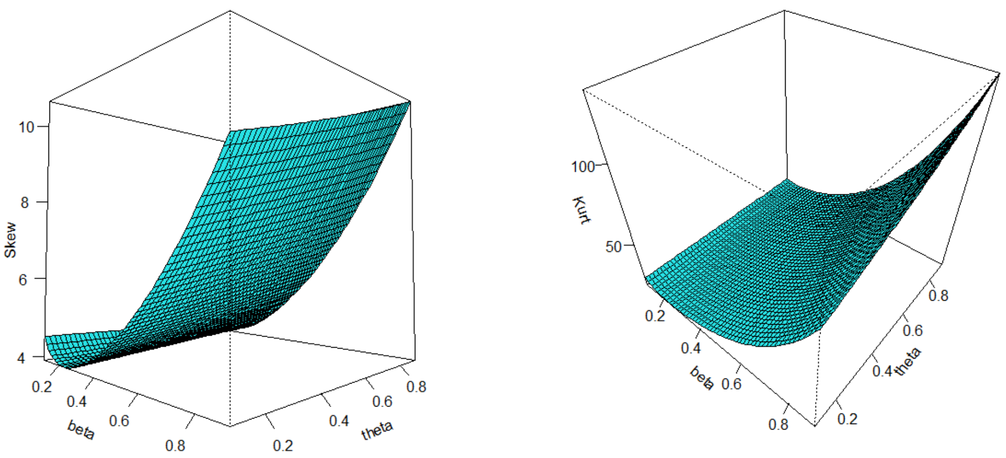


Figure 3: Plots of the pdf and hrf for the GTL-WP distribution

Figures 1 and 3 represent the plots of pdf and hrf of the GTL-LLoGP distribution and GTL-WP distribution, respectively. The pdfs of both distributions can take on right-skewed, reverse-J, unimodal, and almost symmetric shapes, while the hrfs can take on



GTL-WP$(b, 0.1, c, 3.0)$ GTL-WP$(b, 0.1, c, 3.0)$



GTL-WP$(5.0, \beta, 0.1, \theta)$ GTL-WP$(5.0, \beta, 0.1, \theta)$

Figure 4: 3D plots of the skewness and kurtosis of the GTL-WP distribution for some selected parameter values

decreasing, upside down bathtub, bathtub, and upside down bathtub followed by bathtub shapes. Figures 2 and 4, the GTL-LLoGP distribution and GTL-WP distribution can model data sets with a variety of levels of skewness and kurtosis.

7 Simulation Results

The performance of the GTL-WP is examined by conducting various simulations for different sizes (n=25, 50, 100, 200, 400, 800, 1000) via the R package. We simulate $N = 1000$ samples for the true parameters values of $(b, \beta, \lambda, \theta)$ given in Table 2. The tables list the mean MLEs of the model parameters along with the respective average bias (ABIAS) and root mean squared errors (RMSEs). The ABIAS and RMSE for the estimated parameter, say, $\hat{\theta}$, say, are given by:

$$ABIAS(\hat{\theta}) = \frac{\sum_{i=1}^{N} \hat{\theta}_i}{N} - \theta, \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. As we can see from the results, RMSE decreases as the sample size n increases, so the mean estimates of parameter values are closer to the true parameter values. The simulation results suggest that the MLE method is suitable for estimating the parameters of the GTL-WP distribution.

8 Applications

In this section, we provide illustrations of the flexible nature and usefulness of the GTL-LLoGP distributions in data modeling. We fit the GTL-LLoGP distribution to the data set in subsections 7.1 and 7.2. These fits are contrasted with several competing non-nested distributions with the same number of parameters. GTL-LLoGP distribution is compared with Weibull log-logistic logarithmic (WLLoGL) (Mashabe et al., 2019), exponentiated Weibull Poisson (EWP) (Mahmoudi and Sepahdar, 2013), Burr XII Poisson (BXIIP) (da Silva et al., 2015), Type II Topp Leone Power Lomax (TIITLPL) (Al-Marzouki et al., 2019), Transmuted Topp-Leone Weibull (TTLW) (Ibrahim and Yousof, 2020), Topp-Leone-Marshall-Olkin-Weibull (TLMO-W) (Chipepa et al., 2020), Kumaraswamy-Weibull (KwW) (Cordeiro et al., 2010) and exponentiated power generalized Weibull (EPGW) (Peña-Ramírez et al., 2018) distributions. The pdfs of the WLLoGL, EWP, BXIIP, TIITLPL, TTLW, KwW, TLMOW and EPGW distributions are given in the Appendix section.

Our model parameters were estimated using NLmixed in SAS and our goodness-of-fit test was conducted using the package AdequacyModel in R software. The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic ($-2\ln(L)$), Akaike Information Criterion ($AIC = 2p - 2\ln(L)$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$) and Consistent Akaike Information Criterion $\left(AICC = AIC + 2\frac{p(p+1)}{n-p-1}\right)$, where $L = L(\hat{\boldsymbol{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented.

We also obtain the following goodness-of-fit statistics: Crameŕ-von Mises (W^*) and Anderson-Darling Statistics (A^*) described by Chen and Balakrishnan (1995), as well as Kolmogorov-Smirnov (K-S) statistic and its P-value. Note that for the value of the log-likelihood function at its maximum (ℓ_n), larger value is good and preferred, and for

Table 2: Monte Carlo simulation results for GTL-WP distribution: mean, average bias and RMSE

		(0.5, 0.2, 0.5, 0.3)			(0.2, 0.5, 0.1, 0.8)			(0.2, 1.0, 1.5, 0.5)		
Parameter	n	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
b	25	0.7761	0.2761	0.7063	0.3865	0.1865	1.3234	1.0638	0.8638	3.4476
	50	0.6397	0.1397	0.3425	0.2382	0.0382	0.1138	0.2672	0.0672	0.2227
	100	0.5735	0.0735	0.1852	0.2196	0.0196	0.0542	0.2243	0.0243	0.0693
	200	0.5366	0.0366	0.1302	0.2179	0.0179	0.0426	0.2104	0.0104	0.0361
	400	0.5160	0.0160	0.0820	0.2165	0.0165	0.0317	0.2047	0.0047	0.0159
	800	0.5103	0.0103	0.0815	0.2068	0.0068	0.0392	0.2014	0.0014	0.0075
	1000	0.5074	0.0074	0.0698	0.2061	0.0061	0.0208	0.2008	0.0008	0.0049
β	25	0.3899	0.1899	0.3339	0.6896	0.1896	3.3937	2.4664	1.4664	9.5286
	50	0.3656	0.1656	0.3190	0.4673	-0.0327	0.1300	0.9415	-0.0585	0.2980
	100	0.3579	0.1579	0.3221	0.4722	-0.0278	0.1118	0.9687	-0.0313	0.1862
	200	0.3023	0.1023	0.2641	0.4889	-0.0111	0.0781	0.9909	-0.0091	0.1110
	400	0.2359	0.0359	0.1561	0.5029	0.0029	0.0549	0.9958	-0.0042	0.0586
	800	0.2084	0.0084	0.0670	0.5045	0.0045	0.0451	1.0008	0.0008	0.0317
	1000	0.2042	0.0042	0.0437	0.5033	0.0033	0.0259	1.0007	0.0007	0.0256
λ	25	0.7430	0.2430	0.4693	0.3076	0.2076	0.4288	1.4714	-0.0286	0.7834
	50	0.6905	0.1905	0.3946	0.1789	0.0789	0.2243	1.5234	0.0234	0.5265
	100	0.6763	0.1763	0.3586	0.1428	0.0428	0.0697	1.4808	-0.0192	0.3137
	200	0.6066	0.1066	0.2567	0.1410	0.0410	0.0614	1.4802	-0.0198	0.1749
	400	0.5423	0.0423	0.1420	0.1313	0.0313	0.0534	1.4833	-0.0167	0.0560
	800	0.5112	0.0112	0.0610	0.1079	0.0079	0.0261	1.4947	-0.0053	0.0242
	1000	0.5050	0.0050	0.0376	0.1045	0.0045	0.0190	1.4966	-0.0034	0.0170
θ	25	0.4605	0.1605	0.4903	1.3896	0.5896	1.4661	1.5703	1.0703	1.5885
	50	0.4187	0.1187	0.3816	1.3532	0.5532	0.9013	1.2140	0.7140	1.2721
	100	0.3987	0.0987	0.4388	1.1716	0.3716	0.7807	0.8316	0.3316	0.8732
	200	0.3315	0.0315	0.1756	0.9500	0.1500	0.5464	0.6365	0.1365	0.5423
	400	0.3092	0.0092	0.1366	0.8339	0.0339	0.3422	0.5452	0.0452	0.2458
	800	0.2961	-0.0039	0.0671	0.7856	-0.0144	0.1790	0.5071	0.0071	0.1303
	1000	0.2953	-0.0047	0.0489	0.7903	-0.0097	0.1214	0.5021	0.0021	0.1021

AIC, AICC, BIC, and the goodness-of-fit statistics W^*, A^* and $K-S$, smaller values are preferred. The results are shown in Tables 3 and 4.

8.1 Chemotherapy Treatment Data

As the first example, survival times in years of cancer patients receiving chemotherapy treatment alone. The data set is reported in Aalen (1988). The observations are as follows:
0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

The estimated variance-covariance matrix for GTL-LLoGP model on chemotherapy treatment data set is given by

$$\begin{pmatrix} 2.7194 & -0.0009 & 0.0127 & 2.1518 \\ -0.0009 & 2.68\times10^{-7} & -3.98\times10^{-6} & -0.0007 \\ 0.0127 & -3.98\times10^{-6} & 0.0002 & 0.0098 \\ 2.1518 & -0.0007 & 0.0098 & 1.9723 \end{pmatrix}$$

and the 95% two-sided asymptotic confidence intervals for b, β, c and θ are given by $33.931 \pm 3.2322, 11348 \pm 0.001, 0.097 \pm 0.0251$ and 0.441 ± 2.7526, respectively.

Table 3: Parameter estimates and goodness-of-fit statistics for various models fitted for chemotherapy treatment data

	Estimates				Statistics							
Model	$\hat{b}$	$\hat{\beta}$	$\hat{c}$	$\hat{\theta}$	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	p-value
GTLLLoGP	33.9310	11348.0	0.0970	0.4410	115.8	123.8	124.8	131.0	0.0653	0.4475	0.0975	0.7489
	(1.6491)	(0.0005)	(0.0128)	(1.4044)								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{p}$								
WLLoGL	0.7178	1.6267	0.6474	1.24×10^{-8}	116.2	124.2	125.2	131.5	0.0813	0.5436	0.1094	0.6149
	(0.12450)	(12.8080)	(5.0974)	(0.0191)								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\omega}$								
EWP	0.8073	1.1555	1.14×10^{-7}	1.6125	116.1	124.1	125.1	131.3	0.0691	0.4706	0.1004	0.7173
	(0.5277)	(1.5315)	(0.0072)	(1.9800)								
	$\hat{c}$	$\hat{k}$	$\hat{s}$	$\hat{\lambda}$								
BXIIP	1.1326	1610.2	1137.9	0.7906	115.9	123.9	124.9	131.2	0.0744	0.5015	0.1037	0.6804
	(0.0627)	(0.0005)	(0.0008)	(1.3081)								
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$								
TIITLPL	25.2760	44700.0	0.5720	241140.0	116.3	124.3	125.3	131.5	0.0760	0.5115	0.1060	0.6537
	(4.3286)	(0.0045)	(0.0682)	(0.6036)								
	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\lambda}$								
TTLW	0.4255	0.8883	1.3940	0.1166	116.0	124.0	125.0	131.3	0.0706	0.4790	0.1013	0.7070
	(0.4183)	(0.7586)	(2.0105)	(0.5849)								
	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$								
KwW	10.5121	157.1692	0.5989	0.1637	116.1	124.1	125.1	131.3	0.0688	0.4686	0.1021	0.6979
	(4.9910)	(0.0679)	(1.3695)	(0.0464)								
	$\hat{b}$	$\hat{\delta}$	$\hat{\lambda}$	$\hat{\gamma}$								
TLMOW	1.5664	0.8611	0.4921	0.8424	116.1	1124.1	125.1	131.3	0.0688	0.4686	0.1021	0.6979
	(1.9295)	(1.0623)	(0.7341)	(0.6036)								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\omega}$								
EPGW	0.3587	9.6936	3.7253	0.1142	115.9	123.9	124.9	131.1	0.0689	0.4687	0.1005	0.7161
	(0.6288)	(142.8500)	(7.5563)	(1.7184)								

Based on Table 3, GTL-LLoGP distribution has the highest p-value for the K-S statistic and the lowest goodness-of-fit statistics compared to other non-nested models. Thus, we conclude that the GTL-LLoGP model performs better with chemotherapy treatment data than the non-nested WLLoGL, EWP, BXIIP, TIITLPL, TTLW, KwW, TLMOW and EPGW models. Moreover, Figure 5 shows that our model outperforms the competing non-nested models on chemotherapy treatment data.

In Figure 6, we see that the cdf line for the GTL-LLoGP distribution indicated by the blue line is closer to the empirical cdf while the survival function in blue is also close to

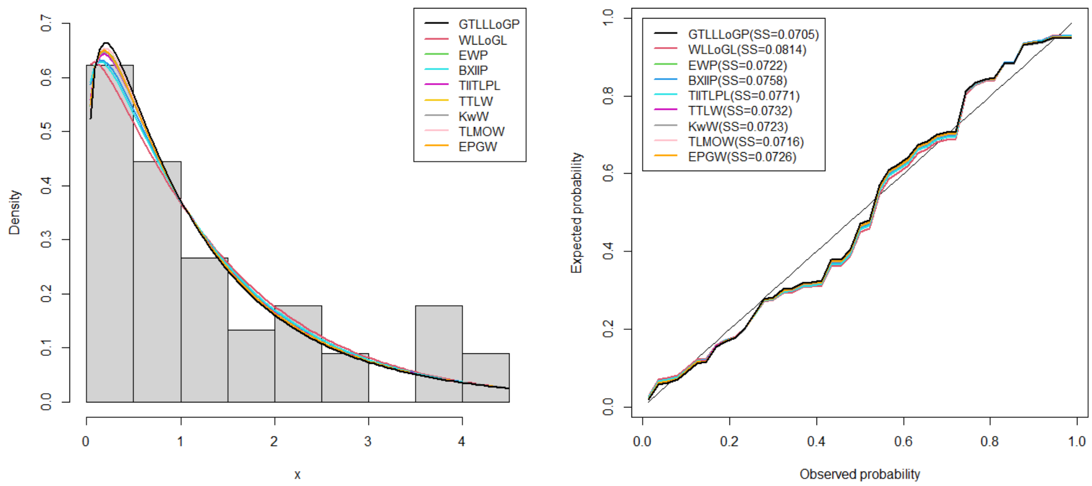


Figure 5: Histogram, fitted density and probability plots for chemotherapy treatment data

the Kaplan-Meier(K-M) curve which indicate that our model is the best in explaining the chemotherapy treatment data. The TTT plot for chemotherapy treatment data indicates a uni-modal hazard rate function, hence the chemotherapy treatment data can be fitted to our model.

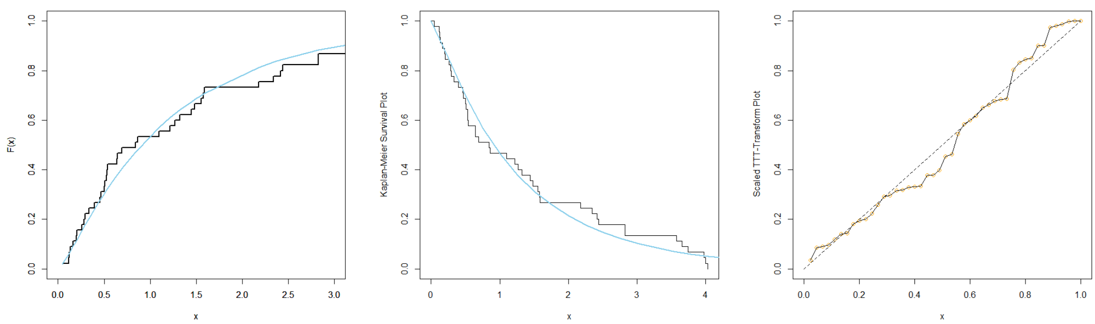


Figure 6: Estimated cdf, Kaplan-Meier survival and scaled TTT-Transform plots for the GTL-LLoGP distribution for chemotherapy treatment data

8.2 Fatigue Life Data

The second data set represent the fatigue life (to the nearest thousand cycles) of 67 specimens of Alloy T7987 that failed before having accumulated 300 thousand cycles of testing obtained from William and Escobar (1998). The observations are as follows:

94, 96, 99, 99, 104, 108, 112, 114, 117, 117, 118, 121, 121, 123, 129, 131, 133, 135, 136, 139, 139, 140, 141, 141, 143, 144, 149, 149, 152, 153, 159, 159, 159, 159, 162, 168, 168, 169, 170, 170, 171, 172, 173, 176, 177, 180, 180, 184, 187, 188, 189, 190, 196, 197, 203, 205, 211, 213, 224, 226, 227, 256, 257, 269, 271, 274, 291.

The estimated variance-covariance matrix for GTL-LLoGP model on fatigue life data set is given by

$$\begin{pmatrix} 6.16\times10^{-5} & 0.0360 & -9.23\times10^{-5} & 0.0002 \\ 0.0360 & 21.1111 & -0.0515 & -0.1566 \\ -9.23\times10^{-5} & -0.0515 & 0.0005 & -0.0310 \\ 0.0002 & -0.1566 & -0.0310 & 2.9812 \end{pmatrix}$$

and the 95% two-sided asymptotic confidence intervals for b, β, c and θ are given by $4601.9 \pm 0.0153, 12.92 \pm 9.006, 0.7111 \pm 0.0429$ and 0.7547 ± 3.3841, respectively.

Table 4: Parameter estimates and goodness-of-fit statistics for various models fitted for fatigue life data

	Estimates				Statistics							
Model	$\hat{b}$	$\hat{\beta}$	$\hat{c}$	$\hat{\theta}$	$-2\log(L)$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	p-value
GTLLLoGP	4601.9	12.9200	0.7111	0.7547	695.78	703.78	704.4	712.6	0.0216	0.1874	0.0571	0.9810
	(0.0078)	(4.5947)	(0.0219)	(1.7266)								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{c}$	$\hat{p}$								
WLLoGL	0.0120	10.5028	0.1143	0.9999	748.2	756.2	756.9	765.1	0.2760	1.8643	0.3043	8.15×10^{-6}
	(0.0826)	(0.0164)	(0.3118)	(0.0010)								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\omega}$								
EWP	0.7316	0.0732	13.7400	20.4800	696.2	704.2	704.9	713.0	0.0303	0.2215	0.0707	0.8913
	(0.2029)	(0.0706)	(15.6512)	(1.9072)								
	$\hat{c}$	$\hat{k}$	$\hat{s}$	$\hat{\lambda}$								
BXIIP	6.4489	0.9552	157.7100	1.38×10^{-8}	698.4	706.4	707.0	715.2	0.0277	0.2378	0.0614	0.9625
	(1.3579)	(0.4904)	(20.0350)	(0.0238)								
	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$								
TIITLPL	353.31	0.1247	1.2660	1226.5	739.9	747.9	748.6	756.8	0.0418	0.3652	0.2469	0.0006
	(8.12×10^{-6})	(0.0441)	(0.0840)	(1.02×10^{-5})								
	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\lambda}$								
TTLW	0.0117	1.0794	33.8639	0.1256	696.0	704.0	704.6	712.8	0.0250	0.1956	0.0913	0.6312
	(0.0012)	(0.0988)	(0.0047)	(0.6828)								
	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$								
KwW	37.0441	0.7188	0.0220	1.2000	695.8	703.8	704.5	712.7	0.0275	0.2050	0.0941	0.5934
	(0.2467)	(2.2727)	(0.0148)	(1.4152)								
	$\hat{b}$	$\hat{\delta}$	$\hat{\lambda}$	$\hat{\gamma}$								
TLMOW	25.3420	4.0042	0.0571	0.7863	695.9	703.9	704.5	712.7	0.0240	0.1926	0.0646	0.9423
	(0.0006)	(0.0040)	(0.0200)	(0.0700)								
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\delta}$	$\hat{\lambda}$								
EPGW	0.3747	114.21	127.85	0.0024	695.80	703.8	704.5	712.7	0.0231	0.1878	0.0600	0.9695
	(0.032)	(9.85×10^{-7})	(2.32×10^{-5})	(0.0004)								

Based on Table 4, GTL-LLoGP distribution has the highest p-value for the K-S statistic and the lowest goodness-of-fit statistics compared to other non-nested models. Thus, we conclude that the GTL-LLoGP model performs better with fatigue life data than non-nested WLLoGL, EWP, BXIIP, TIITLPL, TTLW, KwW, TLMOW and EPGW models.

Moreover, Figure 7 shows that our model outperforms the competing non-nested models on fatigue life data.

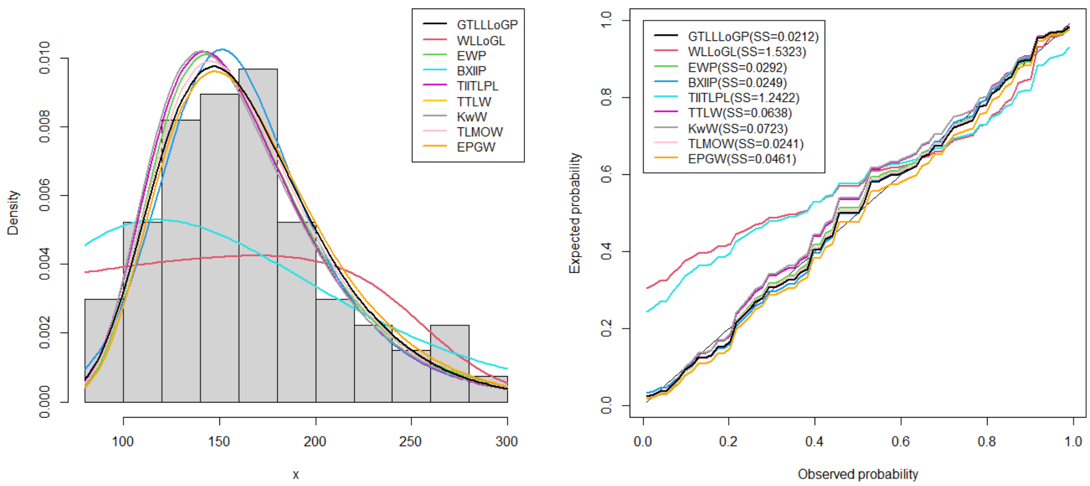


Figure 7: Histogram, fitted density and probability plots for fatigue life data

In Figure 8 again, we see that the cdf line for the GTL-LLoGP distribution indicated by the blue line is closer to the empirical cdf while the survival function in blue is also close to the Kaplan-Meier(K-M) curve which indicate that our model is the best in explaining the fatigue life data. The TTT plot for fatigue life data indicates a increasing hazard rate function, which confirms that indeed GTL-LLoGP distribution is good for fitting fatigue life data.

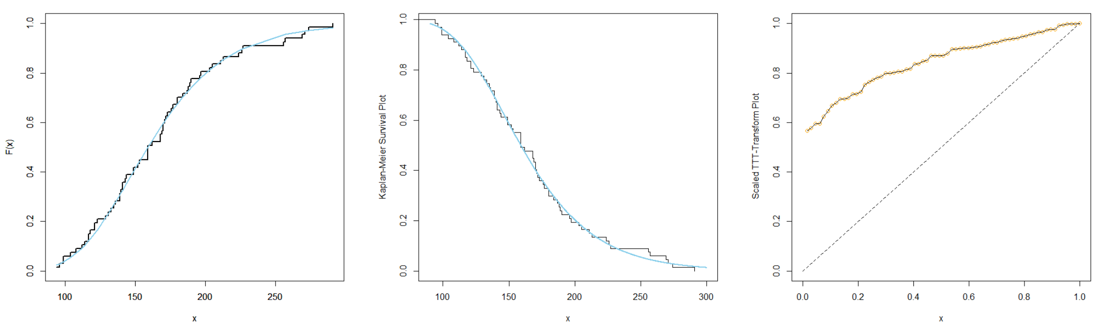


Figure 8: Estimated cdf, Kaplan-Meier survival and scaled TTT-Transform plots for the GTL-LLoGP distribution for fatigue life data

9 Concluding Remarks

In this paper, we presented a new generalized class of distributions called the Generalized Topp-Leone-G power series (GTL-GPS) distribution. The statistical properties and maximum likelihood estimates of the proposed model were derived. Special cases of the new class of distributions were also examined. A simulation study to assess the performance of the maximum likelihood estimates was conducted. As a demonstration of the usefulness of the proposed family of distributions, we examined two real data examples. The new distribution performs better than several non-nested models.

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Appendix

The link below contains the appendix
`https://drive.google.com/file/d/1yZPk91OgaoHajLUxc3dO9p2kf8bKwxJc/view?usp=sharing`