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Estimation of the Parameters of the Generalized Inverted Exponential Distribution with Progressive type I Interval Censored Data

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In this article, we study estimation methodologies for parameters of a generalized inverted exponential distribution based on different estimation methods using progressively type I interval censored data. In this approach, besides conventional maximum likelihood estimation, mid-point method, probability plot method and method of moments are proposed for parameter estimation. To obtain maximum likelihood estimates, we use Newton-Raphson, expectation-maximization and stochastic expectation-maximization methods. Moreover, the approximate confidence intervals of the parameters are obtained via the inverse of the observed information matrix. In addition, percentile bootstrap technique is utilized to compute confidence intervals. Numerical comparisons are presented of the proposed estimators using Monte Carlo simulations. To demonstrate the proposed methodology in a real-life scenario, survival times of guinea pigs injected with different doses of tubercle bacilli data is considered to show the applicability of the proposed methods. Finally, different methods for determining the inspection times and optimal censoring planes are studied.

keywords: The generalized inverted exponential distribution, progressive type I interval censored, optimal censoring, inspection times, stochastic expectation-maximization, expectation-maximization.

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1 Introduction

The generalized inverted exponential distribution (GIED) was first proposed by Abouammoh and Alshingiti (2009) using a shape parameter in inverted exponential distribution. The IED is a continuous transformation of the reciprocal of exponential distribution. Specifically, if a random variable X follows an exponential distribution, then $X = \frac{1}{Y}$ follows an IED with c.d.f. and p.d.f. given by

$$F(y) = e^{-\lambda/y}, y > 0, \lambda > 0$$

$$f(y) = \frac{\lambda}{y^2} \ e^{-\lambda/y}, y > 0, \lambda > 0,$$

respectively. The IED was investigated by many authors, see for example, Prakash (2012) and Singh et al. (2013). A random variable X of the GIED with shape parameter α and scale parameter λ has the following expressions of c.d.f. and p.d.f.

$$F(x) = 1 - (1 - e^{-\lambda/x})^{\alpha}, x > 0, \alpha > 0, \lambda > 0 \tag{1}$$

$$f(x) = \frac{\alpha\lambda}{x^2} \ e^{-\lambda/x} \ (1 - e^{-\lambda/x})^{\alpha-1}, x > 0, \alpha > 0, \lambda > 0, \tag{2}$$

respectively. It can be seen that the hazard function of GIED distribution

$$\frac{f(x)}{1 - F(x)} = \frac{\alpha\lambda}{x^2(e^{\lambda/x} - 1)}$$

can be increasing or decreasing, depending on the shape parameter, α. Abouammoh and Alshingiti (2009) observed that in many situations this distribution may provide a better fit than gamma, Weibull, and generalized exponential distributions. GIED can be used in many applications, for instance; in horse racing, supermarkets queue, sea currents, wind speeds (see Kotz and Nadarajah (2000)).

For more properties an applications of GIED, one can refer to Krishna and Kumar (2013), Dey and Dey (2014a), Dey and Dey (2014b), Singh et al. (2015), and Dube et al. (2016).

The problem of estimating the parameters of GIED under different sampling schemes was considered by many authors. Krishna et al. (2017) estimated of the stress–strength parameter $P(Y < X)$ based on progressively first-failure-censored samples, when X and Y both follow two-parameter generalized inverted exponential distribution with different and unknown shape and scale parameters. Dey and Nassar (2020) estimated the parameters of generalized inverted exponential distribution under constant stress accelerated life test.Hassan et al. (2021) studied the estimation of the reliability of stress–strength reliability model via median ranked set sampling (MRSS) when the stress and the strength variables are modeled by two independent but not identically distributed random variables from the generalized inverted exponential distributions. Garg and Kumar (2021) dealt with the problem of estimation of the stress-strength reliability $P(Y < X)$ when X

and Y both have independent generalized inverted exponential distributions with different shape and common scale parameters based on the hybrid censored samples. Kumari et al. (2022) computed the classical and Bayesian estimates of multicomponent stress-strength reliability from generalized inverted exponential lifetime distributions under a progressively first failure censoring scheme.

In life-testing and reliability studies, the most common censoring schemes are type I and type II censoring. However, it is of great importance in some of these studies that a specific fraction of individuals may be removed from the experiment at each of several ordered failure times (see Cheng et al. (2010)). Clearly, type I and type II schemes do not have the ability of allowing removal of units at points other than the terminal point of the experiment. Aggarwala (2001) proposed the progressive type I interval censored scheme which can be described as follows. Assume n units are put on test at time $t_0 = 0$ and each unit is followed until it fails or is censored. Units are observed at preset times $t_1 < t_2 < \cdots < t_m$, where m is the pre-specified time to the end of the experiment. That is, the time axis is partitioned into intervals $I_j = [t_{j-1}, t_j), j = 1, \cdots, m$, with t_m, is the time at which the experimentation ends. Let d_j denote the number of units which are failed in I_j and r_j denote the number of units which are removed from experiment at time t_j. In specific, if n units are put on test at time t_0 and d_1 are observed at time t_1, at this time r_1 unfailed units are removed from experiment leaving $n - d_1 - r_1$ items still present. At time t_2 when another d_2 items have failed, r_2 of the unfailed items are removed from experiment leaving $n - d_1 - r_1 - d_2 - r_2$ items still present and so on. The experiment terminates after m number of repetitions. Finally, at time t_m, the number of removed unfailed items is r_m. Note that $n = \sum_{i=1}^{m}(r_i + d_i)$. Figure 1 shows a representation of a progressive type I interval censored.

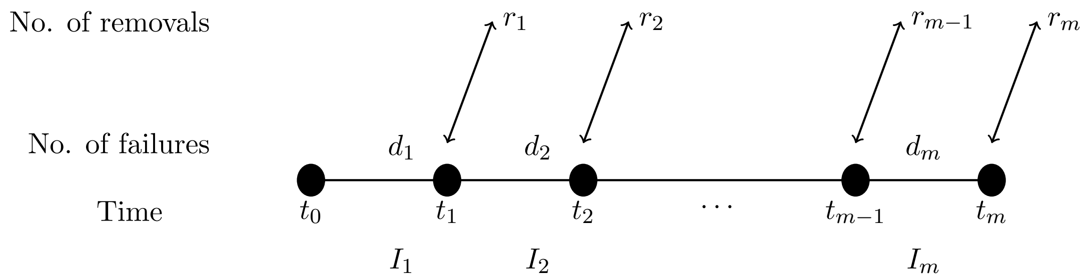

Figure 1: Progressive type I Interval Censored Scheme

Hence our observations consist of $D = \{(t_i, d_i, r_i); i = 1, \cdots, m\}$. The numbers of removal items $r_1, \cdots, r_m$ are expressed as nonnegative integers. Alternatively, the removal numbers may determined by pre-specified percentages of the remaining surviving units as follows. Let $\mathbf{p} = (p_1, p_2, \cdots, p_m)$ be pre-specified percentages with $p_m = 1$. At time t_i, $\lceil p_i \times (\text{number of surviving units at time } t_i) \rceil$ from the remaining surviving units are removed from the experiment where $\lceil w \rceil$ denotes the largest integer, which is smaller than or equal to w.

In this paper, we utilized different estimation procedures for estimating the parameters of GIED under progressive type I interval censored. The remainder of this paper is organized as follows. In Section 2, we obtain the maximum likelihood function estimators (MLEs) of the unknown parameters α and λ. The standard errors for the MLEs and approximated 95% confidence intervals for the parameters are computed as well using the inverse of the observed information matrix. Further, computing the MLE using EM algorithm and stochastic EM algorithm are also investigated. Nonparametric bootstrap percentile technique is utilized to construct 95% confidence intervals of the unknown parameters. Midpoint approximation method, the probability plot and method of moments are studied in Sections 3, 4 and 5, respectively. A Monte Carlo simulation study is presented in Section 6, which provides a comparison of all the estimation procedures in terms of their biases, mean square errors, estimated standard errors, sampled standard error, lengths of 95% confidence intervals and empirical 95% coverage probabilities. An analysis of real data set is presented in Section 7. Inspection times and optimal censoring schemes are studied in Sections 8 and Section 9, respectively. Finally, a conclusion is given in Section 10.

2 Maximum likelihood estimation

Based on the observed progressive type I interval censored sample $D = \{(t_i, d_i, r_i); i = 1, \cdots, m\}$, the likelihood function of α and λ can be written as

$$\begin{aligned} L(\alpha, \lambda|D) &\propto \prod_{i=1}^{m}[F(t_i) - F(t_{i-1})]^{d_i}[1 - F(t_i)]^{r_i} \\ &= \prod_{i=1}^{m}[(1 - e^{-\lambda/t_{i-1}})^{\alpha} - (1 - e^{-\lambda/t_i})^{\alpha}]^{d_i}(1 - e^{-\lambda/t_i})^{\alpha r_i}, \end{aligned} \tag{3}$$

with corresponding log-likelihood function

$$l(\alpha, \lambda|D) \propto \sum_{i=1}^{m} d_i \log\left((1 - e^{-\lambda/t_{i-1}})^{\alpha} - (1 - e^{-\lambda/t_i})^{\alpha}\right) + \alpha \sum_{i=1}^{m} r_i \log(1 - e^{-\lambda/t_i}). \tag{4}$$

Theorem 1 *The MLEs of α and λ for $\alpha > 0$ and $\lambda > 0$ exist and unique.*

Proof : The detailed proof of the theorem is deferred in the appendix. ■
Let, for $i = 1, \cdots, m$,

$$A_i = (1 - e^{-\lambda/t_{i-1}})^{\alpha} - (1 - e^{-\lambda/t_i})^{\alpha} \tag{5}$$

$$B_i = 1 - e^{-\lambda/t_i}. \tag{6}$$

Then the log-likelihood (4) can be expressed as

$$l(\alpha, \lambda|D) = \sum_{i=1}^{m} d_i \log(A_i) + \alpha \sum_{i=1}^{m} r_i \log(B_i). \tag{7}$$

The first order partial derivatives of A_i and B_i with respect to α and λ are given by

$$A_{i,\alpha} := \frac{\partial A_i}{\partial \alpha} = (1-e^{-\lambda/t_{i-1}})^{\alpha}\log(1-e^{-\lambda/t_{i-1}}) - (1-e^{-\lambda/t_i})^{\alpha}\log(1-e^{-\lambda/t_i}) \tag{8}$$

$$A_{i,\lambda} := \frac{\partial A_i}{\partial \lambda} = \frac{\alpha}{t_{i-1}}e^{-\lambda/t_{i-1}}(1-e^{-\lambda/t_{i-1}})^{\alpha-1} - \frac{\alpha}{t_i}e^{-\lambda/t_i}(1-e^{-\lambda/t_i})^{\alpha-1} \tag{9}$$

$$B_{i,\lambda} := \frac{\partial B_i}{\partial \lambda} = \frac{1}{t_i}e^{-\lambda/t_i} \tag{10}$$

and the second order partial derivatives are given by

$$A_{i,\alpha\alpha} := \frac{\partial^2 A_i}{\partial \alpha^2} = \Big(\log(1-e^{-\lambda/t_{i-1}})\Big)^2(1-e^{-\lambda/t_{i-1}})^{\alpha} - \Big(\log(1-e^{-\lambda/t_i})\Big)^2(1-e^{-\lambda/t_i})^{\alpha} \tag{11}$$

$$\begin{aligned} A_{i,\alpha\lambda} := \frac{\partial^2 A_i}{\partial \alpha \partial \lambda} =& \frac{1}{t_{i-1}}e^{-\lambda/t_{i-1}}(1-e^{-\lambda/t_{i-1}})^{\alpha-1}\Big[1+\alpha\log(1-e^{-\lambda/t_{i-1}})\Big] \\ &- \frac{1}{t_i}e^{-\lambda/t_i}(1-e^{-\lambda/t_i})^{\alpha-1}\Big[1+\alpha\log(1-e^{-\lambda/t_i})\Big] \end{aligned} \tag{12}$$

$$\begin{aligned} A_{i,\lambda\lambda} := \frac{\partial^2 A_i}{\partial \lambda^2} =& \frac{\alpha}{t_{i-1}}\Big(\frac{\alpha-1}{t_{i-1}}(e^{-\lambda/t_{i-1}})^2(1-e^{-\lambda/t_{i-1}})^{\alpha-2} - \frac{1}{t_{i-1}}e^{-\lambda/t_{i-1}}(1-e^{-\lambda/t_{i-1}})^{\alpha-1}\Big) \\ &- \frac{\alpha}{t_i}\Big(\frac{\alpha-1}{t_i}(e^{-\lambda/t_i})^2(1-e^{-\lambda/t_i})^{\alpha-2} - \frac{1}{t_i}e^{-\lambda/t_i}(1-e^{-\lambda/t_i})^{\alpha-1}\Big) \end{aligned} \tag{13}$$

$$B_{i,\lambda\lambda} := \frac{\partial^2 B_i}{\partial \lambda^2} = -\frac{1}{t_i^2}e^{-\lambda/t_i}. \tag{14}$$

Hence the first and the second order partial derivatives of the log-likelihood function (7) with respect to α and λ can be computed by

$$l_\alpha := \frac{\partial l(\alpha,\lambda|D)}{\partial \alpha} = \sum_{i=1}^{m} d_i\frac{A_{i,\alpha}}{A_i} + \sum_{i=1}^{m} r_i\log(B_i). \tag{15}$$

$$l_\lambda := \frac{\partial l(\alpha,\lambda|D)}{\partial \lambda} = \sum_{i=1}^{m} d_i\frac{A_{i,\lambda}}{A_i} + \alpha\sum_{i=1}^{m} r_i\frac{B_{i,\lambda}}{B_i}. \tag{16}$$

$$l_{\alpha\alpha} := \frac{\partial^2 l(\alpha,\lambda|D)}{\partial \alpha^2} = \sum_{i=1}^{m} d_i\frac{A_iA_{i,\alpha\alpha} - A_{i,\alpha}^2}{A_i^2} \tag{17}$$

$$l_{\alpha\lambda} := \frac{\partial^2 l(\alpha,\lambda|D)}{\partial \alpha \lambda} = \sum_{i=1}^{m} d_i\frac{A_iA_{i,\alpha\lambda} - A_{i,\alpha}A_{i,\lambda}}{A_i^2} + \sum_{i=1}^{m} r_i\frac{B_{i,\lambda}}{B_i}. \tag{18}$$

$$l_{\lambda\lambda} := \frac{\partial^2 l(\alpha,\lambda|D)}{\partial \lambda^2} = \sum_{i=1}^{m} d_i\frac{A_iA_{i,\lambda\lambda} - A_{i,\lambda}^2}{A_i^2} + \alpha\sum_{i=1}^{m} r_i\frac{B_iB_{i,\lambda\lambda} - B_{i,\lambda}^2}{B_i^2}. \tag{19}$$

To compute the MLEs, $\hat{\alpha}$ and $\hat{\lambda}$, of the unknown parameters, α and λ, we need to solve the normal equations $l_\alpha = 0$ and $l_\lambda = 0$, where l_α and l_λ are given in (15) and (16). It can be seen that there is no closed from of the MLEs. Hence, to obtain the MLEs of α

and λ, we may use a simple numerical procedure like Newton-Raphson method whose iterative equation is given by

$$\begin{pmatrix} \alpha^{(k+1)} \\ \lambda^{(k+1)} \end{pmatrix} = \begin{pmatrix} \alpha^{(k)} \\ \lambda^{(k)} \end{pmatrix} - \begin{pmatrix} l_{\alpha\alpha} & l_{\alpha\lambda} \\ l_{\lambda\alpha} & l_{\lambda\lambda} \end{pmatrix}^{-1} \begin{pmatrix} l_{\alpha} \\ l_{\lambda} \end{pmatrix} \Bigg|_{\alpha=\alpha^{(k)},\lambda=\lambda^{(k)}}$$

or equivalently

$$\alpha^{(k+1)} = \alpha^{(k)} - \frac{l_{\alpha} l_{\lambda\lambda} - l_{\lambda} l_{\alpha\lambda}}{l_{\alpha\alpha} l_{\lambda\lambda} - l_{\alpha\lambda}^2}\Big|_{\alpha=\alpha^{(k)},\lambda=\lambda^{(k)}} \tag{20}$$

$$\lambda^{(k+1)} = \lambda^{(k)} - \frac{l_{\lambda} l_{\alpha\alpha} - l_{\alpha} l_{\alpha\lambda}}{l_{\alpha\alpha} l_{\lambda\lambda} - l_{\alpha\lambda}^2}\Big|_{\alpha=\alpha^{(k)},\lambda=\lambda^{(k)}}, \tag{21}$$

where $\alpha^{(k)}$ and $\lambda^{(k)}$ are the values of α and λ at k-th iteration and $l_{\alpha}, l_{\lambda}, l_{\alpha\alpha}, l_{\alpha\lambda}$ and $l_{\lambda\lambda}$ are given in (15),(16),(17),(18) and (19). The iteration process continues until convergence , i.e., $|\alpha^{(k+1)} - \alpha^{(k)}| + |\lambda^{(k+1)} - \lambda^{(k)}| < \varepsilon$, for some pre-specified $\varepsilon > 0$.

The standard error of the MLEs are computed using the inverse of the observed information matrix. Hence the estimated standard error of α and λ can be calculated by square root of the diagonal elements of the inverting of the observed information matrix evaluated at $(\hat{\alpha}, \hat{\lambda})$ as follows

$$se(\hat{\alpha}) = \sqrt{-\frac{\hat{l}_{\lambda\lambda}}{\hat{l}_{\alpha\alpha}\hat{l}_{\lambda\lambda} - \hat{l}_{\alpha\lambda}^2}} \quad \text{and} \quad se(\hat{\lambda}) = \sqrt{-\frac{\hat{l}_{\alpha\alpha}}{\hat{l}_{\alpha\alpha}\hat{l}_{\lambda\lambda} - \hat{l}_{\alpha\lambda}^2}},$$

where $\hat{l}_{\alpha\alpha}, \hat{l}_{\alpha\lambda}$ and $\hat{l}_{\lambda\lambda}$ are given in (17),(18) and (19) with α and λ are replaced by $\hat{\alpha}$ and $\hat{\lambda}$, respectively. The asymptotic normality of the MLE can be used to compute the approximate confidence intervals for parameters α and λ. Therefore,$100(1-\gamma)\%$ Wald confidence intervals for λ and α are computed by

$$(\hat{\alpha} - z_{\gamma/2} se(\hat{\alpha}), \hat{\alpha} + z_{\gamma/2} se(\hat{\alpha})) \quad \text{and} \quad (\hat{\lambda} - z_{\gamma/2} se(\hat{\lambda}), \hat{\lambda} + z_{\gamma/2} se(\hat{\lambda})),$$

respectively, where z_{γ} is the upper γ-th percentile of the standard normal distribution.

Next, we compute 95% confidence interval for α and λ using nonparametric percentile bootstrap (Boot-p) method. Bootstrap methods are widely used to obtain confidence intervals for the parameters. Boot-p method, proposed by Efron and Tibshirani (1986), is used to construct confidence intervals for the parameters as well as the reliability and hazard functions. To construct the Boot-p confidence interval, we follow the following steps.

Step(1): Compute the MLEs, $\hat{\alpha}$ and $\hat{\lambda}$, based on the original progressively type I interval censored sample $D = \{(t_i, d_i, r_i); i = 1, \cdots, m\}$

Step(2): Based on the computed MLEs in **Step(1)**, $\hat{\alpha}$ and $\hat{\beta}$, generate a bootstrap sample D^* of size m consists of $D^* = \{(t_i, d_i^*, r_i^*); i = 1, \cdots, m\}$ using $\hat{\alpha}$ and $\hat{\lambda}$.

Step(3): Compute the MLEs, $\hat{\alpha}^*$ and $\hat{\beta}^*$, based on the generated bootstrap sample in **Step(2)**.

Step(4): Repeat **Step(2)** and **Step(3)**, for B times, where B is a pre-specified quantity.

Define $\hat{\alpha}_B(x) = G_{\alpha}^{*-1}(x)$, where $G_{\alpha}^*(x)$is the empirical cumulative distribution of $\hat{\alpha}^*$. Similarly, define $\hat{\lambda}_B(x) = G_{\lambda}^{*-1}(x)$, where $G_{\lambda}^*(x)$ is the empirical cumulative distribution of $\hat{\lambda}^*$. Now, compute the approximate $100(1-\gamma)\%$ bootstrap-p confidence interval of α and λ as follows

$$(\hat{\alpha}_B(\gamma/2), \hat{\alpha}_B(1-\gamma/2)) \text{ and } (\hat{\lambda}_B(\gamma/2), \hat{\lambda}_B(1-\gamma/2)),$$

respectively.

2.1 EM Algorithm

It can be seen that utilizing Newton-Raphson method to compute the MLEs requires the computation of the second derivatives of the associated log-likelihood function. In this subsection, we propose EM algorithm to avoid such computations for obtaining the MLEs of α and λ. EM algorithm proposed by Dempster et al. (1977) is a very powerful technique used in parameter estimation based on incomplete or missing information data. The EM algorithm consists of two main steps; Expectation step (E-step) and Maximization step (M-step). In E-step, we compute the conditional expectation of the complete log-likelihood function condition on the observed values and in M-step, we maximize the resulted function with respect to the unknown parameters. Now define $Z_{ij}, j = 1, \cdots, d_i$ to represent the complete survival times within subintervals $I_i = [t_{i-1}, t_i)$ and define $W_{ik}, k = 1, \cdots, r_i$ to represent the complete survival times of those withdrawn items at t_i where $i = 1, \cdots, m$. Using $\mathbf{Z} = (Z_{11}, \cdots, Z_{m,d_m})$ and $\mathbf{W} = (W_{11}, \cdots, W_{m,r_m})$, the complete log-likelihood function can be expressed by

$$\begin{aligned}
l^c(\alpha, \lambda|\mathbf{Z}, \mathbf{W}) \propto & \sum_{i=1}^{m} \Big(\sum_{j=1}^{d_i} \log(f(z_{ij})) + \sum_{k=1}^{r_i} \log(f(w_{ik})) \Big) \\
= & n\log(\alpha) + n\log(\lambda) - 2\sum_{i=1}^{m}\sum_{j=1}^{d_i} \log(z_{ij}) - 2\sum_{i=1}^{m}\sum_{k=1}^{r_i} \log(w_{ik}) \\
& - \lambda \sum_{i=1}^{m}\sum_{j=1}^{d_i} (1/z_{ij}) - \lambda \sum_{i=1}^{m}\sum_{k=1}^{r_i} (1/w_{ik}) \\
& + (\alpha-1)\sum_{i=1}^{m}\sum_{j=1}^{d_i} \log(1-e^{-\lambda/z_{ij}}) + (\alpha-1)\sum_{i=1}^{m}\sum_{k=1}^{r_i} \log(1-e^{-\lambda/w_{ik}}).
\end{aligned} \tag{22}$$

Now, define, for $i = 1, 2, \cdots, m$, the following conditional expectations

$$E_{11i}(\alpha,\lambda) = E(\log(X)|t_{i-1} < X \leq t_i) = \frac{\alpha\lambda\int_{t_{i-1}}^{t_i}\log(x)x^{-2}e^{-\lambda/x}(1-e^{-\lambda/x})^{\alpha-1}dx}{(1-e^{-\lambda/t_{i-1}})^{\alpha}-(1-e^{-\lambda/t_i})^{\alpha}} \tag{23}$$

$$E_{21i}(\alpha,\lambda) = E(\log(X)|t_i < X) = \frac{\alpha\lambda\int_{t_i}^{\infty}\log(x)x^{-2}e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}dx}{(1-e^{-\lambda/t_i})^{\alpha}} \tag{24}$$

$$E_{12i}(\alpha,\lambda) = E(X^{-1}|t_{i-1} < X \leq t_i) = \frac{\alpha\lambda\int_{t_{i-1}}^{t_i}x^{-3}e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}dx}{(1-e^{-\lambda/t_{i-1}})^{\alpha}-(1-e^{-\lambda/t_i})^{\alpha}} \tag{25}$$

$$E_{22i}(\alpha,\lambda) = E(X^{-1}|t_i < X) = \frac{\alpha\lambda\int_{t_i}^{\infty}x^{-3}e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}dx}{(1-e^{-\lambda/t_i})^{\alpha}} \tag{26}$$

$$\begin{aligned} E_{13i}(\alpha,\lambda) &= E(\log(1-e^{-\lambda/X})|t_{i-1} < X \leq t_i) \\ &= \frac{\alpha\lambda\int_{t_{i-1}}^{t_i}\log(1-e^{-\lambda/x})x^{-2}e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}dx}{(1-e^{-\lambda/t_{i-1}})^{\alpha}-(1-e^{-\lambda/t_i})^{\alpha}} \end{aligned} \tag{27}$$

$$E_{23i}(\alpha,\lambda) = E(\log(1-e^{-\lambda/X})|t_i < X) = \frac{\alpha\lambda\int_{t_i}^{\infty}\log(1-e^{-\lambda/x})x^{-2}e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1}dx}{(1-e^{-\lambda/t_i})^{\alpha}}. \tag{28}$$

Then the conditional expectation of the complete log-likelihood function, l^c, given the observed values, D, can be written as

$$\begin{aligned} E(l^c(\alpha,\lambda|\mathbf{Z},\mathbf{W})|D) =& n\log(\alpha) + n\log(\lambda) - 2\sum_{i=1}^{m} d_i E_{11i}(\alpha,\lambda) - 2\sum_{i=1}^{m} r_i E_{21i}(\alpha,\lambda) \\ & - \lambda\sum_{i=1}^{m} d_i E_{12i}(\alpha,\lambda) - \lambda\sum_{i=1}^{m} r_i E_{22i}(\alpha,\lambda) + (\alpha-1)\sum_{i=1}^{m} d_i E_{13i}(\alpha,\lambda) \\ & + (\alpha-1)\sum_{i=1}^{m} r_i E_{23i}(\alpha,\lambda). \end{aligned} \tag{29}$$

By computing the first partial derivatives of the log-likelihood function with respect to the unknown parameters, α and λ, and equating the resulted equations with zero, we get

$$\alpha = -\frac{n}{\sum_{i=1}^{m} d_i E_{13i}(\alpha,\lambda) + \sum_{i=1}^{m} r_i E_{23i}(\alpha,\lambda)} \tag{30}$$

$$\lambda = \frac{n}{\sum_{i=1}^{m} d_i E_{12i}(\alpha,\lambda) + \sum_{i=1}^{m} r_i E_{22i}(\alpha,\lambda)}. \tag{31}$$

Therefore the EM algorithm works as follows. Set initial values of α and λ as $\alpha^{(0)}$ and $\lambda^{(0)}$.

Step(i) At k-th iteration, let $(\alpha^{(k)}, \lambda^{(k)})$ be an estimate of (α, λ).

Step(ii) Using the expressions (25)-(28), compute $E_{12}(\alpha^{(k)}, \lambda^{(k)})$, $E_{22}(\alpha^{(k)}, \lambda^{(k)})$, $E_{13}(\alpha^{(k)}, \lambda^{(k)})$ and $E_{23}(\alpha^{(k)}, \lambda^{(k)})$, where α and λ are replaced by $\alpha^{(k)}$ and $\lambda^{(k)}$, respectively.

Step(iii) Compute $\alpha^{(k+1)}$ and $\lambda^{(k+1)}$ using (30) and (31).

Step(iv) If $|\alpha^{(k+1)} - \alpha^{(k)}| + |\lambda^{(k+1)} - \lambda^{(k)}| < \epsilon$, for some pre-specified quantity ϵ, then set $\alpha^{(k+1)}$ and $\lambda^{(k+1)}$, as the MLEs of α and β, otherwise, set $k = k + 1$ and go to **Step(ii)**.

2.2 Stochastic EM Algorithm

The Stochastic EM (SEM) algorithm is an alternative method of the EM algorithm where the expectation in the E-step is calculated using Monte Carlo simulations. It is useful for the cases when the E-step is hard to calculate exactly. The idea of approximating the E-step in EM algorithm by the Monte-Carlo technique, was first proposed by Wei and Tanner (1990). As mentioned by Wang and Cheng (2010), the approximation have more time-consuming. Later Diebolt and Celeux (1993) modified their idea by replacing the E-step with stochastic step through simulation technique. For more information about SEM, see for example, Tregouet et al. (2004), Zhang and Haenggi (2014) and Arabi Belaghi et al. (2017).

The main idea of SEM method can be described as follows. Observe that the conditional survival functions of X given $a < X \leq b$ can be written as

$$S(t|a < t \leq b) = P(X > t|a < X \leq b) = \frac{S(t) - S(b)}{S(a) - S(b)}. \tag{32}$$

Now, we state the procedure for simulate random variate from the GIED in the interval $[a, b]$. Let $u \sim U(0, 1)$. Observe that, by solving the expression

$$\frac{(1 - e^{-\lambda/t})^{\alpha} - (1 - e^{-\lambda/b})^{\alpha}}{(1 - e^{-\lambda/a})^{\alpha} - (1 - e^{-\lambda/b})^{\alpha}} = u$$

with respect to t, we obtain

$$t = \frac{-\lambda}{\log\left[1 - [u\Big((1 - e^{-\lambda/a})^{\alpha} - (1 - e^{-\lambda/b})^{\alpha}\Big) + (1 - e^{-\lambda/b})^{\alpha}]^{\frac{1}{\alpha}}\right]} \tag{33}$$

Note that, when b approaches to ∞, the above expression reduces to

$$t = \frac{-\lambda}{\log\left(1 - \left[u\Big((1 - e^{-\lambda/a})^{\alpha}\Big)\right]^{1/\alpha}\right)}. \tag{34}$$

Now, we first generate independent d_i number of samples $z_{ij}, i = 1, 2, \cdots, m; j = 1, \cdots, d_i$ from the conditional survival function given in (32) with a and b are replaced by t_{i-1} and t_i, respectively. Next, we generate r_i number of samples of $w_{ij}, i =$

$1, 2, \cdots, m; j = 1, \cdots, r_i$ from the conditional survival function given in (32) with a is replaced by t_i. Using these simulated samples, Equations (30) and (31) reduce to

$$\alpha = -\frac{n}{\sum_{i=1}^{m}\sum_{j=1}^{d_i}\log(1-e^{-\lambda/z_{ij}}) + \sum_{i=1}^{m}\sum_{j=1}^{r_i}\log(1-e^{-\lambda/w_{ij}})} \tag{35}$$

$$\lambda = \frac{n}{\sum_{i=1}^{m}\sum_{j=1}^{d_i}(1/z_{ij}) + \sum_{i=1}^{m}\sum_{j=1}^{r_i}(1/w_{ij})}. \tag{36}$$

Therefore the SEM algorithm works as follows. Set initial values of α and λ as $\alpha^{(0)}$ and $\lambda^{(0)}$.

Step(i) At k-th iteration, let $(\alpha^{(k)}, \lambda^{(k)})$ be the estimate of (α, λ).

Step(ii) Using the expression (33), simulate $z_{ij} \equiv z_{ij}(\alpha^{(k)}, \lambda^{(k)}), i = 1, \cdots, m; j = 1, \cdots, d_i$ and using the expression (34), simulate $w_{ij} \equiv w_{ij}(\alpha^{(k)}, \lambda^{(k)}), i = 1, \cdots, m; j = 1, \cdots, r_i$ where α and λ are replaced by $\alpha^{(k)}$ and $\lambda^{(k)}$, respectively.

Step(iii) Compute $\alpha^{(k+1)}$ and $\lambda^{(k+1)}$ using (35) and (36).

Step(iv) If $|\alpha^{(k+1)} - \alpha^{(k)}| + |\lambda^{(k+1)} - \lambda^{(k)}| < \epsilon$, for some pre-specified quantity ϵ, then set $\alpha^{(k+1)}$ and $\lambda^{(k+1)}$, as the MLEs of α and β, otherwise, set $k = k + 1$ and go to **Step(ii).**

2.3 Midpoint Approximation Method

In this subsection, we estimate unknown parameters of a GIED using the mid point approximation method. The main idea of this method is to approximate the progressive type I interval censored data by type I censored data. We assume that d_i number of failures is observed at the center $a_i = (t_{i-1} + t_i)/2$ of i-th interval $(t_{i-1}, t_i]$ and also r_i number of units are censored at the inspection time $t_i, i = 1, 2, \cdots, m$. The log-likelihood function of α and λ based the this type of observations can be written as

$$\begin{aligned} l^m(\alpha, \lambda|data) =& \sum_{i=1}^{m}\Big[d_i\log[f(a_i)] + r_i\log[1-F(t_i)]\Big] \\ =& \log(\alpha)\sum_{i=1}^{m} d_i + \log(\lambda)\sum_{i=1}^{m} d_i - 2\sum_{i=1}^{m} d_i\log(a_i) - \lambda\sum_{i=1}^{m} d_i/a_i \\ &+ (\alpha-1)\sum_{i=1}^{m} d_i\log(1-e^{-\lambda/a_i}) + \alpha\sum_{i=1}^{m} r_i\log(1-e^{-\lambda/t_i}). \end{aligned} \tag{37}$$

Subsequently, we need to solve the following system of equations to obtain the midpoint estimates of unknown parameters

$$\sum_{i=1}^{m}\frac{d_i}{\alpha} + \sum_{i=1}^{m} d_i\log(1-e^{-\lambda/a_i}) + \sum_{i=1}^{m} r_i\log(1-e^{-\lambda/t_i}) = 0 \tag{38}$$

and

$$\sum_{i=1}^{m} \frac{d_i}{\lambda} - \sum_{i=1}^{m} d_i/a_i + (\alpha-1)\sum_{i=1}^{m} \frac{d_i e^{-\lambda/a_i}/a_i}{1-e^{-\lambda/a_i}} + \alpha \sum_{i=1}^{m} \frac{r_i e^{-\lambda/t_i}/t_i}{1-e^{-\lambda/t_i}} = 0. \tag{39}$$

Likelihood Equations (38) and (39) cannot be solved analytically due to their nonlinear nature. Here we may adopt a numerical method like Newton-Raphson method to obtain the estimates of α and λ.

3 Estimation using Probability Plot

Let $(r_i, d_i, t_i), i = 1, \cdots, m$, with $n = \sum_{i=1}^{m}(d_i + r_i)$ denote a progressive type I interval censored sample from a GIED distribution. The cumulative distribution function at time t_i can be estimated based on this sample as

$$\hat{F}(t_i) = 1 - \prod_{j=1}^{i}(1-\hat{p}_j), \tag{40}$$

where

$$\hat{p}_j = \frac{d_j}{n - \sum_{k=0}^{j-1}(d_k + r_k)}; j = 1, \cdots, m.$$

Estimating the parameters using probability plot method can be performed by finding the values of α and λ that minimize the function

$$\begin{aligned} S &= \sum_{i=1}^{m}(F(t_i) - \hat{F}(t_i))^2 \\ &= \sum_{i=1}^{m}\left(1 - (1-e^{-\lambda/t_i})^{\alpha} - \hat{F}(t_i)\right)^2. \end{aligned}$$

So, we need to solve the following system of equations $\frac{\partial S}{\partial \alpha} = 0$ and $\frac{\partial S}{\partial \lambda} = 0$ where

$$\begin{aligned} \frac{\partial S}{\partial \alpha} &= -2\sum_{i=1}^{m}(1 - (1-e^{-\lambda/t_i})^{\alpha} - \hat{F}(t_i))(1-e^{-\lambda/t_i})^{\alpha}\log(1-e^{-\lambda/t_i}) \\ \frac{\partial S}{\partial \lambda} &= -2\alpha\sum_{i=1}^{m}(1 - (1-e^{-\lambda/t_i})^{\alpha} - \hat{F}(t_i))(1-e^{-\lambda/t_i})^{\alpha-1}\frac{1}{t_i}e^{-\lambda/t_i}. \end{aligned}$$

These estimates can be computed numerically using some nonlinear optimization technique.

4 Method of moments estimation

The kth population moment of a GIED distribution with pdf given in (2) has not an explicit form and can be computed by

$$\begin{aligned}E_{\alpha,\lambda}(X^k) =&\alpha\lambda\int_0^\infty x^{k-2}e^{-\lambda/x}(1-e^{-\lambda/x})^{\alpha-1}dx\\ =&k\int_0^\infty x^{k-1}(1-e^{-\lambda/x})^{\alpha}dx, \quad k\in\mathbb{I}^+,\end{aligned}$$

where $\mathbb{I}^+$ is the set of positive integers. Substituting $w=e^{-\lambda/x}$ in the above integral gives us

$$E_{\alpha,\lambda}(X^k) =\alpha\lambda^k(-1)^k\int_0^1\frac{(1-w)^{\alpha-1}}{(\log w)^k}dw.$$

Clearly the above integral converges if $\alpha > k$. Therefore, we consider the moments with negative integer powers. Let $Y=1/X$. Then Y follows general exponential distribution and consequently

$$\begin{aligned}E_{\alpha,\lambda}(X^{-1}) =&E_{\alpha,\lambda}(Y)=(\psi(\alpha+1)-\psi(1))/\lambda\\ E_{\alpha,\lambda}(X^{-2}) =&E_{\alpha,\lambda}(Y^2)=(\psi'(1)-\psi'(\alpha+1)-(\psi(\alpha+1)-\psi(1))^2)/\lambda^2,\end{aligned}$$

where ψ is the digamma function and ψ' is its derivative (see Gupta and Kundu (1999)). Now, the kth negative population moment of a doubly truncated GIED distribution in the interval $[a,b)$, $0<a<b$ is given by

$$\begin{aligned}E_{\alpha,\lambda}[X^{-k}|X\in[a,b]] =&\frac{\int_a^b x^{-k}f(x;\alpha,\lambda)}{F(b;\alpha,\lambda)-F(a;\alpha,\lambda)}\\ =&\frac{\alpha\lambda\int_a^b x^{-k-2}e^{-\lambda/x}(1-e^{-\lambda/x})^{\alpha-1}dx}{(1-e^{-\lambda/a})^{\alpha}-(1-e^{-\lambda/b})^{\alpha}}.\end{aligned} \tag{41}$$

By equating the first and the second negative sample moments to the corresponding population moments, we obtain the following two equations

$$\frac{(\psi(\alpha+1)-\psi(1))}{\lambda} = \frac{1}{n}\left[\sum_{i=1}^m d_iE_{\alpha,\lambda}[X^{-1}|X\in[t_{i-1},t_i]]+\sum_{i=1}^m r_iE_{\alpha,\lambda}[X^{-1}|X\in[t_i,\infty)]\right] \tag{42}$$

and

$$\begin{aligned}\frac{\psi'(1)-\psi'(\alpha+1)-(\psi(\alpha+1)-\psi(1))^2}{\lambda^2} &= \frac{1}{n}\Big[\sum_{i=1}^m d_iE_{\alpha,\lambda}[X^{-2}|X\in[t_{i-1},t_i]]\\ &+\sum_{i=1}^m r_iE_{\alpha,\lambda}[X^{-2}|X\in[t_i,\infty]]\Big]\end{aligned} \tag{43}$$

Since the closed form of the solution to (42) and (43) could not be obtained, iterative procedure can be employed as follows. Set $\alpha^{(0)}$ and $\lambda^{(0)}$ as initial values of α and λ.

Step(i) At k-th iteration, let $(\alpha^{(k)}, \lambda^{(k)})$ be an estimate of (α, λ).

Step(ii) Compute $\alpha^{(k+1)}$ by solving the following equation for α

$$\frac{n(\psi(\alpha+1)-\psi(1))^2}{\psi'(1)-\psi'(\alpha+1)-(\psi(\alpha+1)-\psi(1))^2}$$
$$= \frac{\left(\sum_{i=1}^m d_i E_{\alpha^{(k)},\lambda^{(k)}}[X^{-1}|X\in[t_{i-1},t_i]] + \sum_{i=1}^m r_i E_{\alpha^{(k)},\lambda^{(k)}}[X^{-1}|X\in[t_i,\infty)]\right)^2}{\sum_{i=1}^m d_i E_{\alpha^{(k)},\lambda^{(k)}}[X^{-2}|X\in[t_{i-1},t_i]] + \sum_{i=1}^m r_i E_{\alpha^{(k)},\lambda^{(k)}}[X^{-2}|X\in[t_i,\infty)]}$$

Step(iii) Compute $\lambda^{(k+1)}$, using

$$\lambda^{(k+1)} = \frac{n(\psi(\alpha^{(k+1)}+1)-\psi(1))}{\sum_{i=1}^m d_i E_{\alpha^{(k+1)},\lambda^{(k)}}[X^{-1}|X\in[t_{i-1},t_i]] + \sum_{i=1}^m r_i E_{\alpha^{(k+1)},\lambda^{(k)}}[X^{-1}|X\in[t_i,\infty)]}.$$

Step(iv) If $|\alpha^{(k)}-\alpha^{(k+1)}|+|\lambda^{(k)}-\lambda^{(k+1)}|<\epsilon$, for ϵ pre-specified quantity, set $\alpha^{(k+1)}$ and $\lambda^{(k+1)}$ as the method of moments estimators of α and λ. Otherwise, set $k=k+1$ and go to **Steps**(ii).

5 Simulation

In this section, a simulation study is conducted in order to explore the performance of the proposed methods to estimate the GIED parameters under progressive type I interval censored data. We considered the parameter values and sample sizes, respectively, as $(\alpha,\lambda)=(0.5,0.5),(1.5,1)$ and $n=25,50,100$ and we consider $m=5$ for all the cases. Four different progressive type I interval censored schemes are adopted here, namely
$\mathbf{p}_1=(0.25,0.25,0.5,0.5,1)$
$\mathbf{p}_2=(0.5,0.5,0.25,0.25,1)$
$\mathbf{p}_3=(0,0,0,0,1)$
$\mathbf{p}_4=(0.25,0,0,0,1)$.
The above schemes are chosen to specify the percentage of surviving units to be withdrawn at the 5 censoring and monitoring points. Observe that, in Scheme 1, the first two intervals the removal is lighter as compared to the last two intervals and the Scheme 2, is the reverse scenario of Scheme 1. Moreover, in Scheme 3, no removal is done prior to termination which is a case similar to conventional type I interval censored and in Scheme 4, removal is conducted at the left-most and right-most ends.

Data is simulated by employing an algorithm proposed by Aggarwal and Jacques (2001) to generate number of failures $d_1,d_2,\cdots,d_m$ in each interval $(t_{i-1},t_i]$, for $i=1,\cdots,m$ from sample of size n. The data generation algorithm is described as follows. Given n,m and $\mathbf{p}=(p_1,\cdots,p_m)$ where $0\le p_i\le 1$ and $p_m=1$.

Step (i) Generate $t_1^*,\cdots,t_m^*$ from GIED(α,λ) using $t_i^*=-\lambda/\log(1-U_i^{1/\alpha})$, where $U_i\sim U(0,1)$.

Step(ii) Arrange $t_1^*,\cdots,t_m^*$ as $t_1<t_2<\cdots<t_m$.

Step(iii) Compute $F_i = F(t_i), i = 1, \cdots, m$ using (1).

Step(iv) Set $d_0 = r_0 = F_0 = 0$ and $i = 1$.

Step(v) Generate

$$d_i|(d_0, \cdots, d_{i-1}, r_0, \cdots, r_{i-1}) \sim \text{binomial}\Big(n - \sum_{j=0}^{i-1}(d_j + r_j), q_i\Big),$$

where $q_i = \frac{F_i - F_{i-1}}{1 - F_{i-1}}$.

Step(vi) Compute

$$r_i = \Big\lceil p_i\Big(n - \sum_{j=0}^{i} d_j - \sum_{j=0}^{i-1} r_j\Big)\Big\rceil,$$

where $\lceil x \rceil$ denotes the largest integer not greater than x.

Step(vii) If $i < m$, replace i by $i + 1$ and go to **Step(v)**, otherwise stop.

For the bootstrap confidence intervals, the size of the bootstrap samples is taken to be 5000.

At each iteration, we estimate the parameters using the MLE via Newton-Raphson, EM and SEM, probability plot (PP), mid-point (MP) and method of moments (MM) methods. For each of these methods, we have computed the absolute average bias (Bias), the root mean square error (RMSE), the sample standard deviation (SSE), the estimated standard deviation (ESE) via the observed information matrix. Moreover, we have evaluated the widths (Len) of 95% Wald's confidence intervals using the observed information matrix (CI) and 95% Boot-p (BT) confidence intervals with their empirical coverage probabilities (CP). The process for the estimation is replicated 1000 times and the results of estimation are reported in Tables 1-7.

From Tables 1-6, it is observed that the Bias for all the estimators, in general, are reasonably small which indicates that the estimated values are close to the true parameter values. However the MP method, as expected, presents more bias estimates than the other methods. In addition, the SEM algorithm performs worse than NR and EM based on this aspect. Clearly, the RMSE of MP is higher than that of the other methods. Moreover, the values of SSE and ESE of NR and EM methods are close, especially for large n. This indicates that ESE based on the inverse of the observed information matrix can be considered as a reasonable estimate of the SSE. As expected, the Bias, RMSE, SSE an ESE of all estimators are decreasing when sample sizes are increasing for all the cases. With respect to 95% confidence interval, from Table 7, the length of the confidence intervals is decreasing when the value of sample size is increasing. Moreover, the empirical coverage probabilities of 95% confidence intervals (CP) are very close to the nominal level for all the cases. Hence, the performances of the all proposed methods are satisfactory in terms of the biases, RMSE, standard errors and 95% confidence intervals of the estimates.

With respect to the censoring scheme, selecting a censoring scheme has a considerable effect on the simulation results. It is easy to see that, based on bias, RMSE, ESE, and confidence intervals, the censoring scheme $\mathbf{p}_3$ which is the traditional right-censored scheme shows better performance among the proposed schemes while scheme $\mathbf{p}_2$ has the worst performance.

Table 1: Simulation results of the proposed methods of estimation for $n = 25$

		$\alpha = 1.5$				$\lambda = 1$			
	Method	Bias	RMSE	ESE	SSE	Bias	RMSE	ESE	SSE
$\mathbf{p}_1$	NR	0.459	1.717	1.026	1.228	0.162	0.277	0.501	0.501
	EM	0.460	1.713	1.027	1.226	0.163	0.276	0.502	0.499
	SEM	0.685	1.734	1.201	1.125	0.317	0.255	0.571	0.394
	PP	0.406	1.661	-	-	0.142	0.272	-	-
	MM	0.369	1.495	-	-	0.117	0.264	-	-
	MP	2.287	10.578	-	-	0.195	0.078	-	-
$\mathbf{p}_2$	NR	0.782	3.768	1.622	1.778	0.222	0.417	0.619	0.607
	EM	0.784	3.745	1.623	1.770	0.225	0.411	0.622	0.601
	SEM	1.067	3.795	1.623	1.631	0.400	0.391	0.601	0.480
	PP	0.726	4.450	-	-	0.189	0.448	-	-
	MM	0.639	3.100	-	-	0.162	0.388	-	-
	MP	1.842	7.026	-	-	0.209	0.071	-	-
$\mathbf{p}_3$	NR	0.387	1.269	0.817	1.059	0.132	0.244	0.439	0.476
	EM	0.387	1.267	0.817	1.058	0.132	0.244	0.439	0.476
	SEM	0.592	1.361	0.926	1.006	0.275	0.219	0.487	0.379
	PP	0.384	1.507	-	-	0.119	0.266	-	-
	MM	0.349	1.268	-	-	0.104	0.259	-	-
	MP	1.669	6.731	-	-	0.114	0.069	-	-
$\mathbf{p}_4$	NR	0.453	1.777	0.970	1.254	0.160	0.268	0.477	0.493
	EM	0.453	1.774	0.971	1.253	0.160	0.267	0.478	0.492
	SEM	0.675	1.878	1.113	1.193	0.313	0.254	0.532	0.396
	PP	0.386	1.824	-	-	0.127	0.273	-	-
	MM	0.339	1.343	-	-	0.106	0.256	-	-
	MP	2.642	12.356	-	-	0.247	0.103	-	-

Table 2: Simulation results of the proposed methods of estimation for $n = 50$

		$\alpha = 1.5$				$\lambda = 1$			
	Method	Bias	RMSE	ESE	SSE	Bias	RMSE	ESE	SSE
	NR	0.228	0.543	0.620	0.701	0.088	0.139	0.349	0.363
	EM	0.229	0.542	0.621	0.700	0.088	0.138	0.349	0.361
$\mathbf{p}_1$	SEM	0.427	0.532	0.621	0.592	0.219	0.118	0.349	0.265
	PP	0.199	0.565	-	-	0.073	0.141	-	-
	MM	0.204	0.556	-	-	0.071	0.149	-	-
	MP	1.673	4.714	-	-	0.143	0.040	-	-
	NR	0.311	1.042	0.860	0.973	0.093	0.188	0.420	0.424
	EM	0.312	1.031	0.860	0.967	0.094	0.186	0.421	0.421
$\mathbf{p}_2$	SEM	0.555	1.010	0.860	0.838	0.246	0.157	0.421	0.311
	PP	0.253	1.001	-	-	-	-	0.069	0.186
	MM	0.259	1.020	-	-	0.063	0.193	-	-
	MP	1.340	2.827	-	-	0.167	0.041	-	-
	NR	0.156	0.305	0.468	0.530	0.069	0.100	0.299	0.309
	EM	0.156	0.304	0.468	0.530	0.069	0.100	0.299	0.309
	SEM	0.311	0.288	0.522	0.438	0.183	0.086	0.324	0.230
$\mathbf{p}_3$	PP	0.139	0.355	-	-	0.057	0.106	-	-
	MM	0.133	0.339	-	-	0.051	0.112	-	-
	MP	1.255	2.432	-	-	0.138	0.030	-	-
	NR	0.188	0.504	0.553	0.685	0.061	0.117	0.322	0.336
	EM	0.188	0.503	0.553	0.684	0.062	0.116	0.322	0.336
$\mathbf{p}_4$	SEM	0.385	0.505	0.635	0.598	0.195	0.097	0.354	0.243
	PP	0.181	0.638	-	-	0.052	0.128	-	-
	MM	0.169	0.546	-	-	0.045	0.129	-	-
	MP	2.118	6.760	-	-	0.203	0.065	-	-

Table 3: Simulation results of the proposed methods of estimation for $n = 100$

		$\alpha = 1.5$				$\lambda = 1$			
	Method	Bias	RMSE	ESE	SSE	Bias	RMSE	ESE	SSE
	NR	0.087	0.171	0.397	0.404	0.033	0.064	0.244	0.251
	EM	0.087	0.170	0.397	0.403	0.034	0.064	0.244	0.250
$\mathbf{p}_1$	SEM	0.268	0.168	0.461	0.311	0.153	0.053	0.267	0.172
	PP	0.076	0.190	-	-	0.027	0.065	-	-
	MM	0.088	0.208	-	-	0.029	0.073	-	-
	MP	1.326	2.250	-	-	0.110	0.020	-	-
	NR	0.145	0.393	0.540	0.610	0.044	0.088	0.294	0.294
	EM	0.146	0.388	0.540	0.606	0.045	0.087	0.295	0.291
$\mathbf{p}_2$	SEM	0.354	0.378	0.661	0.503	0.177	0.075	0.338	0.210
	PP	0.111	0.390	-	-	0.028	0.087	-	-
	MM	0.121	0.417	-	-	0.028	0.097	-	-
	MP	1.162	1.824	-	-	0.148	0.029	-	-
	NR	0.065	0.108	0.304	0.322	0.034	0.045	0.208	0.210
	EM	0.065	0.107	0.303	0.321	0.034	0.045	0.208	0.210
$\mathbf{p}_3$	SEM	0.196	0.104	0.336	0.257	0.127	0.040	0.222	0.156
	PP	0.060	0.128	-	-	0.030	0.049	-	-
	MM	0.056	0.134	-	-	0.025	0.053	-	-
	MP	1.029	1.510	-	-	0.102	0.020	-	-
	NR	0.093	0.156	0.357	0.384	0.036	0.057	0.225	0.236
	EM	0.093	0.156	0.357	0.384	0.036	0.057	0.225	0.236
$\mathbf{p}_4$	SEM	0.241	0.150	0.399	0.303	0.138	0.050	0.241	0.178
	PP	0.077	0.171	-	-	0.027	0.058	-	-
	MM	0.073	0.174	-	-	0.021	0.062	-	-
	MP	1.848	4.199	-	-	0.187	0.047	-	-

Table 4: Simulation results of the proposed methods of estimation for $n = 25$

		$\alpha = 0.5$				$\lambda = 0.5$			
	Method	Bias	RMSE	ESE	SSE	Bias	RMSE	ESE	SSE
	NR	0.086	0.089	0.269	0.286	0.104	0.152	0.374	0.376
	EM	0.090	0.089	0.271	0.284	0.108	0.151	0.377	0.373
$\mathbf{p}_1$	SEM	0.178	0.095	0.269	0.252	0.229	0.151	0.374	0.315
	PP	0.103	0.110	-	-	0.136	0.207	-	-
	MM	0.087	0.111	-	-	0.095	0.158	-	-
	MP	0.416	0.361	-	-	0.285	0.100	-	-
	NR	0.197	0.352	0.411	0.560	0.190	0.271	0.449	0.485
	EM	0.207	0.351	0.417	0.555	0.202	0.269	0.457	0.478
$\mathbf{p}_2$	SEM	0.292	0.372	0.417	0.536	0.312	0.275	0.457	0.422
	PP	0.216	0.807	-	-	0.240	1.016	-	-
	MM	0.180	0.321	-	-	0.171	0.264	-	-
	MP	0.542	0.613	-	-	0.382	0.174	-	-
	NR	0.049	0.038	0.183	0.189	0.078	0.109	0.312	0.321
	EM	0.049	0.038	0.183	0.189	0.078	0.109	0.312	0.320
$\mathbf{p}_3$	SEM	0.114	0.040	0.212	0.165	0.178	0.097	0.368	0.255
	PP	0.051	0.043	-	-	0.085	0.123	-	-
	MM	0.051	0.053	-	-	0.072	0.122	-	-
	MP	0.367	0.272	-	-	0.227	0.068	-	-
	NR	0.064	0.058	0.213	0.233	0.102	0.127	0.336	0.341
	EM	0.066	0.058	0.214	0.232	0.104	0.126	0.337	0.340
$\mathbf{p}_4$	SEM	0.140	0.061	0.253	0.203	0.210	0.124	0.397	0.283
	PP	0.069	0.063	-	-	0.115	0.145	-	-
	MM	0.075	0.082	-	-	0.104	0.146	-	-
	MP	0.398	0.359	-	-	0.274	0.097	-	-

Table 5: Simulation results of the proposed methods of estimation for $n = 50$

		$\alpha = 0.5$				$\lambda = 0.5$			
	Method	Bias	RMSE	ESE	SSE	Bias	RMSE	ESE	SSE
	NR	0.032	0.034	0.177	0.181	0.050	0.075	0.262	0.269
	EM	0.034	0.033	0.178	0.179	0.052	0.074	0.263	0.267
$\mathbf{p}_1$	SEM	0.105	0.031	0.214	0.143	0.153	0.065	0.313	0.203
	PP	0.035	0.037	-	-	0.058	0.094	-	-
	MM	0.038	0.042	-	-	0.052	0.083	-	-
	MP	0.318	0.162	-	-	0.258	0.074	-	-
	NR	0.071	0.071	0.239	0.258	0.079	0.105	0.307	0.315
	EM	0.076	0.069	0.242	0.252	0.085	0.102	0.312	0.307
$\mathbf{p}_2$	SEM	0.158	0.073	0.239	0.220	0.195	0.101	0.307	0.250
	PP	0.083	0.082	-	-	0.104	0.134	-	-
	MM	0.071	0.081	-	-	0.075	0.117	-	-
	MP	0.427	0.297	-	-	0.348	0.133	-	-
	NR	0.019	0.016	0.121	0.127	0.022	0.043	0.211	0.207
	EM	0.019	0.016	0.122	0.126	0.022	0.043	0.212	0.207
$\mathbf{p}_3$	SEM	0.071	0.015	0.138	0.101	0.113	0.034	0.246	0.147
	PP	0.019	0.017	-	-	0.025	0.047	-	-
	MM	0.020	0.023	-	-	0.020	0.053	-	-
	MP	0.299	0.136	-	-	0.201	0.048	-	-
	NR	0.035	0.025	0.143	0.155	0.049	0.057	0.230	0.234
	EM	0.036	0.025	0.143	0.155	0.049	0.057	0.230	0.233
$\mathbf{p}_4$	SEM	0.091	0.025	0.162	0.130	0.131	0.047	0.262	0.174
	PP	0.036	0.027	-	-	0.053	0.061	-	-
	MM	0.041	0.038	-	-	0.049	0.072	-	-
	MP	0.342	0.182	-	-	0.256	0.075	-	-

Table 6: Simulation results of the proposed methods of estimation for $n = 100$

		$\alpha = 0.5$				$\lambda = 0.5$			
	Method	Bias	RMSE	ESE	SSE	Bias	RMSE	ESE	SSE
	NR	0.022	0.017	0.124	0.129	0.029	0.036	0.186	0.187
	EM	0.022	0.017	0.124	0.128	0.030	0.035	0.186	0.185
$\mathbf{p}_1$	SEM	0.080	0.017	0.144	0.104	0.113	0.032	0.212	0.140
	PP	0.028	0.020	-	-	0.041	0.046	-	-
	MM	0.020	0.020	-	-	0.026	0.040	-	-
	MP	0.305	0.123	-	-	0.256	0.070	-	-
	NR	0.018	0.023	0.158	0.152	0.043	0.216	0.206	0.206
	EM	0.019	0.022	0.159	0.149	0.020	0.041	0.217	0.201
$\mathbf{p}_2$	SEM	0.094	0.025	0.193	0.126	0.119	0.038	0.254	0.155
	PP	0.021	0.029	-	-	0.024	0.058	-	-
	MM	0.026	0.031	-	-	0.024	0.051	-	-
	MP	0.364	0.169	-	-	0.327	0.111	-	-
	NR	0.009	0.007	0.084	0.086	0.018	0.023	0.150	0.151
	EM	0.009	0.007	0.084	0.086	0.018	0.023	0.150	0.150
$\mathbf{p}_3$	SEM	0.052	0.007	0.093	0.066	0.092	0.022	0.168	0.115
	PP	0.009	0.008	-	-	0.019	0.024	-	-
	MM	0.012	0.011	-	-	0.020	0.029	-	-
	MP	0.269	0.093	-	-	0.192	0.040	-	-
	NR	0.016	0.011	0.098	0.104	0.025	0.030	0.161	0.172
	EM	0.016	0.011	0.098	0.104	0.025	0.030	0.161	0.172
$\mathbf{p}_4$	SEM	0.062	0.010	0.109	0.079	0.094	0.024	0.179	0.122
	PP	0.017	0.012	-	-	0.027	0.032	-	-
	MM	0.023	0.016	-	-	0.031	0.038	-	-
	MP	0.303	0.118	-	-	0.243	0.064	-	-

Table 7: Widths of 95% confidence interval of α and λ and their coverage probabilities.

			$\alpha = 1.5$		$\lambda = 1$		$\alpha = 0.5$		$\lambda = 0.5$	
n			Len	CP	Len	CP	Len	CP	Len	CP
25	$\mathbf{p}_1$	CI	4.830	97.0	2.265	93.0	1.204	96.0	2.080	94.0
		BT	5.973	92.8	2.081	94.3	1.440	95.5	1.605	95.7
	$\mathbf{p}_2$	CI	8.774	96.0	3.010	92.0	2.008	95.0	2.768	93.0
		BT	7.248	93.3	2.317	93.5	2.911	92.5	2.035	92.1
	$\mathbf{p}_3$	CI	3.649	94.0	1.925	92.0	0.770	96.0	1.529	94.0
		BT	4.908	92.5	1.863	92.9	0.892	95.5	1.335	95.0
	$\mathbf{p}_4$	CI	4.517	96.0	2.122	93.0	0.914	95.0	1.692	93.0
		BT	5.756	92.0	2.013	93.6	1.139	94.0	1.458	95.0
50	$\mathbf{p}_1$	CI	2.644	95.0	1.473	93.0	0.744	96.0	1.226	94.0
		BT	3.294	93.0	1.438	94.0	0.778	96.1	1.050	95.6
	$\mathbf{p}_2$	CI	3.897	96.0	1.837	92.0	1.049	95.0	1.546	92.0
		BT	4.900	93.0	1.716	94.0	1.230	94.2	1.270	94.7
	$\mathbf{p}_3$	CI	1.930	95.0	1.239	93.0	0.493	96.0	0.931	96.0
		BT	2.373	93.0	1.237	93.0	0.517	94.9	0.853	95.9
	$\mathbf{p}_4$	CI	2.327	95.0	1.346	94.0	0.586	95.0	1.025	94.0
		BT	3.031	93.0	1.354	95.0	0.630	94.6	0.930	95.5
100	$\mathbf{p}_1$	CI	1.618	95.0	0.992	95.0	0.504	96.0	0.797	95.0
		BT	1.780	94.0	0.979	94.6	0.512	95.4	0.732	96.0
	$\mathbf{p}_2$	CI	2.270	95.0	1.219	94.0	0.656	96.0	0.961	95.0
		BT	2.632	93.6	1.192	95.0	0.678	96.4	0.847	96.3
	$\mathbf{p}_3$	CI	1.219	95.0	0.837	95.0	0.334	94.0	0.624	96.0
		BT	1.331	94.1	0.835	94.2	0.342	94.0	0.596	95.8
	$\mathbf{p}_4$	CI	1.445	95.0	0.912	94.0	0.392	94.0	0.673	94.0
		BT	1.627	94.0	0.912	94.3	0.402	94.0	0.639	94.4

6 Application

In this section, we analyze a data set as a real life application of the GIED under progressive type I interval censored observations. The data set is provided by Bjerkedal et al. (1960), and it represents the survival times (in days) of guinea pigs injected with different doses of tubercle bacilli. It is known that guinea pigs have a high susceptibility to human tuberculosis and that is why they were used in this particular study. The regimen number is the common logarithm of the number of bacillary units in 0.5ml. of challenge solution; i.e., regimen 6.6 corresponds to 4.0×10.6 bacillary units per 0.5 ml. is $(\log(4.0 \times 106) = 6.6)$. Kundu and Howlader (2010) used this data to fit the inverse Weibull distribution. Corresponding to regimen 6.6, there were 72 observations listed below:

$$12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60,$$

$$60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99,$$

$$109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.$$

First, we check whether the GIED is suitable for the data set based on the complete data set. We propose three measures for fitting the data set with GIED and these measure are Akaike's information criterion (AIC), the Bayesian information criterion (BIC) and the minimum distance of Kolmogorov-Simrnov (KS). These measures are defined by

$$\begin{aligned}\text{AIC} &= -\,2l(\hat{\alpha}, \hat{\lambda}|D) + 4\\ \text{BIC} &= -\,2l(\hat{\alpha}, \hat{\lambda}|D) + 2\log(n)\end{aligned}$$

and

$$\text{KS}(F) = \sup_{0 \le t < \infty} |\hat{F}(t) - F(t; \hat{\alpha}, \hat{\lambda})|,$$

where $\hat{\alpha}$ and $\hat{\lambda}$ are the MLEs of α and λ, l is the log-likelihood function given in (4), $\hat{F}$ is the empirical c.d.f. and F is the population c.d.f. given in (1). The values AIC, BIC and KS of some two-parameter lifetimes distributions, namely; GIED, BurrXII, generalized exponential (GExp), Weibull and inverse Weibull (Iweibull) are reported in Table 9. In addition, the curves of the population c.d.f. of GIED, $F(t; \hat{\alpha}, \hat{\lambda})$, and the empirical c.d.f. data set, $\hat{F}$, is depicted in Figure 2. Clearly, from Table 9 and Figure 2, it is shown that the GIED is the best fitted distribution the data comparing with BurrXII, GExp, Weibull and Iweibull distributions.

Next, we estimate α and λ of GIED based on the real data set using the proposed methodology. For analyzing the above data set, we take $m = 5$ and inspection times $t = (40, 90, 150, 190, 220)$. In addition, we consider the same censoring schemes presented in the simulation section, namely $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ and $\mathbf{p}_4$. According to the censoring schemes, the values of (d_i, r_i) within the intervals $I_0 = (0, t_1]$ and $I_i = (t_{i-1}, t_i], i = 1, 2, \cdots, m$ are reported in Table 8. To propose initial values of the parameters, Cantor plot of the log-likelihood function based on the real data set is plotted and is presented in Figure 3. Table 10 presents the estimates and standard errors while Table 11 presents

Table 8: Values of (r_i, d_i) within each interval I_i for the data set

	$\mathbf{p}_1$		$\mathbf{p}_2$		$\mathbf{p}_3$		$\mathbf{p}_4$	
I	d	r	d	r	d	r	d	r
(0,40]	11	16	11	31	11	0	11	16
(40,90]	20	7	5	13	36	0	20	0
(90,150]	7	6	1	3	14	0	14	0
(150,190]	0	3	0	2	2	0	2	0
(190,220]	0	2	0	6	1	8	1	8

the confidence intervals of the parameters, α and β, for the real data sets. From the obtained results, one can see that the values of the MLEs computed using NR and EM methods are very close except for the censoring scheme p_2. Similar conclusion can be observed for the ESE values. With respect to the length of the confidence intervals, both methods; CI and BT have introduced almost the same lengths except for the scheme p_2.

Table 9: The values of MLEs, AIC, BIC and KS of real data set

Distribution	MLEs (α, λ)	AIC	BIC	KS
GIED	(1.435207,86.308831)	155.063097	159.616430	0.088796
BurrXII	(1.37295,0.1)	221.50750	226.06083	0.24498
GExp	(152.39614,0.1)	454.18751	458.74084	0.45205
Weibull	(0.197891,28.571845)	163.320348	167.873680	0.089201
IWeibull	(1.244539,182.158051)	154.737928	159.291260	0.089019

7 Inspection times

We usually, in progressive type I interval censored, identified inspection times by fixed quantities before the start of the experiment. However, it is important to investigate the effect of different inspection times on the efficiency of obtained estimators. This problem under progressive interval censored observations has not received much attention in the literature. Lin et al. (2009) determined optimally spaced inspection times for the two-parameter lognormal distribution under progressive type I interval censored plan. Recently,Arabi Belaghi et al. (2017), Singh and Tripathi (2018) and Lodhi and Tripathi

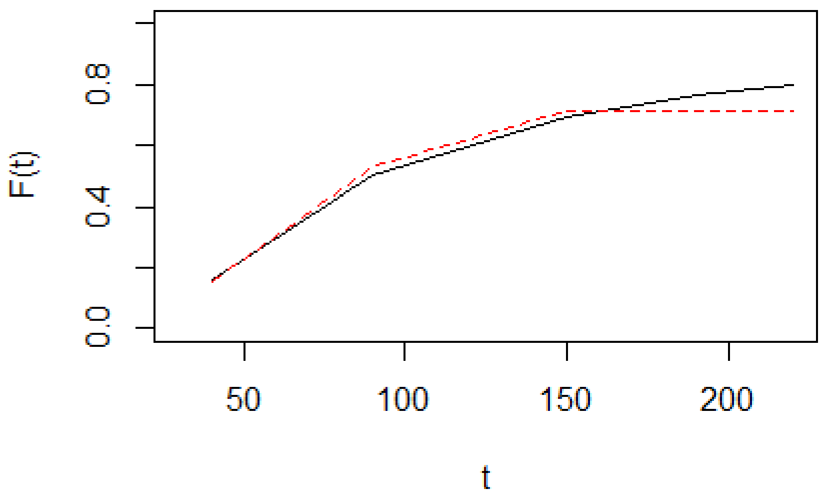

Figure 2: represents the population CDF and Empirical c.d.f. of GIED. Solid line: population c.d.f and dashed lines: empirical c.d.f

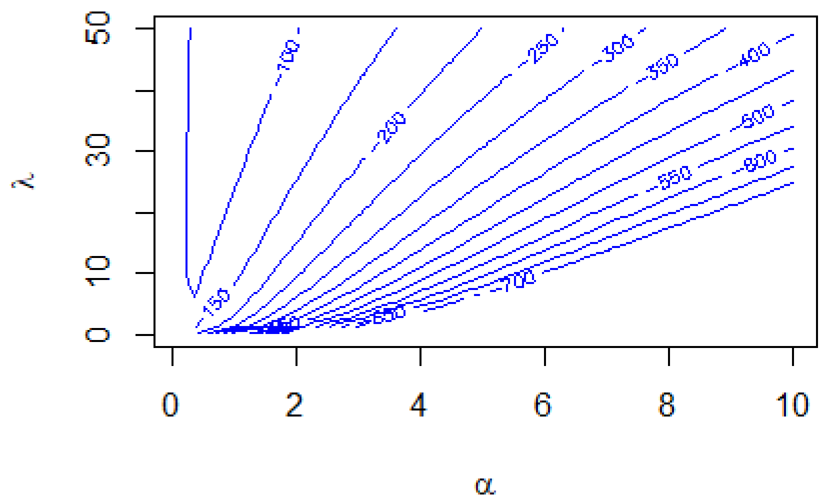

Figure 3: The log-likelihood contour plot of the GIED

Table 10: Estimates of α and λ of the real data set

		α		λ	
	Method	Estim	ESE	Estim	ESE
$\mathbf{p}_1$	NR	1.435	0.438	86.309	18.439
	EM	1.432	0.437	86.162	18.419
	SEM	1.273	0.372	80.390	17.299
	PP	1.126	-	71.319	-
	MM	1.629	-	92.913	-
	MP	0.869	-	36.566	-
$\mathbf{p}_2$	NR	0.229	0.119	26.146	17.041
	EM	0.298	0.169	34.587	20.960
	SEM	0.277	0.149	32.967	19.815
	PP	0.186	-	18.252	-
	MM	0.266	-	30.645	-
	MP	0.254	-	27.562	-
$\mathbf{p}_3$	NR	2.560	0.582	105.410	16.231
	EM	2.557	0.581	105.330	16.218
	SEM	2.329	0.515	99.172	15.406
	PP	2.647	-	106.016	-
	MM	3.085	-	116.689	-
	MP	2.528	-	44.583	-
$\mathbf{p}_4$	NR	1.969	0.507	100.692	17.731
	EM	1.969	0.507	100.659	17.726
	SEM	1.972	0.614	89.278	18.562
	PP	2.070	-	103.790	-
	MM	1.996	-	101.449	-
	MP	1.574	-	42.313	-

Table 11: 95% Wald's confidence intervals and 95%Boot-p confidence intervals α and λ of the real data set.

	Method	α	λ
$\mathbf{p}_1$	CI	(0.789,2.610)	(56.781,131.192)
	BT	(0.775,2.588)	(53.878,130.019)
$\mathbf{p}_2$	CI	(0.083,0.632)	(7.288,93.800)
	BT	(0.100,0.569)	(4.087,65.316)
$\mathbf{p}_3$	CI	(1.639,3.998)	(77.949,142.546)
	BT	(1.644,4.201)	(75.553,142.914)
$\mathbf{p}_4$	CI	(1.189,3.263)	(71.302,142.195)
	BT	(1.213,3.306)	(68.215,140.461)

(2020) obtained various inspection times by using the expected Fisher information matrix for Burr XII, inverse Weibull and truncated normal distributions, respectively.

In the following, we study four different approaches to determine of the inspection times, namely; pre-specified (PS), equally spaced (ES), optimally spaced (OS) and equal probability (EP). In PS approach, time points are commonly pre-determined on the basis of the available knowledge about the experiment. ES inspection times are identified by constructing inspection intervals of equal length in which times points to be included are considered. In specific, if t_m is the termination time of the experiment, time points can be obtained by $t_i = \frac{i}{m} t_m, i = 1, \cdots, m$. Singh and Tripathi (2018) mentioned that, when units on the test has a decreasing failure rate the ES inspection times may provide efficient estimates. In OS approach, time points are obtained in order to achieve some optimality criteria. To study the problem of selecting the inspection times, we consider the following optimality criteria:

Criterion I: Minimizing the trace of the expected variance covariance matrix of the MLEs.

Criterion II: Maximizing the determinant of the expected Fisher information matrix of the MLEs.

It is known that the expected variance covariance matrix of the MLEs can be obtained by inverting expected Fisher information matrix. Let $\mathbf{p} = (p_1, \cdots, p_m)$ be a censoring scheme. Observe that the probability that a unit fails in the interval $(0, t_1]$ is

$$P(0 < T \leq t_1 | T > 0) = \frac{F(t_1) - F(0)}{1 - F(0)} = F(t_1).$$

Then $D_1 \sim Binomial(n, F(t_1))$ and $R_1|D_1 \sim Binomial(n - D_1, p_1)$. Consequently, the expected number of failures in the interval $(0, t_1]$ is $\zeta_1 = E(D_1) = nF(t_1)$ and the expected number of removed units is $\tau_1 = E(R_1|D_1)|_{\zeta_1} = (n - \zeta_1)p_1$. Subsequently, the probability that a unit fails in the interval $(t_{i-1}, t_i]$

$$P(t_{i-1} < T \leq t_i | T > t_{i-1}) = \frac{F(t_i) - F(t_{i-1})}{1 - F(t_{i-1})},\ i = 1, 2, \cdots, m.$$

Then the conditional distributions of D_i and R_i are given by

$$D_i|(D_{i-1}, R_{i-1}, \cdots, D_1, R_1) \sim Binomial\Big(n - \sum_{j=1}^{i-1}(D_j + R_j), \frac{F(t_i) - F(t_{i-1})}{1 - F(t_{i-1})}\Big) \quad (44)$$

$$R_i|(D_i, D_{i-1}, R_{i-1}, \cdots, D_1, r_1) = R_i \sim Binomial\Big(n - \sum_{j=1}^{i} D_j - \sum_{j=1}^{i-1} R_j, p_i\Big) \quad (45)$$

and expected number of failures and the expected number of removed items are respectively computed by

$$\begin{aligned}\zeta_i =& E(D_i|D_{i-1}, R_{i-1}, \cdots, D_1, R_1)|_{(\zeta_{i-1}, \tau_{i-1}, \cdots, \zeta_1, \tau_1)} \\ =& \Big(n - \sum_{j=1}^{i-1}(\zeta_j + \tau_j)\Big)\frac{F(t_i) - F(t_{i-1})}{1 - F(t_{i-1})} \quad (46) \\ \tau_i =& E(R_i|D_i, D_{i-1}, R_{i-1}, \cdots, D_1, R_1)|_{(\zeta_i, \zeta_{i-1}, \tau_{i-1}, \cdots, \zeta_1, \tau_1)} \\ =& \Big(n - \sum_{j=1}^{i-1}(\zeta_j + \tau_j) - \zeta_i\Big)p_i. \quad (47)\end{aligned}$$

Therefore, the expected Fisher information matrix can be obtained from expressions 17,18 and 19 by replacing D_i with ζ_i and R_i with τ_i, see, for example Singh and Tripathi (2018). It is easy to observe that computing OS inspection times is a constraint optimization problem due to the condition $t_i > t_{i-1}, i = 1, 2, \cdots, m$. Hence in order to remove the monotonicity constraints, we consider the transformation of t_i's as $t_i = \sum_{k=1}^{i} e^{s_k}$. With the use of new variables s_i's, genetic algorithm is used for the determination the OS inspection times via GA() package.

In the last approach, EP, we may interest to obtain inspection times for a pre-specified percentage of censoring observations quantity h satisfying the expression $\sum_{i=1}^{m} \tau_i = nh$. Note that $\sum_{i=1}^{m} \zeta_i = n(1-h)$ since $\sum_{i=1}^{m} \zeta_i + \sum_{i=1}^{m} \tau_i = n$. Furthermore, we consider the probability of expected number of failures in each inspection interval is considered to be the same. As a consequence, the problem of finding EP inspection times reduces to compute t_i's such that $\zeta_1 = \zeta_2 = \cdots = \zeta_m$ and $\sum_{i=1}^{m} \zeta_i = n(1-h)$. Observe that, by solving (46) for t_i we obtain

$$t_i = F^{-1}\Big[\frac{\zeta_i[1 - F(t_{i-1})]}{n - \sum_{j=1}^{i-1}(\zeta_j + \tau_j)} + F(t_{i-1})\Big],\ i = 1, 2, \cdots, m.$$

Hence, we propose the following algorithm to obtain EP inspection times (see for example Singh and Tripathi (2018)).

Input: Choose $n \in \mathbb{Z}^+, m \in \mathbb{Z}^+, m \leq n, h \in [0,1]$ and $\mathbf{p} = (p_1, p_2, \cdots, p_m)$, where $\mathbb{Z}^+$ is the set of positive integers and $p_i \in [0,1]$.

Initialize: Set $\zeta_i = \frac{n(1-h)}{m}, i = 1, \cdots, m$. Compute $t_1 = F^{-1}(\frac{\zeta_1}{n})$ and $\tau_1 = (n - \zeta_1)p_1$. Repeat **Step 1** to **Step 3** for $i = 2, \cdots, m$.

Step 1: Obtain

$$t_i = F^{-1}\Big(\frac{\zeta_i(1 - F(t_{i-1}))}{n - \sum_{j=1}^{i-1}(\zeta_j + \tau_j)} + F(t_{i-1})\Big)$$

Step 2: Compute

$$\tau_i = \Big\lceil n - \frac{i * n(1-h)}{m} - \sum_{j=1}^{i-1} \tau_j \Big\rceil p_i.$$

Step 3: If $\sum_{j=1}^{i} \tau_j > nh$ then set $\tau_i = nh - \sum_{j=1}^{i} \tau_j$, $\tau_k = 0$ for $k = i+1, \cdots, m$ and stop.

Here $\lceil x \rceil$ denotes the greatest integer less than or equal to x.

Numerical results concern with different inspection times are reported in Tables 12-18. Some items of these tables are presented as - which represent the situations that experiment can be terminated only after the failure of all remaining units. In Tables 12 and 13, we compare the performance of the MLEs based on PS with the MLEs based ES inspection times in terms on Bias and RMSE. In Tables 14 and 15, we obtain the EP inspection times for $h = 0.3, 0.5, 0.8$ and $m = 5, 10$. For $m = 5$, we consider the schemes $\mathbf{p}_{\mathrm{i}}$,i=1,2,3,4, proposed in the simulation section and for $m = 10$, we adopt the following censoring schemes

$\mathbf{p}_5 = (0.25, 0.25, 0.25, 0.25, 0.25, 0.5, 0.5, 0.5, 0.5, 1)$
$\mathbf{p}_6 = (0.5, 0.5, 0.5, 0.5, 0.5, 0.25, 0.25, 0.25, 0.25, 1)$
$\mathbf{p}_7 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$
$\mathbf{p}_8 = c(0.25, 0, 0, 0, 0, 0, 0, 0, 0, 1)$.

It can be seen that some EP inspection times are not available for censoring schemes $\mathbf{p}_1$ and $\mathbf{p}_2$ (or $\mathbf{p}_5$ and $\mathbf{p}_6$ for $m = 10$) and $h \leq 0.5$. Moreover, the first EP inspection times, t_1 is the same for all the censoring schemes and the scheme $\mathbf{p}_4$ has the largest values of EP inspection times among the other scheme for a fixed value of h. Clearly, the values of inspection times are decreasing with the values of the percentage of censoring, h. Tables 16-18 includes the OS inspection times based on criteria I and II for $m = 5, 10$ and $n = 25, 50, 100$. The main observation from these tables, the first inspection times for criterion I is less than that of criterion II for all the cases.

Table 12: Bias and RMSE of α and λ using PS and ES for $m = 5$

		α=0.5				λ=0.5			
		PS		ES		PS		ES	
n		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
25	$\mathbf{p}_1$	0.032	0.403	0.010	0.402	0.028	0.393	0.017	0.410
	$\mathbf{p}_2$	0.002	0.377	0.019	0.382	0.019	0.354	0.021	0.387
	$\mathbf{p}_3$	0.005	0.308	0.014	0.326	0.018	0.344	0.005	0.373
	$\mathbf{p}_4$	0.008	0.341	0.003	0.362	0.020	0.355	0.011	0.384
50	$\mathbf{p}_1$	0.013	0.294	0.022	0.286	0.014	0.295	0.019	0.298
	$\mathbf{p}_2$	0.024	0.352	0.004	0.351	0.003	0.308	0.007	0.343
	$\mathbf{p}_3$	0.027	0.236	0.034	0.245	0.014	0.259	0.023	0.276
	$\mathbf{p}_4$	0.024	0.256	0.029	0.264	0.016	0.269	0.024	0.288
100	$\mathbf{p}_1$	0.024	0.221	0.036	0.227	0.026	0.226	0.034	0.241
	$\mathbf{p}_2$	0.002	0.245	0.016	0.251	0.017	0.228	0.020	0.255
	$\mathbf{p}_3$	0.034	0.195	0.038	0.201	0.027	0.210	0.034	0.221
	$\mathbf{p}_4$	0.030	0.203	0.036	0.211	0.026	0.214	0.034	0.229
		α=1.5				λ=1			
25	$\mathbf{p}_1$	0.277	1.151	0.318	1.294	0.057	0.440	0.054	0.506
	$\mathbf{p}_2$	0.207	0.997	0.171	1.046	0.011	0.427	0.006	0.506
	$\mathbf{p}_3$	0.206	0.726	0.218	0.912	0.059	0.356	0.061	0.428
	$\mathbf{p}_4$	0.262	0.958	0.512	8.706	0.088	0.385	0.067	0.455
50	$\mathbf{p}_1$	0.122	0.721	0.109	0.649	0.020	0.299	0.017	0.323
	$\mathbf{p}_2$	0.296	1.089	0.193	0.994	0.055	0.373	0.018	0.401
	$\mathbf{p}_3$	0.073	0.456	0.087	0.479	0.026	0.238	0.021	0.261
	$\mathbf{p}_4$	0.010	0.522	0.110	0.567	0.032	0.243	0.022	0.285
100	$\mathbf{p}_1$	0.070	0.414	0.107	0.469	0.019	0.188	0.0277	0.224
	$\mathbf{p}_2$	0.111	0.571	0.097	0.568	0.021	0.233	0.018	0.270
	$\mathbf{p}_3$	0.051	0.300	0.042	0.330	0.018	0.166	0.009	0.192
	$\mathbf{p}_4$	0.049	0.353	0.073	0.387	0.008	0.178	0.022	0.207

Table 13: Bias and RMSE of α and λ using PS and ES for $m = 10$

		α=0.5				λ=0.5			
		PS		ES		PS		ES	
n		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
25	$\mathbf{p}_5$	0.078	0.540	0.036	0.418	0.026	0.425	0.013	0.388
	$\mathbf{p}_6$	0.066	0.412	0.008	0.395	0.094	0.387	0.025	0.339
	$\mathbf{p}_7$	0.008	0.316	0.009	0.285	0.022	0.363	0.010	0.326
	$\mathbf{p}_8$	0.003	0.338	0.003	0.316	0.027	0.369	0.009	0.351
50	$\mathbf{p}_5$	0.001	0.338	0.001	0.290	0.020	0.310	0.010	0.272
	$\mathbf{p}_6$	0.015	0.380	0.028	0.354	0.053	0.332	0.010	0.297
	$\mathbf{p}_7$	0.028	0.237	0.032	0.226	0.009	0.266	0.024	0.249
	$\mathbf{p}_8$	0.017	0.250	0.023	0.239	0.005	0.272	0.019	0.253
100	$\mathbf{p}_5$	0.010	0.256	0.020	0.230	0.021	0.242	0.030	0.222
	$\mathbf{p}_6$	0.032	0.343	0.006	0.294	0.010	0.268	0.034	0.240
	$\mathbf{p}_7$	0.036	0.198	0.042	0.199	0.026	0.215	0.038	0.211
	$\mathbf{p}_8$	0.029	0.202	0.035	0.198	0.022	0.215	0.033	0.207
		α=1.5				λ=1			
25	$\mathbf{p}_5$	0.265	1.152	0.372	1.351	0.027	0.447	0.067	0.452
	$\mathbf{p}_6$	0.036	1.136	0.117	1.090	0.149	0.563	0.055	0.487
	$\mathbf{p}_7$	0.197	0.742	0.245	0.766	0.059	0.350	0.099	0.337
	$\mathbf{p}_8$	0.251	0.864	0.255	0.861	0.070	0.388	0.085	0.355
50	$\mathbf{p}_5$	0.130	0.769	0.127	0.662	0.004	0.315	0.024	0.289
	$\mathbf{p}_6$	0.447	1.756	0.369	1.246	0.016	0.483	0.059	0.380
	$\mathbf{p}_7$	0.091	0.449	0.088	0.425	0.038	0.242	0.031	0.227
	$\mathbf{p}_8$	0.102	0.518	0.107	0.492	0.033	0.248	0.033	0.239
100	$\mathbf{p}_5$	0.082	0.533	0.089	0.455	0.016	0.240	0.024	0.194
	$\mathbf{p}_6$	0.162	0.893	0.166	0.698	0.014	0.337	0.028	0.251
	$\mathbf{p}_7$	0.040	0.310	0.025	0.310	0.017	0.177	0.003	0.181
	$\mathbf{p}_8$	0.049	0.346	0.038	0.345	0.019	0.182	0.008	0.187

Table 14: EP inspection times for $m = 5$

				$(\alpha, \lambda) = (0.5, 0.5)$		
h		t_1	t_2	t_3	t_4	t_5
0.3	$\mathbf{p}_1$	0.372	0.828	-	-	-
	$\mathbf{p}_2$	0.372	1.219	-	-	-
	$\mathbf{p}_3$	0.372	0.684	1.219	2.324	5.302
	$\mathbf{p}_4$	0.372	0.828	1.85	5.302	38.677
0.5	$\mathbf{p}_1$	0.301	0.564	1.174	-	-
	$\mathbf{p}_2$	0.301	0.743	-	-	-
	$\mathbf{p}_3$	0.301	0.489	0.743	1.12	1.738
	$\mathbf{p}_4$	0.301	0.564	0.975	1.738	3.463
0.8	$\mathbf{p}_1$	0.196	0.290	0.417	0.763	2.733
	$\mathbf{p}_2$	0.196	0.336	0.684	1.685	9.873
	$\mathbf{p}_3$	0.196	0.267	0.336	0.409	0.489
	$\mathbf{p}_4$	0.196	0.290	0.384	0.489	0.613
				$(\alpha, \lambda) = (1.5, 1)$		
0.3	$\mathbf{p}_1$	0.426	0.684	-	-	-
	$\mathbf{p}_2$	0.426	0.841	-	-	-
	$\mathbf{p}_3$	0.426	0.615	0.841	1.158	1.682
	$\mathbf{p}_4$	0.426	0.684	1.037	1.682	3.748
0.5	$\mathbf{p}_1$	0.372	0.55	0.825	1.448	8.166
	$\mathbf{p}_2$	0.372	0.644	-	-	-
	$\mathbf{p}_3$	0.372	0.505	0.644	0.805	1.006
	$\mathbf{p}_4$	0.372	0.550	0.748	1.006	1.393
0.8	$\mathbf{p}_1$	0.276	0.362	0.458	0.586	0.779
	$\mathbf{p}_2$	0.276	0.399	0.615	1.277	-
	$\mathbf{p}_3$	0.276	0.343	0.399	0.453	0.505
	$\mathbf{p}_4$	0.276	0.362	0.435	0.505	0.577

Table 15: EP inspection times for $m = 10$

h		t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
$(\alpha, \lambda) = (0.5, 0.5)$											
0.3	$\mathbf{p}_5$	0.250	0.415	-	-	-	-	-	-	-	-
	$\mathbf{p}_6$	0.250	0.511	-	-	-	-	-	-	-	-
	$\mathbf{p}_7$	0.250	0.372	0.511	0.684	0.911	1.219	1.66	2.324	3.396	5.302
	$\mathbf{p}_8$	0.250	0.415	0.622	0.911	1.348	2.069	3.396	6.279	14.625	61.478
0.5	$\mathbf{p}_5$	0.215	0.330	0.501	-	-	-	-	-	-	-
	$\mathbf{p}_6$	0.215	0.390	-	-	-	-	-	-	-	-
	$\mathbf{p}_7$	0.215	0.301	0.390	0.489	0.605	0.743	0.911	1.12	1.388	1.738
	$\mathbf{p}_8$	0.215	0.330	0.455	0.605	0.795	1.045	1.388	1.879	2.622	3.826
0.8	$\mathbf{p}_5$	0.155	0.209	0.271	0.353	0.474	0.675	1.340	-	-	-
	$\mathbf{p}_6$	0.155	0.232	0.372	-	-	-	-	-	-	-
	$\mathbf{p}_7$	0.155	0.196	0.232	0.267	0.301	0.336	0.372	0.409	0.448	0.489
	$\mathbf{p}_8$	0.155	0.209	0.255	0.301	0.348	0.396	0.448	0.504	0.564	0.630
$(\alpha, \lambda) = (1.5, 1)$											
0.3	$\mathbf{p}_5$	0.328	0.457	-	-	-	-	-	-	-	-
	$\mathbf{p}_6$	0.328	0.519	-	-	-	-	-	-	-	-
	$\mathbf{p}_7$	0.328	0.426	0.519	0.615	0.721	0.841	0.983	1.158	1.38	1.682
	$\mathbf{p}_8$	0.328	0.457	0.582	0.721	0.886	1.095	1.38	1.809	2.566	4.461
0.5	$\mathbf{p}_5$	0.295	0.395	0.513	-	-	-	-	-	-	-
	$\mathbf{p}_6$	0.295	0.439	-	-	-	-	-	-	-	-
	$\mathbf{p}_7$	0.295	0.372	0.439	0.505	0.573	0.644	0.721	0.805	0.899	1.006
	$\mathbf{p}_8$	0.295	0.395	0.483	0.573	0.669	0.776	0.899	1.045	1.225	1.457
0.8	$\mathbf{p}_5$	0.232	0.289	0.346	0.412	0.496	0.61	0.883	-	-	-
	$\mathbf{p}_6$	0.232	0.312	0.426	-	-	-	-	-	-	-
	$\mathbf{p}_7$	0.232	0.276	0.312	0.343	0.372	0.399	0.426	0.453	0.479	0.505
	$\mathbf{p}_8$	0.232	0.289	0.333	0.372	0.408	0.444	0.479	0.514	0.55	0.587

Table 16: OS Inspection times for $m = 5$

(α, λ)	n		Crit.I					Crit.II				
(0.5, 0.5)	25	$\mathbf{p}_1$	1.7	3.8	7.0	10.4	14.1	1.8	4.7	8.7	12.4	15.2
		$\mathbf{p}_2$	0.4	2.9	4.9	5.6	7.9	2.1	5.2	8.7	12.4	16.0
		$\mathbf{p}_3$	1.1	1.8	3.3	5.9	9.4	1.5	3.2	6.0	9.9	13.6
		$\mathbf{p}_4$	1.5	2.7	4.7	7.5	11.3	1.7	3.4	5.8	9.2	13.2
	50	$\mathbf{p}_1$	1.7	3.8	7.1	10.8	14.0	1.9	4.8	8.7	12.5	16.1
		$\mathbf{p}_2$	0.3	1.3	3.4	5.0	6.4	2.0	5.1	8.8	12.5	16.3
		$\mathbf{p}_3$	1.0	1.7	3.0	5.3	9.0	1.5	3.3	6.0	9.9	13.7
		$\mathbf{p}_4$	0.2	4.0	6.5	9.0	10.0	1.8	3.4	5.9	9.5	13.5
	100	$\mathbf{p}_1$	1.7	3.9	7.2	11.0	14.8	1.8	4.5	8.5	11.7	15.5
		$\mathbf{p}_2$	0.4	2.9	4.8	6.7	8.6	2.1	5.1	8.9	12.8	16.8
		$\mathbf{p}_3$	1.0	1.7	3.1	5.4	9.2	1.4	3.0	5.6	9.4	13.3
		$\mathbf{p}_4$	0.2	3.2	6.0	9.7	12.4	1.7	3.5	6.2	9.8	13.7
(1.5, 1)	25	$\mathbf{p}_1$	0.9	3.3	4.4	5.7	6.1	1.5	2.4	3.8	5.5	7.8
		$\mathbf{p}_2$	1.2	3.6	5.7	8.2	8.8	1.7	2.8	3.9	5.4	7.4
		$\mathbf{p}_3$	0.6	2.8	4.6	5.9	7.7	1.2	1.7	2.3	3.5	5.6
		$\mathbf{p}_4$	0.9	2.6	4.3	6.1	6.4	1.5	1.7	2.6	4.0	5.8
	50	$\mathbf{p}_1$	0.9	1.2	4.1	4.7	5.6	1.5	2.5	3.8	5.6	7.8
		$\mathbf{p}_2$	1.2	3.1	5.9	8.4	9.9	1.7	2.8	3.9	5.4	7.6
		$\mathbf{p}_3$	0.6	3.3	4.2	5.5	7.3	1.2	1.6	2.3	3.6	5.6
		$\mathbf{p}_4$	0.9	2.7	4.5	7.6	9.9	1.5	1.8	2.7	4.1	6.3
	100	$\mathbf{p}_1$	0.9	2.3	4.6	6.9	8.0	1.5	2.4	3.8	5.5	7.5
		$\mathbf{p}_2$	1.2	3.4	4.4	5.7	7.4	1.7	2.7	3.9	5.2	7.2
		$\mathbf{p}_3$	0.6	3.3	4.3	5.6	7.3	1.2	1.8	2.4	3.8	5.8
		$\mathbf{p}_4$	0.9	2.9	4.3	6.4	8.7	1.5	2.0	2.7	4.3	6.4

Table 17: OS Inspection times for $(\alpha, \lambda) = (0.5, 0.5)$ and $m = 10$

n		Crit	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
25	$\mathbf{p}_5$	I	1.6	3.7	6.6	10.0	13.8	17.7	21.0	23.0	25.1	26.8
		II	1.8	4.6	8.2	12.2	15.9	19.5	22.4	25.0	26.6	28.9
	$\mathbf{p}_6$	I	0.4	3.0	3.9	5.8	7.2	8.8	10.2	11.5	12.6	13.9
		II	2.0	5.1	9.0	12.6	16.2	19.8	21.7	24.2	25.9	27.9
	$\mathbf{p}_7$	I	1.0	1.5	2.2	3.7	5.6	7.3	9.7	12.9	16.6	19.9
		II	1.5	2.7	4.2	6.3	8.7	12.0	15.7	19.2	23.1	26.3
	$\mathbf{p}_8$	I	0.2	3.9	5.9	7.7	9.9	12.4	15.6	18.5	21.7	25.3
		II	1.8	3.1	4.8	6.6	9.3	12.7	16.2	20.1	23.8	27.4
50	$\mathbf{p}_5$	I	1.6	3.7	6.6	10.0	13.8	17.7	21.0	23.0	25.1	26.8
		II	1.8	4.6	8.2	12.2	15.9	19.5	22.4	25.0	26.6	28.9
	$\mathbf{p}_6$	I	0.4	3.0	3.9	5.8	7.2	8.8	10.2	11.5	12.6	13.9
		II	2.0	5.1	9.0	12.6	16.1	19.8	21.7	24.2	25.9	27.9
	$\mathbf{p}_7$	I	1.0	1.5	2.2	3.7	5.6	7.3	9.7	12.9	16.6	19.9
		II	1.5	2.7	4.2	6.3	8.7	12.0	15.7	19.2	23.1	26.3
	$\mathbf{p}_8$	I	0.2	3.9	5.9	7.7	9.9	12.4	15.7	18.5	21.7	25.3
		II	1.8	3.1	4.8	6.6	9.3	12.7	16.2	20.1	23.8	27.4
100	$\mathbf{p}_5$	I	1.6	3.6	6.3	9.6	13.3	17.0	19.9	22.2	24.3	25.9
		II	1.8	4.4	7.9	11.6	15.5	19.0	22.3	23.9	26.9	29.2
	$\mathbf{p}_6$	I	0.4	3.1	4.8	6.7	9.2	10.3	12.3	12.6	13.6	14.3
		II	2.0	5.1	8.8	12.3	16.0	19.4	21.0	22.4	25.3	27.4
	$\mathbf{p}_7$	I	1.0	1.5	2.2	3.5	5.4	7.7	10.3	13.5	16.9	20.7
		II	1.4	2.4	3.8	5.8	8.1	11.1	14.7	18.4	21.6	25.5
	$\mathbf{p}_8$	I	1.5	2.2	3.2	4.7	6.5	8.7	11.6	15.1	18.8	22.4
		II	1.8	2.8	4.2	6.0	8.5	11.2	14.4	18.0	22.0	25.8

Table 18: OS Inspection times for $(\alpha, \lambda) = (1.5, 1)$ and $m = 10$

n		Crit	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
25	$\mathbf{p}_5$	I	0.9	3.9	4.7	6.4	7.5	8.3	8.8	10.9	13.2	15.8
		II	1.5	2.3	3.4	4.7	6.3	8.4	10.7	13.4	15.7	17.4
	$\mathbf{p}_6$	I	1.2	3.3	5.2	7.5	10.0	10.1	11.3	12.8	14.0	14.6
		II	1.7	2.8	4.3	5.9	8.0	9.1	10.9	13.1	14.8	16.8
	$\mathbf{p}_7$	I	0.5	2.3	5.5	7.4	8.7	10.9	12.4	14.2	15.1	18.4
		II	1.2	1.5	2.1	2.9	4.2	5.9	7.9	9.5	11.6	13.9
	$\mathbf{p}_8$	I	0.9	2.7	5.2	8.9	11.8	14.7	18.2	19.8	21.6	23.5
		II	1.5	1.9	2.1	2.8	3.8	5.0	6.4	7.5	9.0	11.5
50	$\mathbf{p}_5$	I	0.9	2.8	4.8	7.0	9.0	10.5	11.9	13.1	15.1	18.1
		II	1.5	2.4	3.6	5.1	6.7	8.6	10.2	12.9	14.8	16.6
	$\mathbf{p}_6$	I	1.2	4.3	4.6	5.0	5.7	7.1	8.2	9.6	11.6	13.9
		II	1.7	2.8	4.1	5.7	7.7	9.3	11.6	12.9	14.4	16.0
	$\mathbf{p}_7$	I	0.6	3.6	5.2	7.3	8.9	11.3	12.8	14.6	15.8	17.6
		II	1.2	1.4	2.0	2.7	3.8	5.4	7.3	9.3	11.5	13.9
	$\mathbf{p}_8$	I	0.9	3.7	5.9	7.7	9.9	12.0	14.8	15.6	16.3	18.4
		II	1.5	1.8	2.6	3.5	4.6	6.3	7.9	9.9	11.8	14.5
100	$\mathbf{p}_5$	I	0.9	2.8	3.6	6.4	7.7	10.2	11.0	11.5	12.3	12.9
		II	1.5	2.3	3.5	4.9	6.8	8.9	10.8	13.1	15.3	17.0
	$\mathbf{p}_6$	I	1.2	2.6	4.2	6.6	8.4	9.9	11.4	12.1	13.5	15.0
		II	1.7	2.9	4.3	6.0	7.7	9.3	11.7	14.2	15.9	17.6
	$\mathbf{p}_7$	I	0.6	3.0	5.6	6.7	9.1	11.2	13.5	14.3	17.0	17.2
		II	1.2	1.8	2.3	2.9	3.7	5.2	6.6	8.3	10.6	11.9
	$\mathbf{p}_8$	I	0.9	3.0	4.2	6.5	9.7	11.9	13.5	15.0	16.4	18.6
		II	1.5	1.7	2.3	3.2	4.6	6.2	8.2	10.3	12.6	14.4

8 Optimal censoring

It is common in the analysis of real life experiment to consider the censoring scheme as a fixed and pre-specified. However, in the estimation problem, we may choose the censoring scheme among a set of possible schemes in order to improve the estimations of parameters. It is known that, under progressive type I interval censored, the number of units removed, R_i, at each inspection time, t_i, can be a constant number or a pre-specified proportion, p_i, of surviving units. Optimal censoring can be described as finding the expected numbers $\mathbf{R} = (R_1, R_2, \cdots, R_m)$ (or proportions $\mathbf{p} = (p_1, p_2, \cdots, p_m)$) which attain to a specific optimality criterion. The issue of identifying the optimal censoring scheme for different distributions under progressive type I interval censored has received little attention in the statistical literature. See Arabi Belaghi et al. (2017) for Burr XII and Singh and Tripathi (2018) for inverse Weibull distribution.

The problem of selecting the optimal censoring method under progressive type I interval censored observation can be described as follows. For given n and h, the optimal censoring scheme is the one among all possible censoring schemes which satisfies the conditions $\sum_i^m R_i = \lceil nh \rceil$ and $\sum_{i=1}^m (\zeta_i + \tau_i) = n$, where ζ_i and τ_i are defined in (46) and (47). Recall that the number of all possible censoring schemes satisfying the relation $\sum_i^m R_i = \lceil nh \rceil$ is $\frac{(\lceil nh \rceil + m-1)!}{(m-1)! \lceil nh \rceil !}$. First, we consider optimal censoring with PS inspection times, i.e. $\mathbf{t}$ includes pre-specified quantities. Assume that $\psi(\zeta, \tau, \mathbf{t})$ is the objective function that needs to be minimized (or maximized). Following Singh and Tripathi (2018), we make use the following algorithm to get the optimal censoring scheme based on PS inspection times.

Step 1. Set the values of n, m, h, and $\mathbf{t}$=$(t_1, t_2, \cdots, t_m)$.

Step 2. Calculate $W = \frac{(\lceil nh \rceil + m-1)!}{(m-1)! \lceil nh \rceil !}$ and set $c = 0$ and $k = 1$.

Step 3. Generate $\sum_{i=1}^m R_i = \lceil nh \rceil$ and consider $\tau_i = R_i$.

Step 4. Compute the $\zeta_i, i = 1, 2, \cdots, m$ using (46).

Step 5. If $\sum_{i=1}^m \zeta_i - n + \lceil nh \rceil \leq \epsilon$, set $c = c+1$ and compute $\psi_k(\zeta, \tau, \mathbf{t})$ else set $k = k+1$ and go to **Step 3.**

Step 6. If $\psi_k(\zeta, \tau, \mathbf{t}) >$ (or $<$) $\psi_{k-1}(\zeta, \tau, \mathbf{t})$, update the optimal censoring scheme $(R_1, R_2, \cdots, R_m)$ and go to **Step 3** with $k = k + 1$ until $k = W$.

Here, ϵ is a pre-specified quantity and $\psi_k(.)$ is the value of $\psi(.)$ at the k-th iteration. Next, we utilize the following algorithm to obtain the optimal censoring scheme based on EP inspection times (see, Singh and Tripathi (2018)).

Step 1. Select the values of n, m, and h.

Step 2. Set $\zeta_i = \frac{n - \lceil nh \rceil}{m}, i = 1, 2, \cdots, m$.

Step 3. Calculate $W = \frac{(\lceil nh \rceil + m-1)!}{(m-1)! \lceil nh \rceil !}$ and set $k = 1$.

Step 4. Generate $(R_1, \cdots, R_m)$ such that $\sum_{i=1}^{m} R_i = \lceil nh \rceil$ and consider $\tau_i = R_i, i = 1, 2, \cdots, m$.

Step 5. Compute

$$t_i = F^{-1}\Big(\frac{\zeta_i(1 - F(t_i - 1)}{n - \sum_{j=1}^{i-1}(\zeta_i + \tau_i)} + F(t_i - 1)\Big), i = 2, 3, \cdots, m,$$

where $t_0 = 0$.

Step 6. Given the values of τ_i, ζ_i and $t_i, i = 1, 2, \cdots, m$, compute $\psi_k(\zeta, \tau, \mathbf{t})$.

Step 7. If $\psi_k(\zeta, \tau, \mathbf{t}) >$ (or $<)\psi_{k-1}(\zeta, \tau, \mathbf{t})$ then update the optimal censoring scheme $(R_1, R_2, \cdots, R_m)$ and EP inspection times $(t_1, t_2, \cdots, t_m)$. Further set $k = k + 1$ and go to **Step 4** until $k = W$.

Based on the above algorithms, we suggest to consider the following two criteria.

Criterion(I): Minimizing the objective function $\psi(.)$ which is the trace of the expected variance covariance matrix of the MLEs.

Criterion(II): Maximizing the objective function $\psi(.)$ which is the determinant of the expected Fisher information matrix of the MLEs.

It is clear that for a large value of m, the total number of sampling schemes can be quite large. For example when $n = 25, m = 10$ and $h = 0.3$ the possible number of censoring schemes is $\binom{\lceil nh \rceil + m - 1}{m-1} = \binom{29}{9} = 10015005$. Following Pradhan and Kundu (2013), we propose to use a sub-optimal censoring problem in which the optimal censoring scheme belongs to the convex hull generated by the points $(\lceil nh \rceil, 0, \cdots, 0)$, $(0, \lceil nh \rceil, 0, \cdots, 0)$ $, \cdots, (0, \cdots, 0,\ \lceil nh \rceil)$. Therefore, the sub-optimal censoring scheme can be obtained by choosing the optimal censoring scheme among these extreme points on the convex hull. In addition, for generating censoring schemes $(R_1, \cdots, R_m)$ satisfies the condition $\sum_{i=1}^{m} R_i = \lceil nh \rceil$, we may utilize the function *c*ompositions() from *p*artition package in R language.

In Table 19, we have reported the optimal censoring schemes for $m = 5$ and in Table 20, we have reported the sub-optimal censoring schemes for $m = 10$. For both tables, we have considered $n = 25, 50, 50$; $h = 0.3, 0.5, 0.8$ and parameters $(\alpha, \lambda) = (0.5, 0.5), (1.5, 1)$. It can be seen that, for the both tables, by changing the sample size, the censoring scheme patterns, in general, do not affected. However, from Table 19, the reported censoring schemes for almost all the cases are same or very close to each other under criteria I and II. Moreover, most of the unites are removed in the first and the last stages. From Table 20, the censoring scheme patterns for both criteria are showed that the units are removed in the i-*th* stage, i=1,2,3, except for few cases for $h = 0.3$. Furthermore, to investigate the optimal proportion of the removed unites instead of optimal number, one may consider the expression (47).

Table 19: Optimal censoring schemes under PS and EP inspection times for $m = 5$

(α, λ)		n	h	Crit.I=$(R_1, \cdots, R_5)$	Crit.II=$(R_1, \cdots, R_5)$
(0.5, 0.5)	PS	25	0.3	(0,0,0,0,7)	(0,0,0,0,7)
			0.5	(7,0,0,0,5)	(6,0,0,2,4)
			0.8	(18,0,0,0,2)	(17,1,0,0,2)
		50	0.3	(1,0,0,0,14)	(0,0,1,1,13)
			0.5	(15,0,0,0,10)	(14,1,0,0,10)
			0.8	(36,0,0,0,4)	(35,1,0,0,4)
		100	0.3	(2,0,0,0,28)	(2,0,0,0,28)
			0.5	(30,0,0,0,20)	(30,0,0,0,20)
			0.8	(30,0,0,0,20)	(30,0,0,0,20)
	EP	25	0.3	(1,0,0,0,14)	(0,0,1,1,13)
			0.5	(15,0,0,0,10)	(14,1,0,0,10)
			0.8	(36,0,0,0,4)	(35,1,0,0,4)
		50	0.3	(1,0,0,0,14)	(0,0,1,1,13)
			0.5	(15,0,0,0,10)	(14,1,0,0,10)
			0.8	(36,0,0,0,4)	(35,1,0,0,4)
		100	0.3	(1,0,0,0,14)	(0,0,1,1,13)
			0.5	(15,0,0,0,10)	(14,1,0,0,10)
			0.8	(36,0,0,0,4)	(35,1,0,0,4)
(1.5, 1)	PS	25	0.3	(6,0,0,0,1)	(6,0,0,0,1)
			0.5	(11,0,0,1,0)	(11,0,0,1,0)
			0.8	(15,4,1,0,0)	(19,1,0,0,0)
		50	0.3	(13,0,0,0,2)	(13,0,0,0,2)
			0.5	(1,21,3,0,0)	(23,0,0,1,1)
			0.8	(30,9,1,0,0)	(39,0,0,1,0)
		100	0.3	(25,0,1,0,4)	(23,3,0,0,4)
			0.5	(47,0,0,0,3)	(47,0,0,0,3)
			0.8	(60,18,2,0,0)	(79,0,0,0,1)
	EP	25	0.3	(6,0,0,0,1)	(6,0,0,0,1)
			0.5	(11,0,0,0,1)	(11,0,0,0,1)
			0.8	(19,0,0,0,1)	(19,0,0,0,1)
		50	0.3	(13,0,0,0,2)	(12,0,0,0,3)
			0.5	(23,0,0,0,2)	(23,0,0,0,2)
			0.8	(39,0,0,0,1)	(39,0,0,0,1)
		100	0.3	(26,0,0,0,4)	(25,0,0,0,5)
			0.5	(47,0,0,0,3)	(46,0,0,0,4)
			0.8	(57,0,0,0,3)	(57,0,0,0,3)

Table 20: Optimal censoring schemes under PS and EP inspection times for $m = 10$

(α, λ)		n	h	Crit.I= $(R_1, \cdots, R_{10})$	Crit.II=$(R_1, \cdots, R_{10})$
			0.3	(0,0,0,0,0,0,0,0,7,0)	(0,0,0,0,0,0,0,0,7,0)
		25	0.5	(0,0,0,12,0,0,0,0,0,0)	(0,0,0,12,0,0,0,0,0,0)
			0.8	(0,20,0,0,0,0,0,0,0,0)	(20,0,0,0,0,0,0,0,0,0)
			0.3	(0,0,0,0,0,0,15,0,0,0)	(0,0,0,0,0,0,15,0,0,0)
	PS	50	0.5	(0,25,0,0,0,0,0,0,0,0)	(0,0,25,0,0,0,0,0,0,0)
			0.8	(0,40,0,0,0,0,0,0,0,0)	(40,0,0,0,0,0,0,0,0,0)
			0.3	(0,30,0,0,0,0,0,0,0,0)	(0,30,0,0,0,0,0,0,0,0)
		100	0.5	(0,0,50,0,0,0,0,0,0,0)	(0,50,0,0,0,0,0,0,0,0)
			0.8	(0,80,0,0,0,0,0,0,0,0)	(80,0,0,0,0,0,0,0,0,0)
$(0.5, 0.5)$			0.3	(7,0,0,0,0,0,0,0,0,0)	(0,7,0,0,0,0,0,0,0,0)
		25	0.5	(12,0,0,0,0,0,0,0,0,0)	(0,12,0,0,0,0,0,0,0,0)
			0.8	(20,0,0,0,0,0,0,0,0,0)	(0,0,20,0,0,0,0,0,0,0)
			0.3	(15,0,0,0,0,0,0,0,0,0)	(0,15,0,0,0,0,0,0,0,0)
	EP	50	0.5	(25,0,0,0,0,0,0,0,0,0)	(0,25,0,0,0,0,0,0,0,0)
			0.8	(40,0,0,0,0,0,0,0,0,0)	(0,0,40,0,0,0,0,0,0,0)
			0.3	(30,0,0,0,0,0,0,0,0,0)	(0,30,0,0,0,0,0,0,0,0)
		100	0.5	(50,0,0,0,0,0,0,0,0,0)	(0,50,0,0,0,0,0,0,0,0)
			0.8	(80,0,0,0,0,0,0,0,0,0)	(0,0,80,0,0,0,0,0,0,0)
			0.3	(7,0,0,0,0,0,0,0,0,0)	(7,0,0,0,0,0,0,0,0,0)
		25	0.5	(0,0,12,0,0,0,0,0,0,0)	(12,0,0,0,0,0,0,0,0,0)
			0.8	(0,20,0,0,0,0,0,0,0,0)	(20,0,0,0,0,0,0,0,0,0)
			0.3	(15,0,0,0,0,0,0,0,0,0)	(15,0,0,0,0,0,0,0,0,0)
	PS	50	0.5	(0,0,25,0,0,0,0,0,0,0)	(25,0,0,0,0,0,0,0,0,0)
			0.8	(0,40,0,0,0,0,0,0,0,0)	(40,0,0,0,0,0,0,0,0,0)
			0.3	(0,30,0,0,0,0,0,0,0,0)	(0,30,0,0,0,0,0,0,0,0)
		100	0.5	(0,0,50,0,0,0,0,0,0,0)	(0,50,0,0,0,0,0,0,0,0)
			0.8	(0,80,0,0,0,0,0,0,0,0)	(0,80,0,0,0,0,0,0,0,0)
$(1.5, 1)$			0.3	(7,0,0,0,0,0,0,0,0,0)	(0,7,0,0,0,0,0,0,0,0)
		25	0.5	(12,0,0,0,0,0,0,0,0,0)	(0,12,0,0,0,0,0,0,0,0)
			0.8	(20,0,0,0,0,0,0,0,0,0)	(0,0,20,0,0,0,0,0,0,0)
			0.3	(15,0,0,0,0,0,0,0,0,0)	(0,15,0,0,0,0,0,0,0,0)
	EP	50	0.5	(25,0,0,0,0,0,0,0,0,0)	(25,0,0,0,0,0,0,0,0,0)
			0.8	(40,0,0,0,0,0,0,0,0,0)	(0,0,40,0,0,0,0,0,0,0)
			0.3	(30,0,0,0,0,0,0,0,0,0)	(0,30,0,0,0,0,0,0,0,0)
		100	0.5	(50,0,0,0,0,0,0,0,0,0)	(0,50,0,0,0,0,0,0,0,0)
			0.8	(80,0,0,0,0,0,0,0,0,0)	(0,0,80,0,0,0,0,0,0,0)

9 Concluding remarks

In this article, statistical inference of the unknown parameters of GIED based on progressively type I interval censored data is considered. The MLEs, probability plot, mid-point and method of moments as well as associated standard error, root mean square error and confidence intervals are obtained. MLEs are obtained by using Netwon-Raphson method, expectation minimization (EM) algorithm and stochastic expectation minimization (SEM) algorithm. The Simulation results showed that all the estimators, except MP method, present reasonably small amounts of biases and RMSEs. Moreover, the ESE based on the inverse of the observed information matrix can be considered as a reasonable estimate of the SSE for NR and EM methods, especially for large n. With respect to 95% confidence interval, the length of the confidence intervals is decreasing when the value of sample size is increasing and the estimated CP of 95% confidence intervals are very close to the nominal level for all the cases.

In real data analysis, we analyze, based on the proposed methodology, the survival times of guinea pigs injected with different doses of tubercle bacilli. Fitting the data set with GIED is first implemented and then the GIED parameters are estimated based on the proposed methods.

Selecting the inspection times is important practical issue to improve the efficiency of the obtained estimators. By considering such an issue, we investigate pre-specified (PS), equally spaced (ES), optimally spaced (OS) and equal probability (EP) methods to determine the inspection times. In regard to optimal censoring, the censoring schemes with most of the removal units are appeared in the first stages (at most the first three stages) is the most preferred ones among the other schemes based on all criteria. However, the considered censoring schemes are almost the same under the criteria I and II for almost the all cases.

We hope that the methodologies proposed in this work will be useful to applied statisticians. It will be interesting to study the methods of estimation under hybrid censored data. The work is in progress and it will be reported later.

References

Abouammoh, A. and Alshingiti, A. M. (2009). Reliability estimation of generalized inverted exponential distribution. *Journal of Statistical Computation and Simulation*, 79(11):1301–1315.

Aggarwal, R. and Jacques, K. T. (2001). The impact of fdicia and prompt corrective action on bank capital and risk: Estimates using a simultaneous equations model. *Journal of Banking & Finance*, 25(6):1139–1160.

Aggarwala, R. (2001). Progressive interval censoring: some mathematical results with applications to inference. *Communications in Statistics-Theory and Methods*, 30(8-9):1921–1935.

Arabi Belaghi, R., Noori Asl, M., and Singh, S. (2017). On estimating the parameters of the burr xii model under progressive type-i interval censoring. *Journal of Statistical Computation and Simulation*, 87(16):3132–3151.

Bjerkedal, T. et al. (1960). Acquisition of resistance in guinea pies infected with different doses of virulent tubercle bacilli. *American Journal of Hygiene*, 72(1):130–48.

Cheng, C., Chen, J., and Li, Z. (2010). A new algorithm for maximum likelihood estimation with progressive type-i interval censored data. *Communications in Statistics - Simulation and Computation®*, 39(4):750–766.

Dempster, A. P., Laird, N. M., and Rubin, D. B. (1977). Maximum likelihood from incomplete data via the em algorithm. *Journal of the Royal Statistical Society: Series B (Methodological)*, 39(1):1–22.

Dey, S. and Dey, T. (2014a). Generalized inverted exponential distribution: Different methods of estimation. *American Journal of Mathematical and Management Sciences*, 33(3):194–215.

Dey, S. and Dey, T. (2014b). On progressively censored generalized inverted exponential distribution. *Journal of Applied Statistics*, 41(12):2557–2576.

Dey, S. and Nassar, M. (2020). Generalized inverted exponential distribution under constant stress accelerated life test: Different estimation methods with application. *Quality and Reliability Engineering International*, 36(4):1296–1312.

Diebolt, J. and Celeux, G. (1993). Asymptotic properties of a stochastic em algorithm for estimating mixing proportions. *Stochastic Models*, 9(4):599–613.

Dube, M., Krishna, H., and Garg, R. (2016). Generalized inverted exponential distribution under progressive first-failure censoring. *Journal of Statistical Computation and Simulation*, 86(6):1095–1114.

Efron, B. and Tibshirani, R. (1986). Bootstrap methods for standard errors, confidence intervals, and other measures of statistical accuracy. *Statistical science*, pages 54–75.

Garg, R. and Kumar, K. (2021). On estimation of p (y¡ x) for generalized inverted exponential distribution based on hybrid censored data. *Statistica*, 81(3):335–361.

Gupta, R. D. and Kundu, D. (1999). Theory & methods: Generalized exponential distributions. *Australian & New Zealand Journal of Statistics*, 41(2):173–188.

Hassan, A. S., Al-Omari, A., and Nagy, H. F. (2021). Stress–strength reliability for the generalized inverted exponential distribution using mrss. *Iranian Journal of Science and Technology, Transactions A: Science*, 45(2):641–659.

Kotz, S. and Nadarajah, S. (2000). *Extreme value distributions: theory and applications.* world scientific.

Krishna, H., Dube, M., and Garg, R. (2017). Estimation of p (y¡ x) for progressively first-failure-censored generalized inverted exponential distribution. *Journal of Statistical Computation and Simulation*, 87(11):2274–2289.

Krishna, H. and Kumar, K. (2013). Reliability estimation in generalized inverted exponential distribution with progressively type ii censored sample. *Journal of Statistical Computation and Simulation*, 83(6):1007–1019.

Kumari, A., Kumar, S., and Kumar, K. (2022). Inference for reliability in a multicomponent stress-strength model from generalized inverted exponential lifetime distribution under progressive first failure censoring. *Journal of Statistical Computation and Simulation*, 0(0):1–25.

Kundu, D. and Howlader, H. (2010). Bayesian inference and prediction of the inverse weibull distribution for type-ii censored data. *Computational Statistics & Data Analysis*, 54(6):1547–1558.

Lin, C.-T., Wu, S. J., and Balakrishnan, N. (2009). Planning life tests with progressively type-i interval censored data from the lognormal distribution. *Journal of Statistical Planning and Inference*, 139:54–61.

Lodhi, C. and Tripathi, Y. M. (2020). Inference on a progressive type i interval-censored truncated normal distribution. *Journal of Applied Statistics*, 47(8):1402–1422.

Pradhan, B. and Kundu, D. (2013). Inference and optimal censoring schemes for progressively censored birnbaum–saunders distribution. *Journal of Statistical Planning and Inference*, 143(6):1098–1108.

Prakash, G. (2012). Inverted exponential distribution under a bayesian viewpoint. *Journal of Modern Applied Statistical Methods*, 11(1):16.

Singh, S. and Tripathi, Y. M. (2018). Estimating the parameters of an inverse weibull distribution under progressive type-i interval censoring. *Statistical Papers*, 59(1):21–56.

Singh, S., Tripathi, Y. M., and Jun, C.-H. (2015). Sampling plans based on truncated life test for a generalized inverted exponential distribution. *Industrial Engineering and Management Systems*, 14(2):183–195.

Singh, S. K., Singh, U., and Kumar, D. (2013). Bayes estimators of the reliability function and parameter of inverted exponential distribution using informative and non-informative priors. *Journal of Statistical computation and simulation*, 83(12):2258–2269.

Tregouet, D., Escolano, S., Tiret, L., Mallet, A., and Golmard, J. (2004). A new algorithm for haplotype-based association analysis: the stochastic-em algorithm. *Annals of human genetics*, 68(2):165–177.

Wang, F.-K. and Cheng, Y. (2010). Em algorithm for estimating the burr xii parameters with multiple censored data. *Quality and Reliability Engineering International*, 26(6):615–630.

Wei, G. C. and Tanner, M. A. (1990). A monte carlo implementation of the em algorithm and the poor man's data augmentation algorithms. *Journal of the American statistical Association*, 85(411):699–704.

Zhang, X. and Haenggi, M. (2014). A stochastic geometry analysis of inter-cell interference coordination and intra-cell diversity. *IEEE Transactions on Wireless Communications*, 13(12):6655–6669.

Appendix: Proof of Theorems 1

Proof : Observe that, for fixed $\lambda > 0$, we have

$$\lim_{\alpha \to 0} l(\alpha, \lambda|D) = \lim_{\alpha \to \infty} l(\alpha, \lambda|D) = -\infty$$

and for fixed $\alpha > 0$, we have

$$\lim_{\lambda \to 0} l(\alpha, \lambda|D) = \lim_{\lambda \to \infty} l(\alpha, \lambda|D) = -\infty.$$

It is easy to see that

$$\frac{\partial^2 l(\alpha, \lambda|D)}{\partial \alpha^2} = -\sum_{i=1}^{m} d_i \frac{(1 - e^{-\lambda/t_{i-1}})(1 - e^{-\lambda/t_i}) \left[\log((1 - e^{-\lambda/t_i})/(1 - e^{-\lambda/t_{i-1}}))\right]^2}{\left[(1 - e^{-\lambda/t_{i-1}})^\alpha - (1 - e^{-\lambda/t_i})^\alpha\right]^2} < 0$$

and

$$\begin{aligned}\frac{\partial^2 l(\alpha, \lambda|D)}{\partial \lambda^2} =& -\alpha \sum_{i=1}^{m} d_i \Bigg\{ \frac{\frac{1}{t_{i-1}^2} e^{-\lambda/t_{i-1}} (1 - e^{-\lambda/t_{i-1}})^{\alpha-2} - \frac{1}{t_i^2} e^{-\lambda/t_i} (1 - e^{-\lambda/t_i})^{\alpha-2}}{(1 - e^{-\lambda/t_{i-1}})^\alpha - (1 - e^{-\lambda/t_i})^\alpha} \\ & - \frac{(1 - e^{-\lambda/t_{i-1}})^{\alpha-2} (1 - e^{-\lambda/t_i})^{\alpha-2}}{\left[(1 - e^{-\lambda/t_{i-1}})^\alpha - (1 - e^{-\lambda/t_i})^\alpha\right]^2} \\ & \times \left[\frac{1}{t_{i-1}} e^{-\lambda/t_{i-1}} (1 - e^{-\lambda/t_{i-1}}) - \frac{1}{t_i} e^{-\lambda/t_i} (1 - e^{-\lambda/t_i})\right]^2 \Bigg\} \\ & - \alpha \sum_{i=1}^{m} r_i \frac{\frac{1}{t_i} e^{-\lambda/t_i}}{(1 - e^{-\lambda/t_i})^2} < 0.\end{aligned}$$

That is for fixed $\lambda > 0$, the log-likelihood function $l(\alpha, \lambda|D)$ is strictly log-concave in α and for fixed $\alpha > 0$, the log-likelihood function $l(\alpha, \lambda|D)$ is strictly log-concave in λ. This completes the proof. ■