Electronic Journal of Applied Statistical Analysis
EJASA, Electron. J. App. Stat. Anal.
http://siba-ese.unisalento.it/index.php/ejasa/index
e-ISSN: 2070-5948
DOI: 10.1285/i20705948v13n2p536

Type II Exponentiated Half Logistic generated family of distributions with applications
By Al-Mofleh et al.

Published: 14 October 2020

Electronic Journal of Applied Statistical Analysis
Vol. 13, Issue 02, October 2020, 536-561
DOI: 10.1285/i20705948v13n2p536

Type II Exponentiated Half Logistic generated family of distributions with applications

Hazem Al-Mofleh*[a], Mohamed Elgarhy[b], Ahmed Z. Afify[c], and Mohammad Zannon[a]

[a] *Department of Mathematics, Tafila Technical University, Tafila, Jordan*
[b] *Valley High Institute for Management Finance and Information Systems Obour, Qaliubia, 11828, Egypt*
[c] *Department of Statistics, Mathematics and Insurance, Benha University, Benha 13511, Egypt*

Published: 14 October 2020

A new family of distributions called type II exponentiated half logistic is introduced. Four new special models are presented. Some mathematical properties of the new family are derived including explicit expressions for the quantile function, moments, probability weighted moments, mean deviation, order statistics and Rényi entropy. Parameter estimation of the proposed family are explored by the maximum likelihood. Two real data sets are employed to show the usefulness of the new family.

keywords: Exponentiated half logistic distribution; Order statistics; Maximum likelihood method.

1 Introduction

Most popular classical distributions often do not fit and predict data in several applied areas such as biological and environmental sciences, engineering, medical sciences, finance and economics. Hence, several generalized families are considered an improvement for creating and extending the usual classical distributions. The newly generated families have been broadly studied in several areas as well as yield more flexibility in applications. These extended families have been utilized in modeling data in many

*Corresponding author: almof1hm@cmich.edu

ISSN: 2070-5948
http://siba-ese.unisalento.it/index.php/ejasa/index

applied sciences due to their flexibility. Recently, several generated families have constructed by many authors. For example, the Marshall-Olkin-G (Marshall and Olkin, 1997) beta-G (Eugene et al., 2002), Kumaraswamy-G (Cordeiro and de Castro, 2011), gamma-G (Ristić and Balakrishnan, 2012), Kummer beta-G (Pescim et al., 2012), exponentiated generalized (Cordeiro et al., 2013), Weibull-G (Bourguignon et al., 2014), exponentiated half-logistic (Cordeiro et al., 2014), type I half-logistic (Cordeiro et al., 2016), Kumaraswamy Weibull-G (Hassan and Elgarhy, 2016b), exponentiated Weibull-G (Hassan and Elgarhy, 2016a), Kumaraswamy transmuted-G (Afify et al., 2016), exponentiated extended-G (Elgarhy et al., 2017), type II half-logistic-G (Hassan et al., 2017), generalized odd Lindley-G (Afify et al., 2019), odd Lomax-G (Cordeiro et al., 2019), odd Dagum-G (Afify and Alizadeh, 2020) and arcsine exponentiated-X (He et al., 2020), among others.

Furthermore, these families also employed by many authors to propose new extended models, such as Darna distribution by Al-Omari and Shraa (2019), Kumaraswamy moment exponential distribution by Hashmi et al. (2019), power length-biased Suja distribution by Al-Omari et al. (2019), extended odd Weibull exponential distribution by Afify and Mohamed (2020) and weighted Burr-XII distribution by Shakhatreh and Al-Masri (2020).

In the current paper, we introduce a new extended family of distributions based on the exponentiated half-logistic (EHL) distribution called type II exponentiated half-logistic (TIIEHL-G) family. The TIIEHL-G family provides greater flexibility in fitting data as well as their special models can provide left-skewed, right-skewed, symmetrical, and reversed-J shaped densities. These models also have upside-down bathtub, bathtub, increasing, decreasing, constant and reversed-J hazard rate functions.

This paper can be outlined as follows: In the next section, the TIIEHL-G family is defined. Section 3 concerns with some general mathematical properties of the TIIEHL-G family. In Section 4, four special models of the new family are considered. Estimation of the TIIEHL parameters is implemented by the maximum likelihood approach in Section 5. Section 6 deals with a Monte-Carlo simulation study. Two real data sets are analyzed for illustrative purpose in Section 7. Finally, concluding remarks are handled in Section 8.

2 TIIEHL-G Family

The EHL distribution is a member of the family of logistic distributions which has the following cumulative distribution function (cdf)

$$F(t) = \left[\frac{1-e^{-\lambda t}}{1+e^{-\lambda t}}\right]^{a}; \; t > 0, \quad a, \lambda > 0. \tag{1}$$

Its associated probability density function (pdf) takes the form

$$f\left(t\right)=\frac{2a\lambda e^{-\lambda t}\left(1+e^{-\lambda t}\right)^{a-1}}{\left(1+e^{-\lambda t}\right)^{a+1}}. \tag{2}$$

On the basis of the gamma generated fmily (Ristić and Balakrishnan, 2012), we obtain the TIIEHL-G family with the following cdf

$$F\left(x\right)=1-\int_{0}^{-\log G(x,\boldsymbol{\zeta})}\frac{2a\lambda e^{-\lambda t}\left(1+e^{-\lambda t}\right)^{a-1}}{\left(1+e^{-\lambda t}\right)^{a+1}}dt=1-\left[\frac{1-\left[G\left(x,\boldsymbol{\zeta}\right)\right]^{\lambda}}{1+\left[G\left(x,\boldsymbol{\zeta}\right)\right]^{\lambda}}\right]^{a};\ x>0,\ \ a,\lambda>0, \tag{3}$$

where λ is a scale parameter, a is a shape parameter and $G(x,\boldsymbol{\zeta})$ is a baseline cdf, which depends on a parameter vector $\boldsymbol{\zeta}$.

The pdf of the TIIEHL-G family reduces to

$$f\left(x\right)=\frac{2a\lambda g\left(x,\zeta\right)\left[G\left(x,\boldsymbol{\zeta}\right)\right]^{\lambda-1}\left(1-\left[G(x,\boldsymbol{\zeta})\right]^{\lambda}\right)^{a-1}}{\left(1+\left[G(x,\boldsymbol{\zeta})\right]^{\lambda}\right)^{a+1}};\ x>0,\ \ a,\lambda>0. \tag{4}$$

Hereafter, we denote by $X\sim$TIIEHL-G$(a,\lambda,\boldsymbol{\zeta})$ for a random variable X has pdf (4). The hazard rate function has the form

$$h\left(x\right)=2a\lambda\ g\left(x,\boldsymbol{\zeta}\right)\frac{\left[G\left(x,\boldsymbol{\zeta}\right)\right]^{\lambda-1}}{1-\left[G\left(x,\boldsymbol{\zeta}\right)\right]^{2\lambda}}.$$

The quantile function (qf), $Q(u)$, of X is

$$Q\left(u\right)=G^{-1}\left[\frac{1-(1-u)^{\frac{1}{a}}}{1+(1-u)^{\frac{1}{a}}}\right]^{\frac{1}{\lambda}},\ 0<u<1, \tag{5}$$

where U is a uniform random variable, and $G^{-1}(\cdot)$ is the inverse function of $G(\cdot)$. The three quartiles follows simply by setting $u=0.25,\ 0.5$ and 0.75, respectively, in (5).

3 Some Mathematical Properties

This section provides some mathematical general properties of TIIEHL-G family.

3.1 Linear Representations

Now , we present important expansions for the pdf and cdf of the TIIEHL-G fmily. Using the following two generalized binomial series

$$(1+z)^{-\beta}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta+i-1}{i}z^{i} \tag{6}$$

and

$$(1-z)^{\beta}=\sum_{j=0}^{\infty}(-1)^{j}\binom{\beta}{j}z^{j}, \tag{7}$$

for $|z|<1$, and β is a positive real non-integer, the pdf of TIIEHL-G family becomes

$$f(x)=\sum_{i,j=0}^{\infty}\eta_{i,j}g(x;\boldsymbol{\zeta})\,G(x;\boldsymbol{\zeta})^{\lambda(i+j+1)-1}, \tag{8}$$

where

$$\eta_{i,j}=2a\lambda(-1)^{i+j}\binom{a+i}{i}\binom{a-1}{j}.$$

Another formula can be extracted from (8), as follows

$$f(x)=\sum_{i=0}^{\infty}W_{i,j}\ h_{\lambda(i+j+1)}(x), \tag{9}$$

where $W_i{=}\eta_{i,j}/\lambda\,(i{+}j{+}1)$, and $h_a(x)=a\ g(x;\boldsymbol{\zeta})\,G(x;\boldsymbol{\zeta})^{a-1}$, is the exponentiated-G (exp-G) density with power parameter a.

Furthermore, an expansion for the $(F(x))^h$ is derived for h is integer, again, the binomial expansion is worked out

$$(F(x))^{h}=\sum_{k=0}^{h}(-1)^{k}\binom{h}{k}\left[\frac{1-[G(x,\zeta)]^{\lambda}}{1+[G(x,\zeta)]^{\lambda}}\right]^{ak}.$$

Again, using the binomial expansions (6) and (7) in the last equation, it will be reduced to

$$(F(x))^{h}=\sum_{k=0}^{h}\sum_{m,l=0}^{\infty}(-1)^{k+l+m}\binom{h}{k}\binom{ak+m-1}{m}\binom{ak}{l}[G(x,\zeta)]^{\lambda(l+m)}.$$

Again, the binomial expansion is applied to $G(x,\zeta)^{\lambda(l+m)}$ by adding and subtracting 1, then $[F(x)]^h$ can be expressed as follows

$$(F(x))^h=\sum_{k=0}^{h}\sum_{m,l,\upsilon=0}^{\infty}(-1)^{k+l+m+\upsilon}\binom{h}{k}\binom{ak+m-1}{m}\binom{ak}{l}\binom{\lambda(l+m)}{\upsilon}[1-G(x,\zeta)]^{\upsilon}.$$

Hence, $(F(x))^h$ can be rewritten as follows

$$(F(x))^h=\sum_{z=0}^{\infty}s_z(G(x;\zeta))^z, \tag{10}$$

where,

$$s_z=\sum_{k=0}^{h}\sum_{m,l,\upsilon=0}^{\infty}(-1)^{k+l+m+\upsilon+z}\binom{h}{k}\binom{ak+m-1}{m}\binom{ak}{l}\binom{\lambda(l+m)}{\upsilon}\binom{\upsilon}{z}.$$

3.2 The Probability Weighted Moments

The probability-weighted moments (PWMs) has been proposed by Greenwood et al. (1979). For a random variable X, the PWMs can be defined by

$$\tau_{r,s}=E\left[X^r(F(x))^s\right]=\int_{-\infty}^{\infty}x^r f(x)\,(F(x))^s dx. \tag{11}$$

The PWMs of TIIEHL-G is obtained by substituting (8) and (10) into (11), and replacing h with s, as follows

$$\tau_{r,s}=\int_{-\infty}^{\infty}\sum_{i,j,z=0}^{\infty}s_z\eta_{i,j}x^r g(x;\zeta)\,(G(x;\zeta))^{z+\lambda(i+j+1)-1}dx.$$

Then,

$$\tau_{r,s}=\sum_{i,j,z=0}^{\infty}s_z\eta_{i,j}\tau_{r,z+\lambda(i+j+1)-1},$$

where

$$\tau_{r,z+\lambda(i+j+1)-1}=\int_{-\infty}^{\infty}x^r g(x;\zeta)(G(x;\zeta))^{z+\lambda(i+j+1)-1}dx.$$

Additionally, another formula can be calculated using qf as follows

$$\tau_{r,s}=\int_0^1\sum_{i,j,z=0}^{\infty}s_z\eta_{i,j}(Q_G(u))^r u^{z+\lambda(i+j+1)-1}du.$$

3.3 Moments

If X has the pdf (8), then the rth moment takes the form

$$\mu_r' = \int_{-\infty}^{\infty} x^r f(x)dx = \int_{-\infty}^{\infty} \sum_{i,j=0}^{\infty} \eta_{i,j} x^r g(x;\zeta) G(x;\zeta)^{\lambda(i+j+1)-1} dx.$$

Then,

$$\mu_r' = \sum_{i,j=0}^{\infty} \eta_{i,j} \tau_{r,\lambda(i+j+1)-1} \ ,$$

where, $\tau_{r,\lambda(i+j+1)-1}$ is the PWMs.
Furthermore, another formula can be deduced, based on the parent qf, as follow

$$\mu_r' = \sum_{i,j=0}^{\infty} \eta_{i,j} \int_0^1 (Q_G(u))^r u^{\lambda(i+j+1)-1} du.$$

The moment generating function (mgf) of X is defined by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r' = \sum_{r,i,j=0}^{\infty} \frac{t^r}{r!} \eta_{i,j} \tau_{r,\lambda(i+j+1)-1}.$$

A different form for the mgf of the TIIEHL-G class based on qf is given by

$$M_u(t) = \sum_{i,j=0}^{\infty} \eta_{i,j} \int_0^1 e^{(tQ_G(u))} u^{\lambda(i+j+1)-1} du.$$

3.4 Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample from the TIIEHL-G family. Let $X_{1:n} < X_{2:n} < ... < X_{n:n}$ be the corresponding order statistics. According to David and Nagaraja (2004), the pdf of the kth order statistic is defined as

$$f_{k:n}(x) = \frac{f(x)}{B(k, n-k+1)} \sum_{v=0}^{n-k} (-1)^v \binom{n-k}{v} F(x)^{v+k-1} \ , \tag{12}$$

where $B(\cdot,\cdot)$ stands for beta function. The pdf of the kth order statistic for TIIEHL-G family follows by substituting (8) and (10) in (12), and replacing h with $v+k-1$, we get

$$f_{k:n}(x) = \frac{g(x;\zeta)}{B(k, n-k+1)} \sum_{v=0}^{n-k} \sum_{z=0}^{\infty} \sum_{i,j=0}^{\infty} \eta_{i,j} p_{z,v} G(x;\zeta)^{z+\lambda(i+j+1)-1}, \tag{13}$$

where $p_{z,v}$=$(-1)^v \binom{n-k}{v}$ s_z and $g(\cdot)$ and $G(\cdot)$ are the pdf and cdf of a baseline model, respectively.
Furthermore, the rth moment of kth order statistic for TIIEHL-G family is

$$E\left(X_{k:n}{}^r\right)=\int_{-\infty}^{\infty} x^r f_{k:n}\left(x\right)dx. \tag{14}$$

Substituting (13) in (14), leads to

$$E\left(X_{k:n}{}^r\right)=\frac{1}{B\left(k,n-k+1\right)}\sum_{v=0}^{n-k}\sum_{z=0}^{\infty}\sum_{i,j=0}^{\infty}\eta_{i,j}\ p_{z,v}\int_{-\infty}^{\infty} x^r g\left(x;\zeta\right)G\left(x;\zeta\right)^{z+\lambda(i+j+1)-1}dx.$$

Thus,

$$E\left(X_{k:n}{}^r\right)=\frac{1}{B\left(k,n-k+1\right)}\sum_{v=0}^{n-k}\sum_{z=0}^{\infty}\sum_{i,j=0}^{\infty}\eta_{i,j}\ p_{z,v}\tau_{r,z+\lambda(i+j+1)-1}.$$

3.5 Rényi Entropy

The entropy of a random variable X is a measure of variation of uncertainty. The Rényi entropy (Rényi, 1961) is defined by

$$I_\delta\left(X\right)=\frac{1}{1-\delta}\log\left[\int_{-\infty}^{\infty}\left(f\left(x\right)\right)^{\delta}dx\right];\quad \delta>0\ and\ \delta\neq 1.$$

Using the pdf (4), we can write

$$\left(f\left(x\right)\right)^{\delta}=\sum_{i,j=0}^{\infty} t_{i,j}g\left(x;\zeta\right)^{\delta}G\left(x;\zeta\right)^{\lambda(i+j+\delta)-\delta},$$

where

$$t_{i,j}=\left(2a\lambda\right)^{\delta}(-1)^{i+j}\binom{a\delta+\delta+i-1}{i}\binom{(a-1)\delta}{j}.$$

Hence, the Rényi entropy of TIIEHL-G family reduces to

$$I_\delta\left(X\right)=\frac{1}{1-\delta}\log\left[\sum_{i,j=0}^{\infty} t_{i,j}\int_{-\infty}^{\infty} g\left(x;\zeta\right)^{\delta}G\left(x;\zeta\right)^{\lambda(i+j+\delta)-\delta}dx\right].$$

4 Four Special Models

In this section, we define four special models of the TIIEHL-G family namely, TIIEHL-uniform (TIIEHLU), TIIEHL-Burr XII (TIIEHLBXII), TIIEHL-Weibull (TIIEHLW) and TIIEHL-quasi Lindley (TIIEHLQL) distributions.

4.1 TIIEHLU Distribution

The pdf of the TIIEHLU distribution follows from (4), by taking $g(x,\theta) = 1/\theta;\ \ 0 < x < \theta$, and $G\left(x,\theta\right) = x/\theta$, as

$$f\left(x;a,\lambda,\theta\right) = 2a\lambda\theta^{\lambda}x^{\lambda-1}\frac{\left(\theta^{\lambda}-x^{\lambda}\right)^{a-1}}{\left(\theta^{\lambda}+x^{\lambda}\right)^{a+1}};\quad a,\lambda>0,\ 0<x<\theta.$$

The corresponding cdf takes the form

$$F(x) = 1-\left[\frac{\theta^{\lambda}-x^{\lambda}}{\theta^{\lambda}+x^{\lambda}}\right]^{a}.$$

The HRF and qf are

$$HRF\left(t\right) = 2a\lambda\theta^{\lambda}t^{\lambda-1}\frac{\left(\theta^{\lambda}-t^{\lambda}\right)^{-1}}{\left(\theta^{\lambda}+t^{\lambda}\right)^{2a+1}}$$

and

$$Q\left(q\right) = \theta\left[\frac{1-\left(1-q\right)^{\frac{1}{a}}}{1+\left(1-q\right)^{\frac{1}{a}}}\right]^{\frac{1}{\lambda}};\qquad 0<q<1.$$

Figure 1 illustrates possible shapes of the pdf of TIIEHLU distribution for some values of $a,\ \lambda$ and θ. Figure 2 shows possible shapes of the HRF of TIIEHLU distribution for some parametric values.

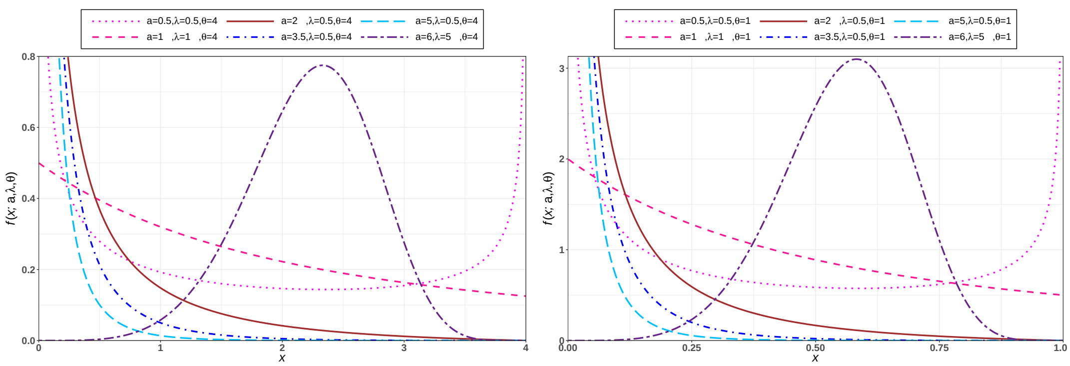

Figure 1: Plots of the pdf of TIIEHLU distribution for some parameter values.

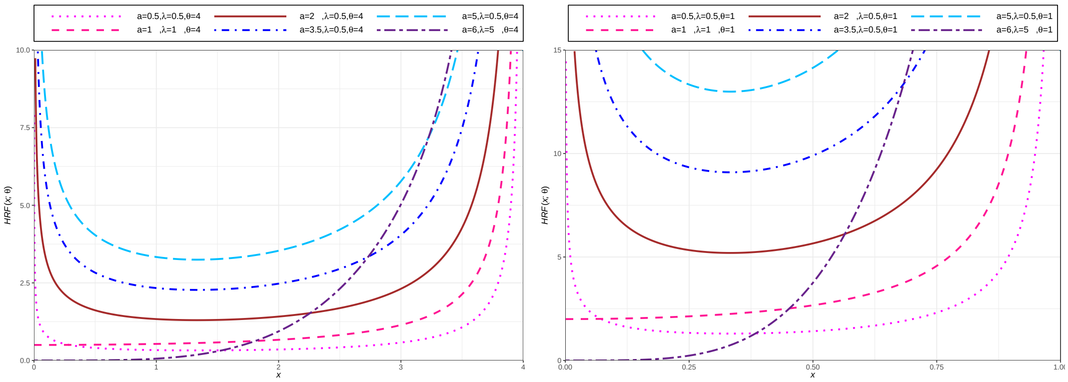

Figure 2: Plots of the HRF of TIIEHLU distribution for some parameter values.

4.2 TIIEHLBXII Distribution

Let us consider the Burr XII distribution with pdf and cdf given, respectively, by

$$g\left(x,c,\mu,\ \sigma\right)=c\sigma\mu^{-c}x^{c-1}\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-\sigma-1},c,\mu,\ \sigma>0$$

and

$$G\left(x,c,\mu,\ \sigma\right)=1-\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-\sigma}.$$

Then, the cdf, pdf, HRF and qf of the TIIEHLBXII distribution are

$$F\left(x\right)=1-\left[\frac{1-\left[1-\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-\sigma}\right]^{\lambda}}{1+\left[1-\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-\sigma}\right]^{\lambda}}\right]^{a};\quad a,\lambda,\ c,\mu,\ \sigma>0,x>0,$$

$$f\left(x\right)=2a\lambda c\sigma\mu^{-c}x^{c-1}\frac{\left[1-\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-\sigma}\right]^{\lambda-1}}{\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{\sigma+1}}\frac{\left[1-\left[1-\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-\sigma}\right]^{\lambda}\right]^{a-1}}{\left[1+\left[1-\left[1+\left(\frac{x}{\mu}\right)^{c}\right]^{-\sigma}\right]^{\lambda}\right]^{a+1}},$$

$$HRF\left(t\right)=2a\lambda c\sigma\mu^{-c}x^{c-1}\frac{\left[1-\left[1+\left(\frac{t}{\mu}\right)^{c}\right]^{-\sigma}\right]^{\lambda-1}}{\left[1+\left(\frac{t}{\mu}\right)^{c}\right]^{\sigma+1}}\frac{\left[1-\left[1-\left[1+\left(\frac{t}{\mu}\right)^{c}\right]^{-\sigma}\right]^{\lambda}\right]^{-1}}{\left[1+\left[1-\left[1+\left(\frac{t}{\mu}\right)^{c}\right]^{-\sigma}\right]^{\lambda}\right]^{2a+1}}$$

and

$$Q\left(q\right)=\mu\left[\left(1-\left[\frac{1-\left(1-q\right)^{\frac{1}{a}}}{1+\left(1-q\right)^{\frac{1}{a}}}\right]^{\frac{1}{\lambda}}\right)^{-\frac{1}{\sigma}}-1\right]^{\frac{1}{c}};\ 0<q<1.$$

Figure 3 illustrates some possible shapes of the pdf of TIIEHLBXII distribution for several selected values of $a,\ \lambda,\ c,\ \mu$ and σ. Figure 4 displays some shapes of the TIIEHLBXII HRF for selected values of $a,\ \lambda,\ c,\ \mu$ and σ.

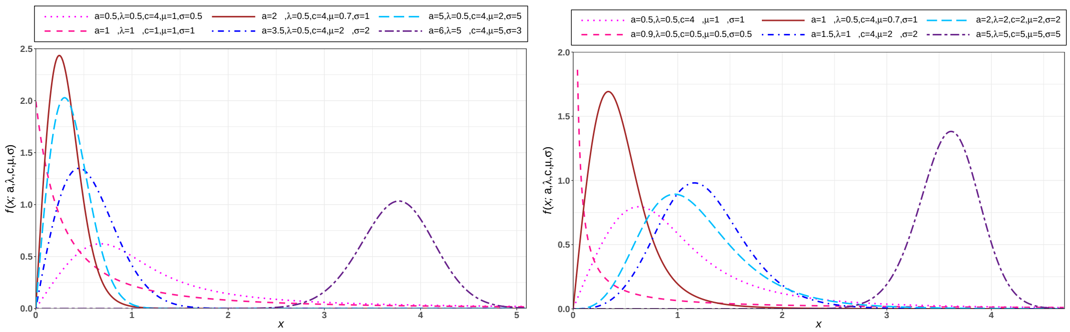

Figure 3: Plots of the pdf of TIIEHLBXII distribution for some parameter values.

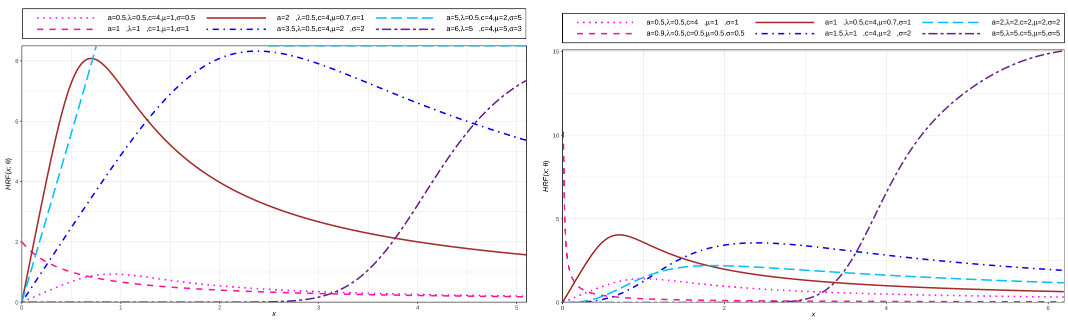

Figure 4: Plots of the HRF of TIIEHLBXII distribution for some parameter values.

4.3 TIIEHLW Distribution

The cdf and pdf of the TIIEHLW distribution are obtained from (3) and (4), by taking $G\left(x,\delta,\gamma\right)=1-e^{-\delta x^{\gamma}}$, as the following

$$F\left(x\right)=1-\left[\frac{1-\left[1-e^{-\delta x^{\gamma}}\right]^{\lambda}}{1+\left[1-e^{-\delta x^{\gamma}}\right]^{\lambda}}\right]^{a};\quad a,\lambda,\ \delta,\ \gamma>0,\ x>0$$

and

$$f\left(x\right)=2a\lambda\delta\gamma x^{\gamma-1}e^{-\delta x^{\gamma}}\left[1-e^{-\delta x^{\gamma}}\right]^{\lambda-1}\frac{\left[1-\left[1-e^{-\delta x^{\gamma}}\right]^{\lambda}\right]^{a-1}}{\left[1+\left[1-e^{-\delta x^{\gamma}}\right]^{\lambda}\right]^{a+1}}.$$

Its HRF and qf are given by

$$HRF\left(t\right)=2a\lambda\delta\gamma t^{\gamma-1}e^{-\delta t^{\gamma}}\left[1-e^{-\delta t^{\gamma}}\right]^{\lambda-1}\frac{\left[1-\left[1-e^{-\delta t^{\gamma}}\right]^{\lambda}\right]^{-1}}{\left[1+\left[1-e^{-\delta t^{\gamma}}\right]^{\lambda}\right]^{2a+1}}$$

and

$$Q\left(q\right)=\left[-\frac{1}{\delta}\log\left(1-\left[\frac{1-(1-q)^{\frac{1}{a}}}{1+(1-q)^{\frac{1}{a}}}\right]^{\frac{1}{\lambda}}\right)\right]^{\frac{1}{\gamma}};\ 0<q<1. \tag{15}$$

For γ= 1, we get TIIEHL-exponential distribution.
Figure 5 depicts possible shapes of the TIIEHLW denisty for some values of $a,\ \lambda,\ \delta$ and γ. Figure 6 illustrates the TIIEHLW HRF shapes for some values of $a,\ \lambda,\ \delta$ and γ.

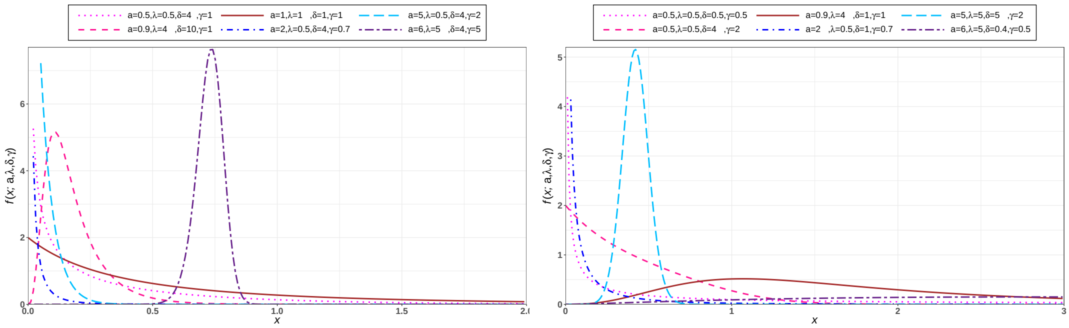

Figure 5: Plots of the pdf of TIIEHLW distribution for some parameter values.

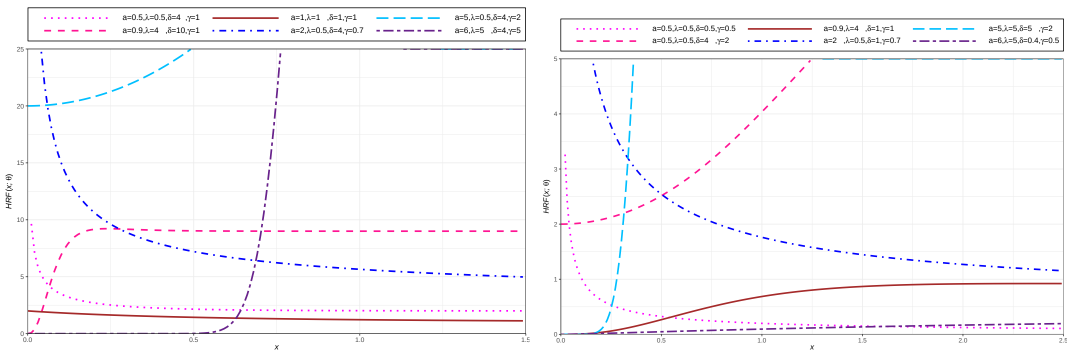

Figure 6: Plots of the HRF of TIIEHLW distribution for some parameter values.

4.4 TIIEHLQL Distribution

The quasi Lindley (QL) distribution is suggested by Shanker and Mishra (2013). The cdf and pdf of the QL distribution are defined by

$$G\left(x,\theta,p\right)=1-e^{-\theta x}\left[1+\frac{\theta x}{p+1}\right]$$

and

$$g\left(x,\theta,p\right)=\frac{\theta}{p+1}\left(p+\theta x\right)e^{-\theta x}.$$

Then, the cdf and pdf TIIEHLQL model take the forms

$$F\left(x\right)=1-\left[\frac{1-\left[1-e^{-\theta x}\left[1+\frac{\theta x}{p+1}\right]\right]^{\lambda}}{1+\left[1-e^{-\theta x}\left[1+\frac{\theta x}{p+1}\right]\right]^{\lambda}}\right]^{a}\ ;\ a,\lambda,\ \theta>0,\ p>-1\ ,\ x>0$$

and

$$f\left(x\right)=\frac{2a\lambda\theta\left(p+\theta x\right)}{\left(p+1\right)}e^{-\theta x}\left[1-e^{-\theta x}\left[1+\frac{\theta x}{p+1}\right]\right]^{\lambda-1}\frac{\left[1-\left[1-e^{-\theta x}[1+\frac{\theta x}{p+1}]\right]^{\lambda}\right]^{a-1}}{\left[1+\left[1-e^{-\theta x}[1+\frac{\theta x}{p+1}]\right]^{\lambda}\right]^{a+1}}.$$

Its HRF and qf reduce to

$$HRF\left(t\right)=\frac{2a\lambda\theta\left(p+\theta t\right)}{\left(p+1\right)}e^{-\theta t}\left[1-e^{-\theta t}\left[1+\frac{\theta t}{p+1}\right]\right]^{\lambda-1}\frac{\left[1-\left[1-e^{-\theta t}[1+\frac{\theta t}{p+1}]\right]^{\lambda}\right]^{-1}}{\left[1+\left[1-e^{-\theta t}[1+\frac{\theta t}{p+1}]\right]^{\lambda}\right]^{2a+1}}$$

and

$$Q\left(q\right)=-\frac{1}{\theta}\left(W_{-1}\left[-\left(p+1\right)e^{-p-1}\left(1-\left[\frac{1-\left(1-q\right)^{\frac{1}{a}}}{1+\left(1-q\right)^{\frac{1}{a}}}\right]^{\frac{1}{\lambda}}\right)\right]+p+1\right)\quad ;\ 0<q<1,$$

where $W_{-1}\left(\cdot\right)$ is the negative branch of the Lambert function.
For p=θ, the TIIEHLQL model reduces to the TIIEHL-Lindley distribution.
Figure 7 shows possible shapes of the TIIEHLQL density for some values of $a,\ \lambda,\ p$ and θ. Figure 8 illustrates possible shapes of the TIIEHLQL HRF for some values of $a,\ \lambda,\ p$ and θ.

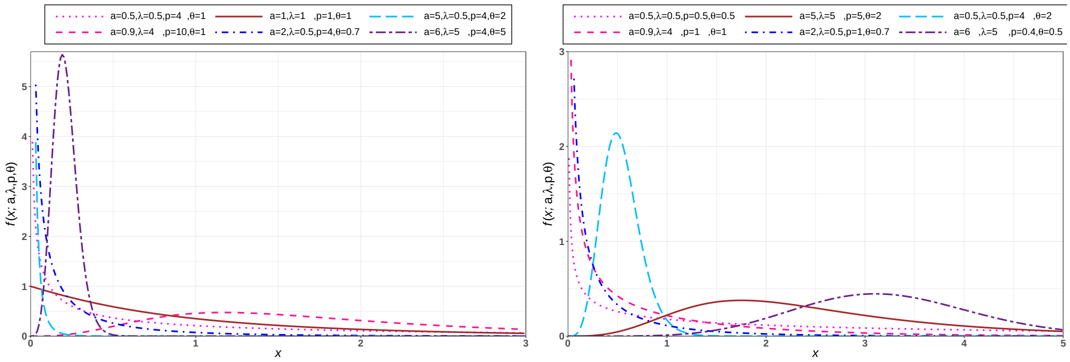

Figure 7: Plots of the pdf of TIIEHLQL distribution for some parameter values.

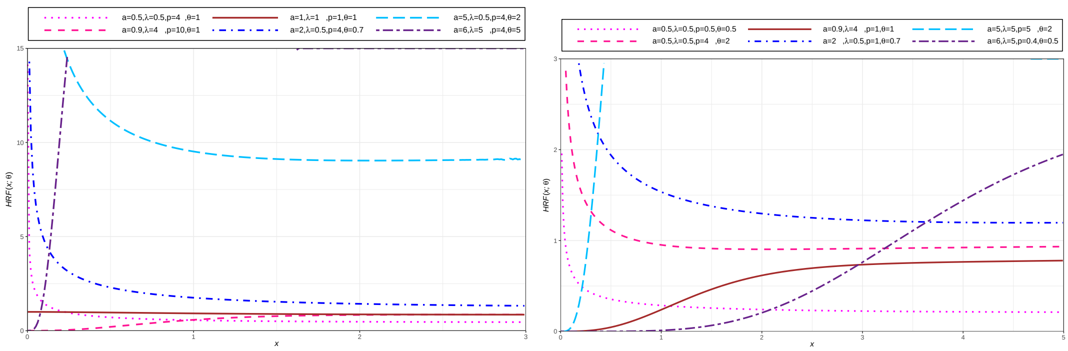

Figure 8: Plots of the HRF of TIIEHLQL distribution for some parameter values.

5 Maximum Likelihood Method

This section deals with the maximum likelihood estimation of the TIIEHL-G parameters on the basis of complete samples. Let $X_1, X_2, ..., X_n$ be a random sample from the TIIEHL-G family with parameters $\mathbf{\Phi}=\left(a, \lambda, \boldsymbol{\zeta}^T\right)^T$. The log-likelihood function, for $\mathbf{\Phi}=\left(a, \lambda, \boldsymbol{\zeta}^T\right)^T$, reduces to

$$\ln L\left(\mathbf{\Phi}\right) = n \ln(2\lambda) + n \ln\left(a\right) + \sum_{i=1}^{n} \ln\left[g(x_i, \boldsymbol{\zeta})\right] + (\lambda-1) \sum_{i=1}^{n} \ln\left[G(x_i, \boldsymbol{\zeta})\right]$$

$$- \left(a+1\right) \sum_{i=1}^{n} \ln\left[1+\left[G\left(x_i, \boldsymbol{\zeta}\right)\right]^{\lambda}\right] + \left(a-1\right) \sum_{i=1}^{n} \ln\left[1-\left[G\left(x_i, \boldsymbol{\zeta}\right)\right]^{\lambda}\right].$$

The elements of the score function $U\left(\mathbf{\Phi}\right) = \left(U_a, U_\lambda, \zeta_k\right)$ are given by

$$U_a = \frac{n}{a} - \sum_{i=1}^{n} \ln[1+[G(x_i, \boldsymbol{\zeta})]^{\lambda}] + \sum_{i=1}^{n} \ln[1-[G(x_i, \boldsymbol{\zeta})]^{\lambda}],$$

$$U_{\lambda}=\frac{n}{\lambda}+\sum_{i=1}^{n}\ln\left[G\left(x_i,\boldsymbol{\zeta}\right)\right]-(a+1)\sum_{i=1}^{n}\frac{\left[G\left(x_i,\boldsymbol{\zeta}\right)\right]^{\lambda}\ln\left[G\left(x_i,\boldsymbol{\zeta}\right)\right]}{\left[1+\left[G\left(x_i,\boldsymbol{\zeta}\right)\right]^{\lambda}\right]}$$
$$-(a-1)\sum_{i=1}^{n}\frac{\left[G\left(x_i,\boldsymbol{\zeta}\right)\right]^{\lambda}\ln\left[G\left(x_i,\boldsymbol{\zeta}\right)\right]}{\left[1-\left[G\left(x_i,\boldsymbol{\zeta}\right)\right]^{\lambda}\right]}$$

and

$$U_{\zeta_k}=\sum_{i=1}^{n}\frac{\frac{\partial g(x_i,\boldsymbol{\zeta})}{\partial\zeta_k}}{g\left(x_i,\boldsymbol{\zeta}\right)}+(\lambda-1)\sum_{i=1}^{n}\frac{\frac{\partial G(x_i,\boldsymbol{\zeta})}{\partial\zeta_k}}{G\left(x_i,\boldsymbol{\zeta}\right)}-\lambda\left(a+1\right)\sum_{i=1}^{n}\frac{\left[G(x_i,\boldsymbol{\zeta})\right]^{\lambda-1}\partial G(x_i,\boldsymbol{\zeta})/\partial\zeta_k}{1+\left[G\left(x_i,\boldsymbol{\zeta}\right)\right]^{\lambda}}$$
$$-\lambda(a-1)\sum_{i=1}^{n}\frac{\left[G(x_i,\boldsymbol{\zeta})\right]^{\lambda-1}\partial G(x_i,\boldsymbol{\zeta})/\partial\zeta_k}{1-\left[G(x_i,\boldsymbol{\zeta})\right]^{\lambda}}.$$

Setting U_a, U_λ and U_{ζ_k} equal to zero and solving them simultaneously give the maximum likelihood estimators (MLEs) $\widehat{\boldsymbol{\Phi}}=\left(\hat{a},\widehat{\lambda},\,\widehat{\boldsymbol{\zeta}}^{T}\right)^{T}$ of $\boldsymbol{\Phi}=\left(a,\lambda,\,\boldsymbol{\zeta}^{T}\right)^{T}$. These equations cannot be solved analytically and can be solved numerically using statistical softwares.
As example, we applied this on the aforementioned TIIEHLW distribution, were $\boldsymbol{\zeta}=(\delta,\gamma)^{T}$, so $\boldsymbol{\Phi}=(a,\lambda,\,\delta,\gamma)^{T}$ and its log-likelihood function has the form

$$\ln L\left(\Phi\right)=n\ln\left(2\lambda a\delta\gamma\right)\ +(\gamma-1)\sum_{i=1}^{n}\ln(x_i)-\delta\sum_{i=1}^{n}x_i^{\gamma}+(\lambda-1)\sum_{i=1}^{n}\ln\left[1-e^{-\delta x_i^{\gamma}}\right]$$
$$-(a+1)\sum_{i=1}^{n}\ln\left[1+\left[1-e^{-\delta x_i^{\gamma}}\right]^{\lambda}\right]+(a-1)\sum_{i=1}^{n}\ln\left[1-\left[1-e^{-\delta x_i^{\gamma}}\right]^{\lambda}\right].$$

This equation can be maximized either by using the different programs like SAS (PROC NLMIXED), **R** software (optim function) or by solving the nonlinear likelihood equations obtained by its differentiating simultaneously.

6 Simulation Study

To investigate the behavior of the MLEs of the TIIEHLW parameters, we present some simulations in terms of the sample size n. The TIIEHLW random variable can be simulated by using $X=Q\left(U\right)$ in Equation (15), where U is a uniformly random variable on the interval $(0,1)$.
By using **R** software (version 3.6.1) (Team, 2019), 2000 random samples from the TIIEHLW distribution has been generated at some sample sizes $n=30,\ 50,\ 100$ and $n=1000$. We set the true values of the parameters as follows: $a=(0.5,\ 1.2)$, $\lambda=(0.5,\ 1)$, $\delta=(0.75,\ 1.5)$ and $\gamma=(1.5,\ 2)$.
Tables 1 and 2 show the average maximum likelihood estimates (AVEs) and mean squared errors (MSEs) were computed for each sample size and each parameter combination, it can be seen that the estimates are stable and close the true parameter values

for these sample sizes. Furthermore, as the sample size increases the MSEs decreases in all cases.

Table 1: Average values of AVEs and the corresponding MSEs $a = 0.5$.

n	$\hat{a}$	MSEs	λ	$\hat{\lambda}$	MSEs	δ	$\hat{\delta}$	MSEs	γ	$\hat{\gamma}$	MSEs
30	0.63572	0.10310	0.5	0.26994	0.09131	0.75	0.17991	0.54227	1.5	2.37179	0.84905
50	0.65371	0.09702		0.34393	0.06883		0.24781	0.53581		1.99194	0.49333
100	0.68347	0.08003		0.37395	0.04384		0.26243	0.51549		1.96274	0.31727
1000	0.55081	0.03329		0.48499	0.00777		0.74382	0.24768		1.62347	0.05798
30	0.63375	0.10097		0.29355	0.09067		0.21486	0.53747	2	2.90239	1.28374
50	0.61389	0.08789		0.34318	0.07000		0.31063	0.53494		2.56972	0.80230
100	0.64160	0.07691		0.38902	0.04059		0.34215	0.51306		2.46117	0.43653
1000	0.53944	0.02918		0.47466	0.00698		0.68903	0.27351		2.10466	0.09109
30	0.64293	0.10091		0.28050	0.08978	1.5	0.63669	2.00441	1.5	2.28660	0.77429
50	0.67122	0.09391		0.33493	0.07107		0.55552	1.97796		2.07391	0.53889
100	0.69900	0.08283		0.37189	0.04382		0.58144	1.90536		1.96509	0.30433
1000	0.53039	0.03106		0.47735	0.00665		1.45622	0.93386		1.56435	0.04730
30	0.62423	0.09734		0.28219	0.09273		0.60929	2.02689	2	3.04988	1.39679
50	0.63988	0.08819		0.32879	0.06798		0.58617	1.97505		2.70827	0.92338
100	0.65323	0.07964		0.38118	0.04466		0.65368	1.92313		2.49718	0.48013
1000	0.52489	0.02913		0.47872	0.00631		1.38017	0.97386		2.07076	0.07774
30	0.43298	0.10916	1	1.12117	0.57945	0.75	1.05068	0.56096	1.5	1.47587	0.21806
50	0.47025	0.10126		1.10392	0.28471		0.94879	0.54493		1.46695	0.13529
100	0.47808	0.08261		1.06640	0.14115		0.90383	0.44573		1.48731	0.07508
1000	0.49665	0.02721		1.02665	0.01572		0.78090	0.10923		1.50299	0.00925
30	0.43777	0.10744		1.13026	0.57059		1.09020	0.56100	2	1.97969	0.38291
50	0.44848	0.09972		1.11204	0.29050		1.00309	0.55284		1.96126	0.22335
100	0.48176	0.08153		1.04339	0.10580		0.84870	0.43103		1.97490	0.11555
1000	0.49967	0.02801		1.01871	0.01593		0.79892	0.13310		1.99067	0.01691
30	0.48466	0.10846		1.06723	0.59581	1.5	1.74535	2.19969	1.5	1.53335	0.24295
50	0.47443	0.09949		1.06245	0.28370		1.65598	2.11115		1.47687	0.13678
100	0.48564	0.07966		1.05653	0.11847		1.56149	1.65669		1.47467	0.06291
1000	0.50834	0.0284		1.01228	0.01492		1.50108	0.41931		1.50544	0.00976
30	0.42569	0.10932		1.15618	0.59659		1.99029	2.21596	2	1.95493	0.38961
50	0.46571	0.10033		1.07271	0.30074		1.82837	2.11205		1.98271	0.24044
100	0.48147	0.07763		1.07081	0.11574		1.62003	1.65411		1.95600	0.12207
1000	0.49385	0.02962		1.01097	0.01459		1.52179	0.54791		1.98901	0.01542

Table 2: Average values of AVEs and the corresponding MSEs $a = 1.2$.

n	$\hat{a}$	MSEs	λ	$\hat{\lambda}$	MSEs	δ	$\hat{\delta}$	MSEs	γ	$\hat{\gamma}$	MSEs
30	0.95961	0.22345	0.5	0.34796	0.09418	0.75	1.02817	0.40981	1.5	2.03397	0.86092
50	1.03137	0.22275		0.40513	0.07380		0.83557	0.36410		1.80222	0.56869
100	1.11198	0.23831		0.44084	0.04690		0.67349	0.32718		1.67140	0.31410
1000	1.16744	0.16494		0.49409	0.00836		0.76199	0.21341		1.52436	0.07497
30	0.95149	0.23470		0.35894	0.09388		1.02869	0.44675	2	2.68201	1.51025
50	1.01726	0.21217		0.42188	0.07462		0.89464	0.37495		2.29088	0.91346
100	1.12507	0.24190		0.44703	0.04903		0.70526	0.33601		2.21443	0.57582
1000	1.18199	0.17336		0.48838	0.00860		0.75660	0.23018		2.05215	0.13392
30	1.05141	0.19914		0.35814	0.09606	1.5	2.55249	1.67681	1.5	2.03601	0.92241
50	1.09037	0.19718		0.4087	0.07172		1.94104	1.28603		1.79473	0.57691
100	1.16910	0.19148		0.44636	0.04823		1.57242	0.99861		1.67173	0.32420
1000	1.18962	0.16351		0.48462	0.00908		1.51158	0.69318		1.54614	0.08252
30	1.02810	0.18997		0.33182	0.09470		2.74590	1.83289	2	2.84372	1.61749
50	1.08354	0.17978		0.40080	0.07540		2.03834	1.26791		2.44080	1.03804
100	1.12731	0.19317		0.43294	0.04863		1.65568	1.03577		2.27856	0.60987
1000	1.19802	0.16944		0.48790	0.00914		1.50227	0.70788		2.04960	0.14589
30	0.77223	0.54040	1	1.26759	0.62946	0.75	1.45120	0.55649	1.5	1.44076	0.33218
50	0.84229	0.63308		1.25207	0.39221		1.37588	0.53900		1.40332	0.19117
100	0.89965	0.58082		1.18451	0.18498		1.19021	0.49206		1.40654	0.10553
1000	1.13553	0.21421		1.06120	0.01704		0.82097	0.13614		1.44980	0.01249
30	0.75521	0.57221		1.21207	0.61835		1.55884	0.65422	2	1.95455	0.51706
50	0.84110	0.61638		1.18992	0.36845		1.22143	0.52254		1.91846	0.35257
100	0.90735	0.66465		1.21932	0.20635		1.31136	0.50929		1.83487	0.18721
1000	1.16107	0.21965		1.05759	0.01767		0.79834	0.13823		1.93038	0.02635
30	0.81773	0.52958		1.19983	0.62041	1.5	2.92469	2.15297	1.5	1.47693	0.32055
50	0.90843	0.63566		1.24426	0.50605		2.34801	1.79762		1.38275	0.22786
100	0.96864	0.59435		1.17827	0.18125		2.07882	1.53937		1.39637	0.10848
1000	1.17331	0.22696		1.07340	0.01928		1.56724	0.55094		1.44320	0.01616
30	0.79904	0.52044		1.18786	0.63204		2.87139	2.11669	2	1.93256	0.63115
50	0.92746	0.62774		1.20057	0.45184		2.23384	1.74622		1.88551	0.38646
100	0.92544	0.60722		1.21356	0.20433		2.13488	1.59409		1.84881	0.19795
1000	1.14180	0.23911		1.07114	0.01752		1.63393	0.58799		1.92295	0.02559

7 Applications to Real Data

In this section, two real data sets are analyzed to illustrate the potentiality of the TIIEHL-G family. Applications of the TIIEHLW distribution is compared with five distributions; namely, type I exponentiated half-logistic Weibull (TIEHLW) (Cordeiro et al., 2014), type I half-logistic Weibull (TIHLW) (Kumar et al., 2015), exponentiated half-logistic (EHL), half-logistic (HL) (Johnson and Kotz, 1994) and Weibull distributions (W). The corresponding densities of the selected models are:

$$f_{TIEHLW}(x) = 2a\lambda\delta\gamma x^{\gamma-1}e^{-\lambda\delta x^{\gamma}}\frac{\left[1-e^{-\lambda\delta x^{\gamma}}\right]^{a-1}}{\left[1+e^{-\lambda\delta x^{\gamma}}\right]^{a+1}},$$

$$f_{TIHLW}(x) = \frac{2\lambda\delta\gamma x^{\gamma-1}e^{-\lambda\delta x^{\gamma}}}{\left[1+e^{-\lambda\delta x^{\gamma}}\right]^{2}},$$

$$f_{W}(x) = \frac{a}{\lambda}\left(\frac{x}{\lambda}\right)^{a-1}e^{-\left(\frac{x}{\lambda}\right)^{a}},$$

$$f_{EHL}(x) = \frac{2a\lambda e^{-\lambda x}}{1-e^{-2\lambda x}}\left[\frac{1-e^{-\lambda x}}{1+e^{-\lambda x}}\right]^{a}$$

and

$$f_{HL}(x) = 2\lambda\frac{e^{-\lambda x}}{(1+e^{-\lambda x})^{2}},$$

Data set 1: Represents the excesses of flood peaks (in m^3/s) Wheaton river near Carcross in the Yukon Territory, Canada. 72 exceedances of the years 1958 to 1984 were recorded, rounded to one decimal place. These data were analyzed by Choulakian and Stephens (2001), Aljouiee et al. (2018), Mansour et al. (2018), Mead et al. (2019) and Al-Mofleh et al. (2020).

Data set 2: Represents the gauge lengths of 10 mm from Kundu and Raqab (2009). This data set consists of 63 observations, and it was analyzed by Afify et al. (2015).

The estimates of the unknown parameters of each distribution is obtained by the maximum likelihood method. In order to compare the four distributions, various criteria were used. Criteria like: -2ℓ, Akaike information criterion (AIC), Bayesian information criterion (BIC), the correct Akaike information criterion ($CAIC$), Hannan information criterion ($HQIC$), Cramér-von Mises test statistic (W), Anderson-Darling test statistic (A), Kolmogorov-Smirnov ($K-S$) and its corresponding $p-$value statistics are considered for the two data sets.

Tables 3 and 5 display the MLEs of the model parameters and its standard error (S.E) (in parentheses) for the tow data sets, respectively.

Figures 9 and 11 provides the plots of the fitted pdfs and cdfs and HRF of the TIIEHLW, TIEHLW, TIHLW, W, EHL and HL models, and the TTT plot, for the two data sets, respectively. Figures 10 and 12 provides the plots of the quantile-quantile (qq) and the probability-probability (pp) of the TIIEHLW, TIEHLW, TIHLW, W, EHL and HL models for the two data sets, respectively. The values in Tables 4 and 6, and the plots indicate that the TIIEHLW distribution is a strong competitor to other distributions used here for fitting the two data sets, respectively.

Table 3: MLEs and the corresponding SEs (given in parentheses) for the data set 1.

Model	Estimates			
$TIIEHLW$	0.2477	0.8413	0.1906	1.1246
$(\hat{a}, \hat{\lambda}, \hat{\delta}, \hat{\gamma})$	(0.2240)	(0.5905)	(0.4176)	(0.3785)
$TIEHLW$	0.5662	0.7743	0.0614	1.1618
$(\hat{a}, \hat{\lambda}, \hat{\delta}, \hat{\gamma})$	(0.2871)	(4.8535)	(0.3712)	(0.4460)
$TIHLW$	2.8131	0.0760	0.7769	—
$(\hat{\lambda}, \hat{\delta}, \hat{\gamma})$	(15.5574)	(0.4197)	(0.0782)	
W	11.6322	0.9012	—	—
$(\hat{a}, \hat{\lambda})$	(1.6017)	(0.0856)		
EHL	0.6872	0.0885	—	—
$(\hat{\lambda}, \hat{\theta})$	(0.0962)	(0.0122)		
HL	0.1082	—	—	—
$(\hat{\lambda})$	(0.0109)			

Table 4: Measurements for all models based on for the data set 1.

Model	-2ℓ	AIC	CAIC	BIC	HQIC	W	A	K-S (stat)	K-S (p-vlaue)
TIIEHLW	**500.2723**	**508.2723**	**508.8693**	**517.3790**	**511.8977**	**0.1002**	**0.5829**	**0.1012**	**0.4522**
TIEHLW	502.3864	510.3864	510.9834	519.4931	514.0118	0.1014	0.6425	0.1106	0.3421
TIHLW	503.3775	509.3775	509.7304	516.2075	512.0965	0.1322	0.7768	0.1138	0.3085
W	502.9973	506.9973	507.1712	511.5506	508.8100	0.1380	0.7854	0.1052	0.4029
EHL	502.5315	506.5315	506.7054	511.0848	508.3442	0.1104	0.6793	0.1099	0.3493
HL	510.3328	512.3328	512.3900	514.6095	513.2392	0.1174	0.7195	0.1969	0.0075

Table 5: MLEs and the corresponding SEs (given in parentheses) for the data set 2.

Model	Estimates			
$TIIEHLW$	0.2090	15.5103	0.1611	3.2200
$(\hat{a}, \hat{\lambda}, \hat{\delta}, \hat{\gamma})$	(0.0538)	(0.4199)	(0.0255)	(0.1405)
$TIEHLW$	26.5167	0.6352	1.5625	1.3530
$(\hat{a}, \hat{\lambda}, \hat{\delta}, \hat{\gamma})$	(50.9272)	(5.6670)	(13.9783)	(0.6247)
$TIHLW$	0.1523	0.0599	4.2578	—
$(\hat{\lambda}, \hat{\delta}, \hat{\gamma})$	(0.6753)	(0.2641)	(0.4082)	
W	3.3147	5.0494	—	—
$(\hat{a}, \hat{\lambda})$	(0.0878)	(0.4557)		
EHL	111.0110	1.9508	—	—
$(\hat{\lambda}, \hat{\theta})$	(55.7608)	(0.1909)		
HL	0.5006	—	—	—
$(\hat{\lambda})$	(0.0489)			

Table 6: Measurements for all models based on for the data set 2.

Model	-2ℓ	AIC	CAIC	BIC	HQIC	W	A	K-S	p-vlaue
TIIEHLW	**111.4801**	**119.4801**	**120.1698**	**128.0526**	**122.8517**	**0.0403**	**0.2299**	**0.0698**	**0.9186**
TIEHLW	112.6982	120.6982	121.3879	129.2708	124.0698	0.0637	0.3362	0.0830	0.7786
TIHLW	125.9383	131.9383	132.3451	138.3677	134.4670	0.1521	1.0386	0.0992	0.5653
W	123.9140	127.9140	128.1140	132.2002	129.6000	0.1284	0.8921	0.0876	0.7192
EHL	113.0276	117.0276	117.2276	121.3139	118.7134	0.0707	0.3663	0.0887	0.7041
HL	243.8286	245.8286	245.8941	247.9717	246.6715	0.0615	0.3896	0.4723	1.2428×10^{-12}

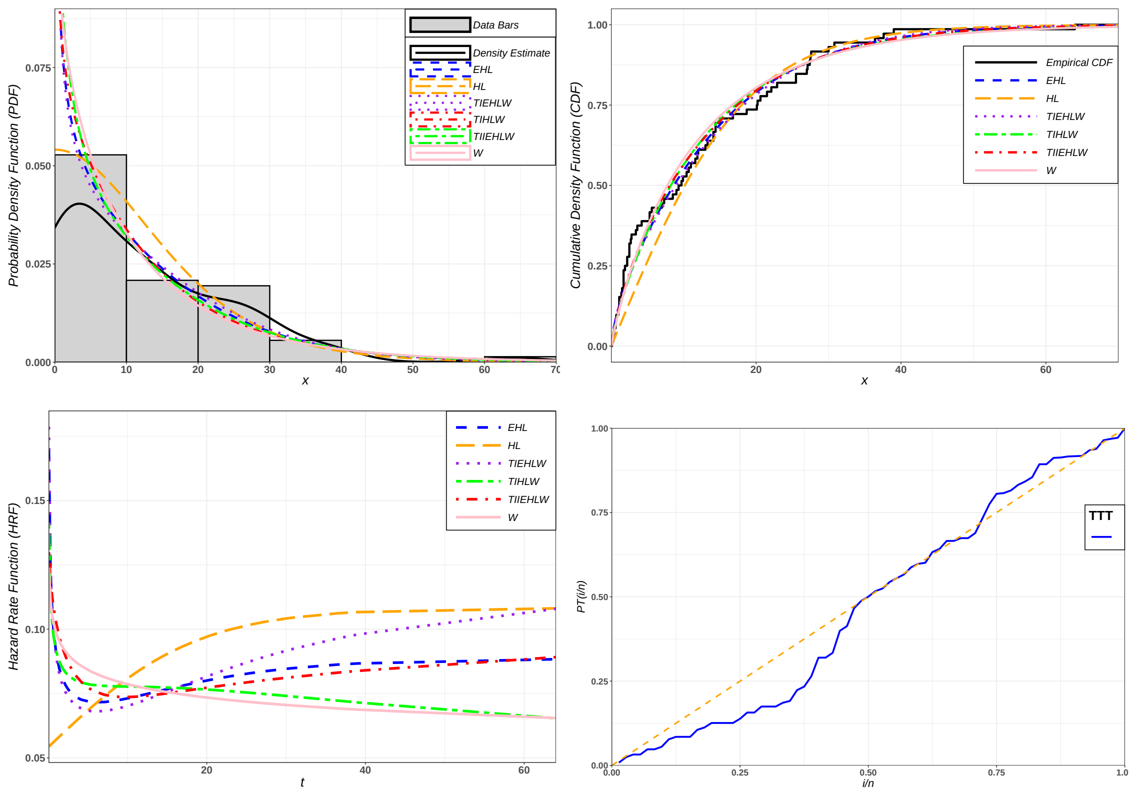

Figure 9: The fitted pdfs, cdfs and HRF plots of the TIIEHLW distribution and other fitted distributions, and the TTT plot, for data set 1.

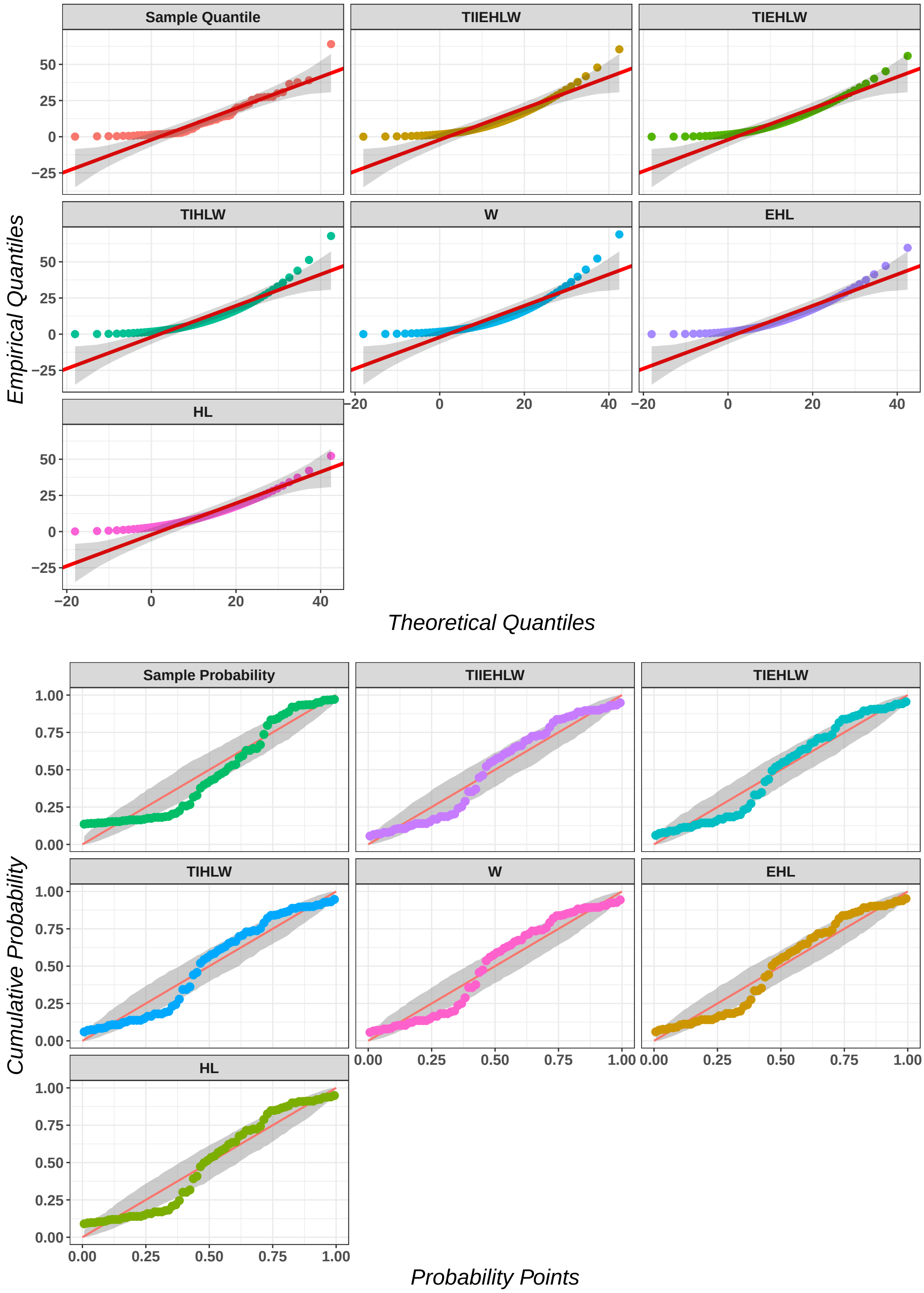

Figure 10: The quantile-quantile (qq) plot and the probability-probability (pp) plot of the TIIEHLW distribution and other fitted distributions for data set 1.

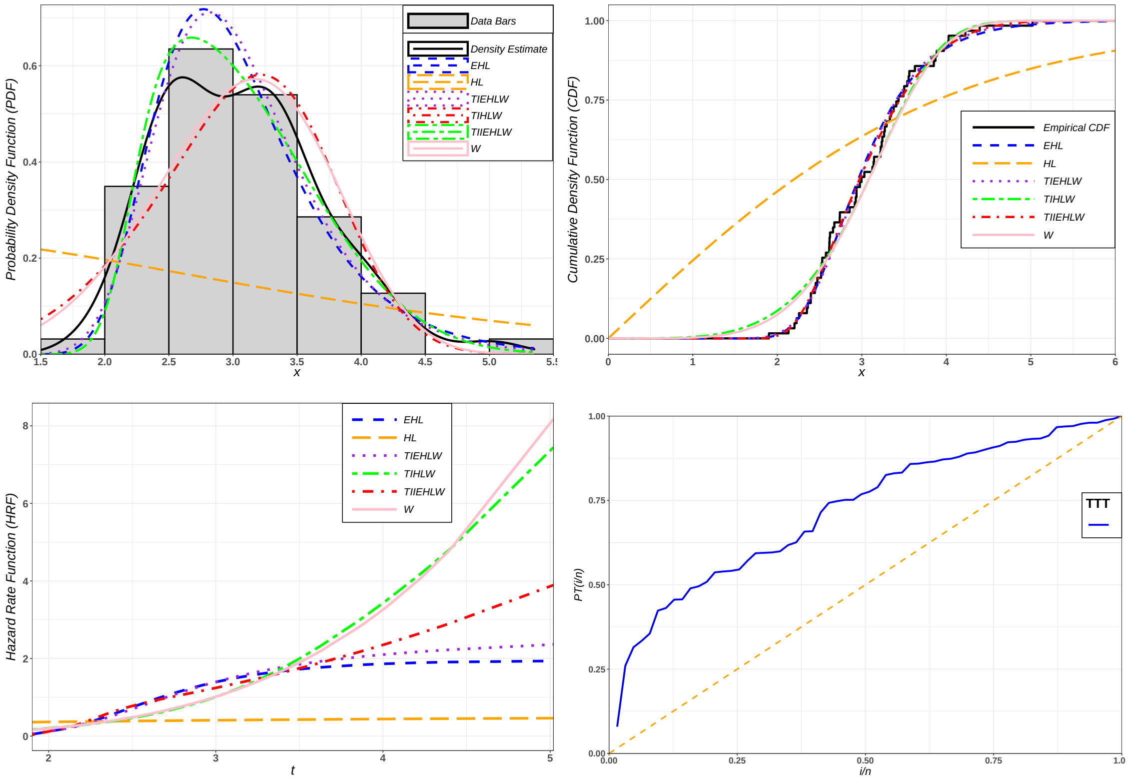

Figure 11: The fitted pdfs, cdfs and HRF plots of the TIIEHLW distribution and other fitted distributions, and the TTT plot, for data set 2.

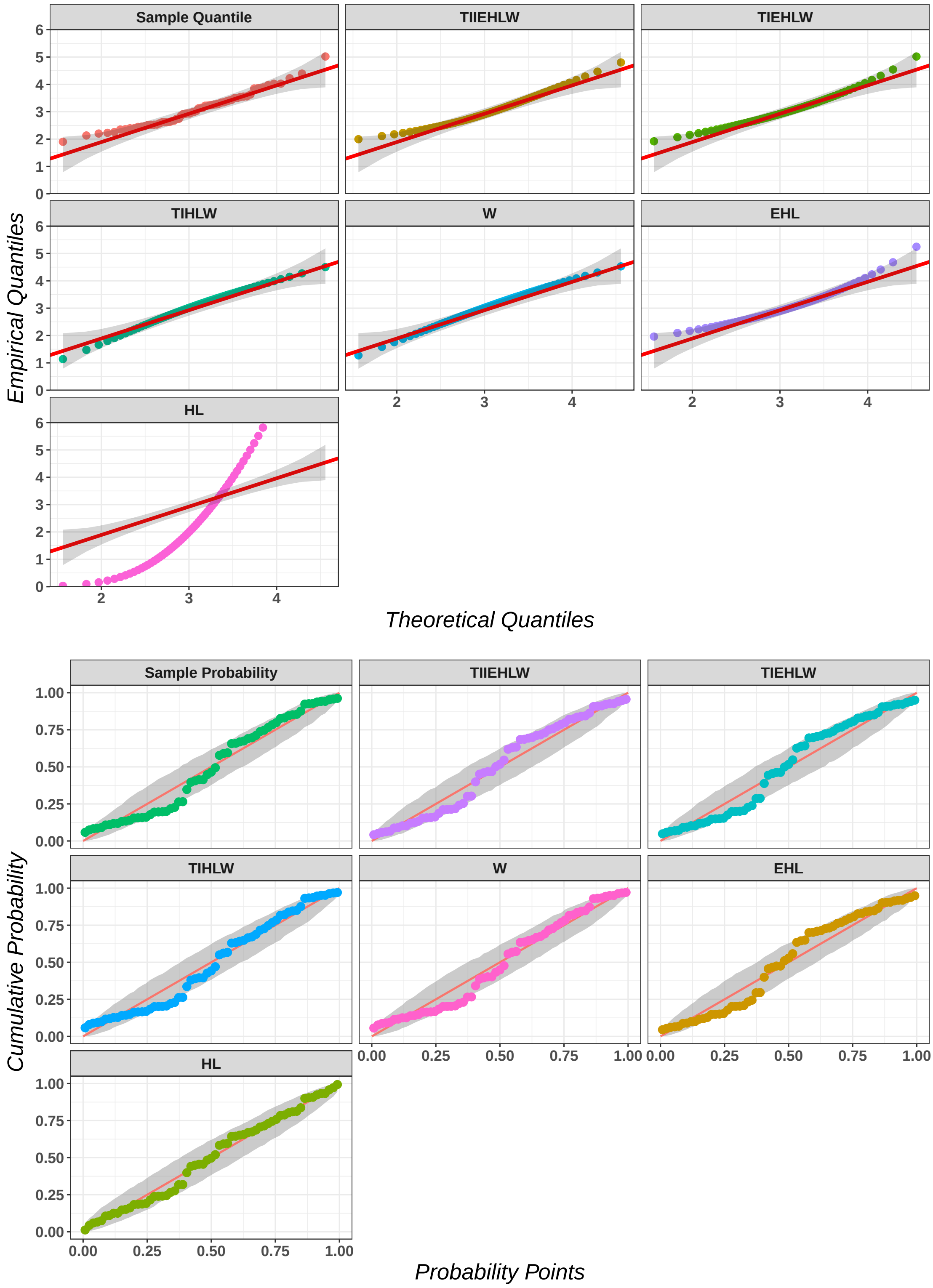

Figure 12: The quantile-quantile (qq) plot and the probability-probability (pp) plot of the TIIEHLW distribution and other fitted distributions for data set 2.

8 Conclusion

In the present paper, the new type II exponentiated half logistic generated family of distributions is proposed. More specifically, the type II exponentiated half logistic generated family covers several new distributions. We wish a broadly statistical application in some area for this new generation. Some characteristics of the TIIEHL-G, such as, expressions for the density function, moments, mean deviation, quantile function and order statistics are discussed. The maximum likelihood method is employed for estimating the model parameters. Type II half logistic uniform, type II exponentiated half logistic Weibull, type II exponentiated half logistic Burr XII and type II exponentiated half logistic quasi Lindley selected models are provided. Applications to real data sets validate the priority of the new family.

References

Afify, A. and Alizadeh, M. (2020). The odd Dagum family of distributions: Properties and applications. *Journal of Applied Probability and Statistics*, 15:45–72.

Afify, A., Cordeiro, G., Yousof, H. M., Alzaatreh, A., and Nofal, Z. (2016). The Kumaraswamy transmuted-G family of distributions: Properties and applications. *Journal of data science: JDS*.

Afify, A. and Mohamed, O. (2020). A new three-parameter exponential distribution with variable shapes for the hazard rate: Estimation and applications. *Mathematics*, 8:135.

Afify, A., Nofal, Z., and Ebraheim, H. (2015). Exponentiated transmuted generalized Raleigh distribution: A new four parameter Rayleigh distribution. *Pakistan Journal of Statistics and Operation*, 11:115–134.

Afify, A. Z., Cordeiro, G. M., Maed, M. E., Alizadeh, M., Al-Mofleh, H., and Nofal, Z. M. (2019). The generalized odd Lindley-G family: Properties and applications. *Anais da Academia Brasileira de Ciências*, 91(3):1–22.

Al-Mofleh, H., Afify, A. Z., and Ibrahim, N. A. (2020). A new extended two-parameter distribution: Properties, estimation methods, and applications in medicine and geology. *Mathematics*, 8(9):1578.

Al-Omari, A. I., Alhyasat, K., Ibrahim, K., and Abu Bakar, M. A. (2019). Power length-biased Suja distribution: properties and application. *Electronic Journal of Applied Statistical Analysis*, 12(2):429–452.

Al-Omari, A. I. and Shraa, D. (2019). Darna distribution: properties and application. *Electronic Journal of Applied Statistical Analysis*, 12(2):520–541.

Aljouiee, A., Elbatal, I., and Al-Mofleh, H. (2018). A new five-parameter lifetime model: Theory and applications. *Pakistan Journal of Statistics and Operation Research*, pages 403–420.

Bourguignon, M., Silva, R. B., and Cordeiro, G. M. (2014). The Weibull-G family of probability distributions. *Journal of Data Science*, 12(1):53–68.

Choulakian, V. and Stephens, M. A. (2001). Goodness-of-fit tests for the generalized Pareto distribution. *Technometrics*, 43(4):478–484.

Cordeiro, G. M., Afify, A. Z., Ortega, E. M., Suzuki, A. K., and Mead, M. E. (2019). The odd Lomax generator of distributions: Properties, estimation and applications. *Journal of Computational and Applied Mathematics*, 347:222–237.

Cordeiro, G. M., Alizadeh, M., and Diniz Marinho, P. R. (2016). The type I half-logistic family of distributions. *Journal of Statistical Computation and Simulation*, 86(4):707–728.

Cordeiro, G. M., Alizadeh, M., and Ortega, E. M. M. (2014). The exponentiated half-logistic family of distributions: Properties and applications. *Journal of Probability and Statistics*, 2014(1):21.

Cordeiro, G. M. and de Castro, M. (2011). A new family of generalized distributions. *Journal of statistical computation and simulation*, 81(7):883–898.

Cordeiro, G. M., Ortega, E. M., and da Cunha, D. C. (2013). The exponentiated generalized class of distributions. *Journal of Data Science*, 11(1):1–27.

David, H. and Nagaraja, H. (2004). *Order Statistics*. Wiley, New York, 3ed edition edition.

Elgarhy, M., Haq, M., and Ozel, G. (2017). A new exponentiated extended family of distributions with applications. *Gazi University Journal of Science*, 30(3):101–115.

Eugene, N., Lee, C., and Famoye, F. (2002). Beta-normal distribution and its applications. *Communications in Statistics-Theory and methods*, 31(4):497–512.

Greenwood, J. A., Landwehr, J. M., Matalas, N. C., and Wallis, J. R. (1979). Probability weighted moments: definition and relation to parameters of several distributions expressable in inverse form. *Water resources research*, 15(5):1049–1054.

Hashmi, S., Usman, R. M., et al. (2019). A generalized exponential distribution with increasing, decreasing and constant shape hazard curves. *Electronic Journal of Applied Statistical Analysis*, 12(1):223–244.

Hassan, A. and Elgarhy, M. (2016a). A new family of exponentiated Weibull-generated distributions. *International Journal of Mathematics And its Applications*, 4:135–148.

Hassan, A., Elgarhy, M. A. E., and Shakil, M. (2017). Type II half logistic family of distributions with applications. *Pakistan Journal of Statistics and Operation Research*, 13:245–264.

Hassan, A. S. and Elgarhy, M. (2016b). Kumaraswamy Weibull-generated family of distributions with applications. *Advances and Applications in Statistics*, 48(3):205–239.

He, W., Ahmad, Z., Afify, A. Z., and Goual, H. (2020). The arcsine exponentiated-X family: Validation and insurance application. *Complexity*, 2020:1–18.

Johnson, N. and Kotz, S. (1994). *Continuous univariate distributions (Vol. 2)*. Wiley.

Kumar, D., Jain, N., and Gupta, S. (2015). The type I generalized half-logistic distribution based on upper record values. *Journal of Probability and Statistics*, 2015:1–11.

Kundu, D. and Raqab, M. Z. (2009). Estimation of $R = P(Y < X)$ for three-parameter

Weibull distribution. *Statistics & Probability Letters*, 79(17):1839–1846.

Mansour, M. M., Abd Elrazik, E. M., Altun, E., Afify, A. Z., and Iqbal, Z. (2018). A new three-parameter Fréechet distribution: properties and applications. *Pakistan Journal of Statistics*, 34(6):441–458.

Marshall, A. W. and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 84(3):641–652.

Mead, M. E., Cordeiro, G. M., Afify, A. Z., and Al-Mofleh, H. (2019). The alpha power transformation family: properties and applications. *Pakistan Journal of Statistics and Operation Research*, 15(3):525–545.

Pescim, R. R., Cordeiro, G. M., Demétrio, C. G., Ortega, E. M., and Nadarajah, S. (2012). The new class of Kummer beta generalized distributions. *Statistics and Operations Research Transactions*, 36(2):153–180.

Rényi, A. (1961). On measures of entropy and information. In *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*, pages 547–561. The Regents of the University of California, Berkeley.

Ristić, M. M. and Balakrishnan, N. (2012). The gamma-exponentiated exponential distribution. *Journal of Statistical Computation and Simulation*, 82(8):1191–1206.

Shakhatreh, M. and Al-Masri, A.-Q. (2020). On the weighted BurrXII distribution: theory and practice. *Electronic Journal of Applied Statistical Analysis*, 13(1):229–255.

Shanker, R. and Mishra, A. (2013). A quasi Lindley distribution. *African Journal of Mathematics and Computer Science Research*, 6(4):pp–64.

Team, R. C. (2019). *R: A Language and Environment for Statistical Computing.* R Foundation for Statistical Computing, Vienna, Austria.