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A robust confidence interval based on modified trimmed standard deviation for the mean of positively skewed populations

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In this study, we propose a robust confidence interval which is adjustment of the Student-t confidence interval based on the trimmed mean and the modified trimmed standard deviation for the mean of skewed populations. The proposed confidence interval is compared with existing confidence intervals in terms of coverage probability and average width. The simulation study showed that the proposed robust confidence interval performed better than the others. Also, proposed confidence interval has narrowest average width in all sample sizes. In addition to the simulation, two real data sets were analyzed to illustrate the findings of the simulation study and the simulation results were verified. Consequently, we recommend the confidence interval based on trimmed mean and modified trimmed standard deviation to estimate the mean of positively skewed populations.

keywords: average width, confidence interval, coverage probability, modified trimmed standard deviation, trimmed mean.

1 Introduction

The confidence intervals (CIs) provide much more information than a point estimate about the population characteristic. The problem of dealing with skewed distributions

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and outliers to estimate population parameter has recently attracted the attention of researchers. CIs are often used for this purpose. There are various methods in the literature in which CIs are obtained for the population mean. The most useful CIs are the classical normal CI and the Student-t CI. The sample size has to be more than or equal to 30 ($n \geq 30$) for the classical CI. In practice, it is often possible to work with smaller sample sizes. In such cases, Student- t CI can be preferred instead of the classical CI, but it requires an assumption of normality. In such cases, it is essential to use robust estimators which are less affected by outliers or small departures from the model assumptions (Sindhumol et al., 2016).

Johnson (1978) proposed a modification of the Student-t CI for skewed distributions. Since Johnson (1978), many researchers obtained CIs for population mean (Kleijnen et al., 1986; Meeden, 1999; Willink, 2005; Kibria, 2006; Shi and Kibria, 2007; AbuShawiesh et al., 2018). Hui et al. (2005) studied for the regression estimation of the population mean. Wang (2008) derived the Bayesian credible interval and likelihood ratio interval for the mean of a normal distribution with general restricted parameter space. Withers and Nadarajah (2011) proposed the CI for the length of a vector mean.

In this study, we propose a robust CI which is simple adjustment of the Student-t CI, namely, modified trimmed standard deviation t CI (MS_T-t CI). The proposed robust CI uses the modification to trimmed standard deviation given by Sindhumol et al. (2016). The performance of the proposed robust method to estimate the population mean (μ) is compared with the existing CIs via a Monte-Carlo simulation study. The coverage probability (CP) and the average width (AW) are considered as a comparison criterion. The CPs were determined as the proportion of cases to the number of replications where the mean was between the lower and upper limits. The AWs were obtained by dividing the total differences of the lower and upper limits found for each replication to the number of replications. The smaller AW indicates better CI when CPs are the same.

The rest of this paper is organized as follows: The classical CIs for the population mean are discussed in Section 2. The robust estimators and sample standard deviation is given in Section 3. The trimmed mean and modified trimmed standard deviation are presented in Section 4. In Section 5, the various modifications of the Student-t CI are detailed. The proposed robust CI is given in Section 6. A Monte-Carlo simulation study is conducted in Section 7. As an application, some real life data are analyzed in Section 8. Section 9 concludes and summarizes the findings and outcomes of the paper.

2 The classical Student-t confidence intervals for the population mean

The classical method to construct the $(1 - \alpha)$ 100 % CI for the population mean is still the most used approach because it is a well understood, simple, and widely used to construct such CI. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a normal distribution with population mean (μ) and population variance (σ^2), that is,$X_1, X_2, ..., X_n \sim N(\mu, \sigma^2)$. Then, the $(1 - \alpha)$ 100% CI for the population mean (μ), can be constructed as follows (Student, 1908):

$$C.I. = \bar{X} \pm Z_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \tag{1}$$

where σ is the known population standard deviation and $Z_{1-\alpha/2}$ is the upper $(\alpha/2)th$ percentile of the standard normal distribution. In real life, however, it is unlikely that the population standard deviation (σ) is known, and then an estimate of σ is used. When the sample size n is large ($n \geq 30$), we can use the sample standard deviation instead of σ and apply the normal distribution to construct the $(1 - \alpha)$ 100% CI for the population mean (μ) as follows:

$$C.I. = \bar{X} \pm Z_{1-\frac{\alpha}{2}} \frac{S}{\sqrt{n}} \tag{2}$$

where$S = \sqrt{(n-1)^{-1} \sum_{i=1}^{n}(X_i - \bar{X})^2}$, $\bar{X} = n^{-1}\sum_{i=1}^{n} X_i$is the sample mean and $Z_{1-\alpha/2}$ is the upper $(\alpha/2)th$ percentile of the standard normal distribution. On the other hand, for the small sample sizen ($n < 30$) and unknown population standard deviation (σ), the $(1 - \alpha)$ 100% CI for the population mean (μ) due to Student (1908) and known as the Student-t CI can be constructed as follows:

$$C.I. = \bar{X} \pm t_{(\frac{\alpha}{2}, n-1)} \frac{S}{\sqrt{n}} \tag{3}$$

where $t_{(\alpha/2, n-1)}$ is the upper $\alpha/2$ percentage point of the student-t distribution with$(n-1)$ degrees of freedom, i.e. $P(t > t_{(\alpha, n-1)}) = \alpha$. Since the classical Student-t CI depends on the normality assumption, it may not be the best CI for skewed distributions. It is well known that if the data are from a normal distribution or the sample size n is large ($n \geq 30$), the CP will be exact or close to 1-α, but when the population distribution is skewed, the Student-t CI has a poor CP. According to the Boos and Hughes-Oliver (2000), the classical Student-t CI is not very robust under extreme deviations from normality.

3 Robust estimators and sample standard deviation

A robust estimator is an estimator that is insensitive to changes in the underlying distribution and also resistant against the presence of outliers. Also, an estimator is said to be robust if it is fully efficient or nearly so for an assumed distribution but maintains high efficiency for plausible alternatives (Tiku and Akkaya, 2004).

The sample standard deviation is considered meaningful and efficient measure of variability only for a normal distribution (AbuShawiesh, 2008). As the calculation of sample standard deviation based on all data points in a sample, then this makes it non-robust to slight deviations from normality and can be easily influenced by the presence of outliers. Therefore, the CIs defined in Equations (1), (2) and (3) would not be reliable and may give erroneous and misleading results. Thereby, it is necessary to take into account the non-normality to prevent the loss of resources, money and time in order that the practitioners made an accurate result. In case of non-normal or skewed distributions, there are other measures that performed better than the sample standard deviation because of their robustness properties. Among those robust scale measures is the trimmed standard deviation (S_T).

4 The trimmed mean and modified trimmed standard deviation

One problem with the mean is that the tails of a distribution can dominate its value. In order to reduce the effect of tails of a distribution, it can be simply removed. The trimmed mean and its standard error are more appealing because of its computational simplicity. Apart from that, these measures are less affected by departures from normality than the usual mean and standard deviation, as observations in the tail are removed. Standard error of trimmed mean is not sufficient to estimate distribution dispersion because of trimming samples (Dixon and Yuen, 1974). Standard estimator of variance of trimmed mean is obtained through winsorization (Wilcox, 2012). Huber (1981) showed a jackknife estimator for its variance. Caperaa and Rivest (2000) derived an exact formula for variance of the trimmed mean as a function of order statistics, when trimming percentage is small. Johnson et al. (1986)compared Bayesian estimator and trimmed means. As trimming makes a reduction in dispersion, estimating population dispersion based standard error of trimmed mean will not give a clear picture of actual dispersion. Variance of trimmed mean which is a function of order statistics or its variance modification based on winsorization, are not helping in this regard. Hence trimmed standard deviation has a limited exposure to applications in literature.

Sindhumol et al. (2016) made a modification to improve the variance of the trimmed mean by multiplying it with a tuning constant to reduce the effect of loss due to trimming so that its robust qualities are not much disturbed. In this paper, a robust CI for the population mean of skewed populations, that is simple adjustment of the Student-t CI, is developed based on this modified measure, namely, modified trimmed standard deviation t CI (MS$_T$ - t CI).

Let$X_{(1)} \leq X_{(2)} \leq \ ... \ \leq X_{(n)}$ denote an order statistics sample of size n, from a population having symmetric distribution. The r-times symmetrically trimmed sample is obtained by dropping both r-lowest and r-highest values. Here $r = [\alpha \text{n}]$ is the greatest integer and trimming is done for $\alpha\,\%$ $(0 \leq \alpha \leq 0.5)$ of n. The trimmed mean $(\bar{X}_T)$ can be calculated as follows:

$$\bar{X}_T = \frac{1}{n-2r} \sum_{i=r+1}^{n-r} X_{(i)} \tag{4}$$

and the sample standard deviation of observations from the trimmed mean($\bar{X}_T$) is denoted byS_T which can be calculated as follows:

$$S_T = \sqrt{\frac{1}{n-2r-1} \sum_{i=r+1}^{n-r} (X_{(i)} - \bar{X}_T)^2} \tag{5}$$

and by assuming symmetric trimming and normal distribution, that is, $X_1,\ X_2,\ \ldots,\ X_n \sim N(\mu\,,\,\sigma^2)$, we get the following modified estimator for the population standard deviation (Sindhumol et al., 2016):

$$P\left\{\left|\frac{X-\mu}{\sigma}\right| \le \frac{S_T}{\sigma}\right\} = 1-2\alpha \tag{6}$$

$$\Phi\left(-\frac{S_T}{\sigma}\right) \;=\; \alpha = 1-\Phi\left(\frac{S_T}{\sigma}\right) \tag{7}$$

$$\hat{\sigma} \;=\; \left[\frac{1}{\phi^{-1}(1-\alpha)}\right] S_T \tag{8}$$

$$\hat{\sigma}_T \;=\; S_T^* \;=\; 1.4826\, S_T \tag{9}$$

where Φ is the distribution function of standard normal distribution.

5 The various modifications of the student-t confidence interval

In this section, let X_1, X_2, ... , X_n be a random sample of small size n ($n \;<\; 30$) from a non-normal or skewed distribution with population mean (μ) and unknown population standard deviation (σ). We will consider various modifications of the Student-t CI that are the most popular CIs in literature.

5.1 The Johnson's-t confidence interval

Johnson (1978) proposed the following CI for the population mean (μ) of a skewed distribution:

$$C.I. \,=\, (\bar{X} + \frac{\hat{\mu}_3}{6S^2 n}) \;\pm\; t_{(\frac{\alpha}{2}\,,\,n-1)}\,\frac{S}{\sqrt{n}} \tag{10}$$

where $\hat{\mu}_3 \,=\, \frac{n}{(n-1)(n-2)}\sum_{i=1}^{n}\left(X_i-\bar{X}\right)^{\,3}$ is the unbiased estimator of the third central moment μ_3. According to Kibria (2006), it appears that the width of Student's-t and Johnson's-t CIs are same**.**

5.2 The Median-t confidence interval

When the distribution is skewed or non-normal, the sample median (MD) describes the center of the distribution better than that of the sample mean which is preferable to other estimators of center for a distribution that is symmetric or relatively homogeneous. Therefore, for a skewed distribution, it is reasonable to define the sample standard deviation in terms of the MD than the sample mean. Kibria (2006) proposed the following CI for the population mean (μ) of a skewed distribution:

$$C.I. \,=\, \bar{X} \;\pm\; t_{(\frac{\alpha}{2}\,,\,n-1)}\,\frac{\tilde{S}_1}{\sqrt{n}} \tag{11}$$

where

$$\tilde{S}_1 = \sqrt{(n-1)^{-1}\sum_{i=1}^{n}(X_i - MD)^2}$$

and

$$MD = \begin{cases} X_{(\frac{n+1}{2})} & if \quad n \quad is \quad odd\, number \\ \frac{X_{(\frac{n}{2})}+X_{(\frac{n}{2}+1)}}{2} & if \quad n \quad is \quad even\, number \end{cases}$$

5.3 The Mad-t confidence interval

In terms of the MD than the sample mean for the defining of the sample standard deviation, Kibria (2006) proposed another CI for the population mean (μ) of a skewed distribution given as follows:

$$C.I. = \bar{X} \pm t_{(\frac{\alpha}{2}, n-1)} \frac{\tilde{S}_2}{\sqrt{n}} \tag{12}$$

where$\tilde{S}_2 = \frac{1}{n}\sum_{i=1}^{n} | X_i - MD |$ is the sample mean absolute deviation (Mad).

5.4 The AADM-t confidence interval

AbuShawiesh et al. (2018) proposed a modification of the Student-t CI for the population mean of a skewed distribution called AADM-t CI and given as follows:

$$C.I. = \bar{X} \pm t_{(\frac{\alpha}{2}, n-1)} \frac{AADM}{\sqrt{n}} \tag{13}$$

where $AADM = \frac{\sqrt{\pi/2}}{n}\sum_{i=1}^{n} | X_i - MD |$ given by Gastwirth (1982) is the average absolute deviation from the sample median. As stated by Gastwirth (1982), the AADM is asymptotically normally distributed, consistent estimator of σ, and converges to σ almost surely.

6 The proposed robust confidence interval

The trimmed mean is a more robust measure for describing the center than the mean and more efficient than the median. We thought that for a skewed distribution with a longer left or right tail, it is reasonable to define the standard deviation in terms of the trimmed mean. Therefore, we propose a robust modification of the Student-t CI for the population mean of skewed populations. It is a simple adjustment of the Student-t CI developed based on the modified trimmed standard deviation given by Equation (1), we refer to as MS_T-t CI.

Thus, the $(1-\alpha)100\%$ MS_T-t CI for the population mean of a skewed distribution is given as:

$$C.I. = \bar{X} \pm t_{(\frac{\alpha}{2}, n-1)} \frac{S_T^*}{\sqrt{n}} \tag{14}$$

where $S_T^* = 1.4826 * S_T$. In Eq. (5), it was obtained as $S_T = \sqrt{\frac{1}{n-2r-1}\sum_{i=r+1}^{n-r}(X_{(i)} - \bar{X}_T)^2}$. Thus, S_T^* is can be obtained as:

$$S_T^* = 1.4826 * S_T = 1.4826 * \sqrt{\frac{1}{n-2r-1}\sum_{i=r+1}^{n-r}(X_{(i)} - \bar{X}_T)^2}. \tag{15}$$

When trimming is made on both ends for symmetrical distributions, trimmed mean is as:

$$\bar{X}_T = \frac{1}{n-2r}\sum_{i=r+1}^{n-r} X_{(i)} \tag{16}$$

When trimming is made only on the high end for positively skewed distributions, it is defined as follows (Tiku and Akkaya, 2004 ; Akyüz et al., 2017):

$$\bar{X}_T = \frac{1}{n-r}\sum_{i=1}^{n-r} X_{(i)} \tag{17}$$

7 Simulation study

In this section, the performance of the various CIs described in this paper is compared by a Monte Carlo simulation study. All simulations were performed using programs written in the MATLAB. The samples are simulated from standard normal, gamma, chisquare and lognormal distributions.

In order to make the comparisons among various CIs, the following criteria are considered: CP and AW of the resulting CIs. The most common 95% CI (α =0.05) for the confidence level is used. The simulation study was designed as:

- Sample sizes n=5, 10, 20, 30, 50, 100 and 500,
- Trimming ratio= 5%, 10%, 20%,
- Replication= 50000,
- Standard normal distribution,
- Gamma (16, 0.0625) with skewness 0.5,
- Gamma (4,0.25) with skewness 1,

- Gamma (1,1) with skewness 2,
- χ^2 (1) with skewness 2.82,
- Lognormal (0,1) with skewness 6.18.

Trimming is made from the high end of the consecutive data in positively skewed distributions and both ends in symmetric distributions. The performances of the simulations for each distribution in terms of CP and AW are reported in Table 1- 6. The first line for each sample size (n) is CP, the second line is AW.

Table 1: Coverage probability (CP) and average width (AW) of 95% CIs for N(0, 1)

n	Student-t	Johnson-t	Median-t	Mad-t	AADM-t	MS_T - t		
						5%	10%	20%
5	0.9485	0.9459	0.9475	0.8733	0.9260	0.9418	0.9418	0.8478
	2.3339	2.0875	2.4544	1.9978	2.0653	1.9224	1.9224	1.9224
10	0.9725	0.9456	0.9473	0.9046	0.9259	0.9422	0.9422	0.8970
	1.8133	1.8305	1.8183	1.7574	1.7513	1.7263	1.7263	1.2492
20	0.9498	0.9442	0.9465	0.9128	0.9409	0.9460	0.9420	0.8437
	0.9246	0.9011	0.9897	0.9975	0.9493	0.9006	0.8928	0.6790
30	0.9484	0.9441	0.9441	0.9024	0.9434	0.9483	0.9478	0.8365
	0.7410	0.7286	0.7775	0.6518	0.7542	0.6421	0.6025	0.5338
50	0.9516	0.9495	0.9407	0.8955	0.9433	0.9488	0.9476	0.8327
	0.5655	0.5599	0.5828	0.5556	0.5710	0.5503	0.5067	0.3998
100	0.9505	0.9493	0.9482	0.8879	0.9498	0.9490	0.9474	0.8264
	0.3957	0.3937	0.4020	0.3872	0.3975	0.3677	0.3326	0.2759
500	0.9490	0.9488	0.9485	0.8807	0.9498	0.9490	0.9474	0.8202
	0.1756	0.1754	0.1762	0.1602	0.1757	0.1559	0.1426	0.1117

In Table 1, it was obtained CPs and AWs of existing and MS_T-t CI. Thus, the MS_T-t CI performance was comparable to existing CIs. Trimming operation for MS_T-t CI was made on both ends of the consecutive data. Random samples produced from standard normal distribution for sample sizes n= 5, 10, 20, 30, 50, 100 and 500. Trimming ratio and Type I error were considered as 5%, 10%, 20% and α =0.05, respectively . It is determined that CPs of the both existing and MS_T-t CI are close to the nominal confidence level in all sample sizes. However, as the trimming ratio increased, CPs of

MS_T-t CI decreased. Furthermore, MS_T-t CI have the smallest AW for all sample sizes. It is observed that the average widths of CIs are reduced as the sample size increases. When the amount of trimming in the data set increased, AWs also decreased.

Table 2: Coverage probability (CP) and average width (AW) of 95% CIs for G(16, 0.0625) with Skewness γ_1 =0.5

n	Student-t	Johnson-t	Median-t	Mad-t	AADM-t	MS_T - t		
						5%	10%	20%
5	0.9466	0.9280	0.9427	0.8710	0.9232	0.9491	0.9491	0.9421
	0.5800	0.5188	0.6109	0.5080	0.5113	0.4379	0.4379	0.4379
10	0.9690	0.9622	0.9473	0.9046	0.9259	0.9490	0.9490	0.9421
	0.4025	0.3819	1.7382	1.6010	2.0066	0.3929	0.3929	0.2993
20	0.9477	0.9431	0.9465	0.9128	0.9409	0.9499	0.9490	0.9431
	0.3304	0.3246	0.9528	0.9003	1.1283	0.3013	0.2759	0.2401
30	0.9499	0.9465	0.9471	0.9024	0.9434	0.9492	0.9490	0.9444
	0.2449	0.2818	0.7534	0.7177	0.8995	0.2323	0.2195	0.1912
50	0.9477	0.9455	0.9487	0.8955	0.9433	0.9510	0.9495	0.9440
	0.1712	0.1898	0.5691	0.5456	0.6838	0.1384	0.1365	0.1250
100	0.9494	0.9484	0.9482	0.9079	0.9498	0.9510	0.9495	0.9442
	0.1389	0.1384	0.3952	0.3807	0.4771	0.1266	0.1158	0.1009
500	0.9505	0.9503	0.9485	0.9087	0.9498	0.9510	0.9495	0.9446
	0.0639	0.0638	0.1742	0.1684	0.2111	0.0559	0.0511	0.0446

In Table 2, random samples produced from gamma distribution for parameter α=16 and β=0.0625 with skewness γ_1 =0.5. Simulation results in Table 2 showed that MS_T-t CI have CP closer to nominal confidence level than existing CIs for small sample sizes (n = 5, 10, 20). It was seen that AWs were nearly same for very small sample size (n=5). However, AWs of MS_T-t CI were narrower than others in all sample sizes. It is obtained that even if the trimming ratio is large enough (20%), AWs of MS_T-t CI is narrower than the other CIs. This result indicates that the MS_T-t CI is quite robust.

Table 3: Coverage probability (CP) and average width (AW) of 95% CIs for G(4, 0.25) with Skewness γ_1 =1

n	Student-t	Johnson-t	Median-t	Mad-t	AADM-t	MS_T - t		
						5%	10%	20%
5	0.9321	0.9137	0.9393	0.8572	0.9086	0.9207	0.9207	0.9207
	1.1828	1.2221	1.2096	0.7976	0.9997	1.1627	1.1627	1.1627
10	0.9611	0.9549	0.9495	0.9488	0.9497	0.9468	0.9468	0.9244
	0.9972	0.9563	1.7380	1.5168	1.9011	0.9221	0.9221	0.7725
20	0.9399	0.9355	0.9488	0.9470	0.9494	0.9469	0.9426	0.9249
	0.5581	0.6465	0.9444	0.8409	1.0539	0.5249	0.4923	0.4276
30	0.9429	0.9401	0.9493	0.9481	0.9498	0.9427	0.9424	0.9246
	0.3681	0.3619	0.7438	0.6663	0.8351	0.3377	0.3063	0.2886
50	0.9453	0.9429	0.9493	0.9483	0.9496	0.9465	0.9433	0.9228
	0.2817	0.2788	0.5617	0.5058	0.6339	0.2364	0.2071	0.2060
100	0.9473	0.9465	0.9495	0.9488	0.9499	0.9496	0.9432	0.9211
	0.2074	0.2064	0.3893	0.3518	0.4409	0.1999	0.1934	0.1779
500	0.9496	0.9493	0.9498	0.9494	0.9499	0.9497	0.9433	0.9296
	0.1277	0.1276	0.1714	0.1553	0.1946	0.1057	0.0940	0.0784

In Table 3 was obtained the simulation results with random samples from gamma distribution for parameter α=4 and β=0.25 with skewness γ_1 =1. CPs and AWs of CIs have good results, even if the value of skewness coefficient increases. CPs are close to nominal level and increase as sample size increases. However, MS_T-t CI have the narrowest AWs in all sample sizes.

Table 4: Coverage probability (CP) and average width (AW) of 95% CIs for G(1, 1) with Skewness γ_1 =2

n	Student-t	Johnson-t	Median-t	Mad-t	AADM-t	MS_T - t		
						5%	10%	20%
5	0.8827	0.8667	0.8928	0.8046	0.8559	0.8372	0.8372	0.8372
	2.1489	1.9220	2.3192	1.8479	1.8447	1.8216	1.8216	1.8216
10	0.9244	0.9198	0.9488	0.9337	0.9418	0.9285	0.9285	0.8732
	1.5541	1.5553	1.8986	1.5684	1.9281	1.5506	1.5506	1.1732
20	0.9158	0.9130	0.9480	0.9344	0.9431	0.9440	0.9103	0.8225
	0.8961	0.8934	1.0373	0.8509	1.0665	0.8925	0.8532	0.6378
30	0.9267	0.9258	0.9526	0.9378	0.9483	0.9489	0.9143	0.8188
	0.7545	0.7593	0.8185	0.7743	0.8451	0.7573	0.6729	0.5015
50	0.9356	0.9345	0.9525	0.9490	0.9425	0.9476	0.9146	0.8117
	0.5585	0.5529	0.6177	0.5094	0.6384	0.5454	0.5061	0.3764
100	0.9419	0.9418	0.9529	0.9475	0.9464	0.9477	0.9140	0.8036
	0.3928	0.3909	0.4278	0.3534	0.4429	0.3218	0.3097	0.2598
500	0.9493	0.9495	0.9529	0.9472	0.9468	0.9475	0.9148	0.7999
	0.1753	0.1752	0.1886	0.1559	0.1954	0.1551	0.1404	0.1081

In Table 4, we wanted to examine the effect of the degree of skewness on CPs and AWs of CIs. CPs and AWs were obtained from gamma distribution for parameter α=1 and β=1 with skewness γ_1 =2. In this case, CPs were closer to nominal confidence level but slightly less high than previous case. Also, MS_T-t CI have narrower AWs than others.

Table 5: Coverage probability (CP) and average width (AW) of 95% CIs for $\chi^2(1)$ with Skewness γ_1 =2.82

n	Student-t	Johnson-t	Median-t	Mad-t	AADM-t	MS_T - t		
						5%	10%	20%
5	0.8627	0.8667	0.8628	0.8346	0.8259	0.8252	0.8252	0.8252
	4.8416	4.3305	5.5160	2.4661	3.0908	1.0583	1.0583	1.0583
10	0.9144	0.9098	0.9188	0.9137	0.9118	0.9185	0.9185	0.8232
	4.011	3.8059	4.3542	1.6512	2.0695	1.5287	1.5287	0.4594
20	0.9158	0.9130	0.9480	0.9344	0.9431	0.9440	0.9103	0.8225
	2.7059	2.6374	2.8734	0.9312	1.1672	1.5480	0.7116	0.1661
30	0.9267	0.9258	0.9526	0.9378	0.9483	0.9489	0.9143	0.8388
	2.3531	2.3135	2.4766	0.7462	0.9352	0.8784	0.5058	0.1037
50	0.9356	0.9345	0.9525	0.9490	0.9425	0.9476	0.9146	0.8417
	1.9310	1.9116	2.0176	0.5682	0.7121	0.6953	0.3420	0.0622
100	0.9419	0.9418	0.9529	0.9475	0.9464	0.9477	0.9140	0.8436
	1.4476	1.4404	1.5032	0.3968	0.4974	0.5438	0.2120	0.0348
500	0.9493	0.9495	0.9529	0.9472	0.9468	0.9475	0.9148	0.8499
	0.6873	0.6866	0.7096	0.1755	0.2200	0.2226	0.0832	0.0121

In Table 5, it is obtain that CPs of CIs for $\alpha = 0.05$ are especially close to nominal confidence levels except where the trimming ratio is high. AWs reduced with increasing sample size for type I error levels. Comparing CIs with similar CPs, the MS_T-t CI exhibited the smallest AW for all sample sizes. Thus, it produced good results for chisquare distribution.

Table 6: Coverage probability (CP) and average width (AW) of 95% CIs for Lognormal (0,1) with Skewness γ_1 =6.18

n	Student-t	Johnson-t	Median-t	Mad-t	AADM-t	MS_T - t		
						5%	10%	20%
5	0.8527	0.8567	0.8528	0.8546	0.8559	0.8189	0.8189	0.8189
	8.6998	8.6628	8.5478	8.0010	8.6542	3.5687	3.5687	3.5687
10	0.8944	0.8998	0.8988	0.8937	0.8918	0.8585	0.8585	0.8132
	6.2546	6.2045	6.2525	6.1042	6.2269	2.5536	2.5536	2.0034
20	0.9058	0.9030	0.9080	0.9044	0.9031	0.9240	0.9003	0.8125
	4.5986	4.2479	4.2888	4.0708	4.2201	2.0027	1.6378	1.4826
30	0.9067	0.9058	0.9026	0.9078	0.9083	0.9389	0.9143	0.8288
	3.5968	3.5578	3.5963	3.1185	3.4698	1.5721	1.0269	0.1647
50	0.9256	0.9245	0.9225	0.9290	0.9225	0.9276	0.9146	0.8417
	1.9697	1.9423	1.9524	1.9402	1.9287	0.9645	0.4100	0.0836
100	0.9319	0.9318	0.9329	0.9375	0.9364	0.9377	0.9140	0.8536
	0.8654	0.8325	0.8523	0.8241	0.8596	0.3547	0.2045	0.0498
500	0.9393	0.9395	0.9329	0.9372	0.9368	0.9375	0.9148	0.8599
	0.3145	0.3095	0.3177	0.3125	0.3199	0.1104	0.0627	0.0289

Table 6 shows the CIs based on data generated from the Lognormal (0,1) distribution. In this table, we have reviewed again the effect of skewness coefficient. It is obtained that even if the skewness coefficient is large, the MS_T-t CI has performed better in terms of CP and AW.

8 Real data analysis

In this section. we provide two real-life examples in order to illustrate and compare the performance of MS_T-t CI in relation to the existing popular alternatives which have been considered in this paper, when the samples are assumed to come from normal and positively skewed distributions.

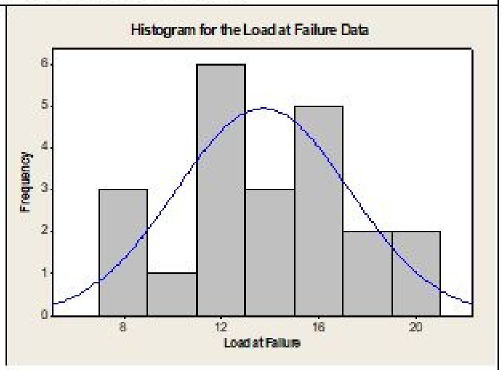
8.1 Real data I: Load at failure data

The first data set was obtained from Montgomery and Runger (2003). The data describes the results of tensile adhesion tests (*in megapascals*) on 22 U-700 alloy specimens.

We are interested to determine the 95% CI for the population mean for load specimen failure. A summary with descriptive statistics, Box-and-Whisker plot, the histogram, density plot, and normal probability plot for the data was obtained using Minitab and the results are shown in Figure 1.

Summary for Load at Failure (*in megapascals*)

Descriptive Statistics	
Sample Size (n)	22
Mean	13.741
Median	13.850
Mode	14.068
Standard Deviation	3.554
Skewness	-0.02
Kurtosis	-0.75
Minimum	7.500
1st Quartile (Q_1)	11.400
3rd Quartile (Q_3)	16.025
Maximum	19.800



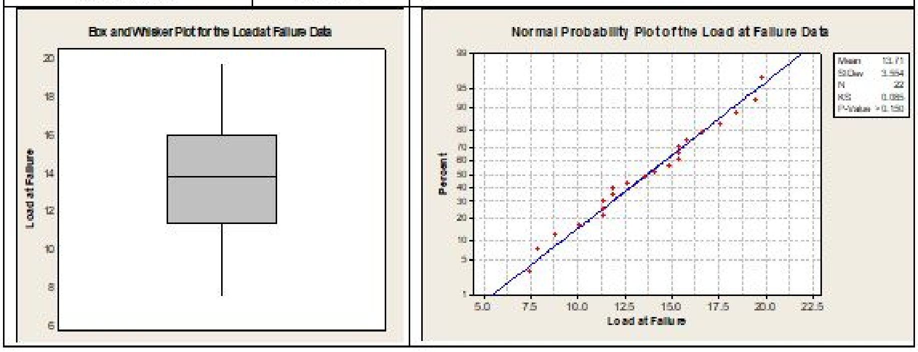


Figure 1: Descriptive statistics for real data I

As can be observed, the Kolmogorov-Smirnov (K-S) goodness-of-fittest for normality have a p-value (p-value>0.150) greater than α= 0.05. We conclude that the data are in excellent agreement with a normal distribution. Additionally, the histogram, the box plot and the normal probability plot of the tensile adhesion test data provide good support for the assumption that the population is normally distributed.

Table 7: The 95% confidence intervals for real data I

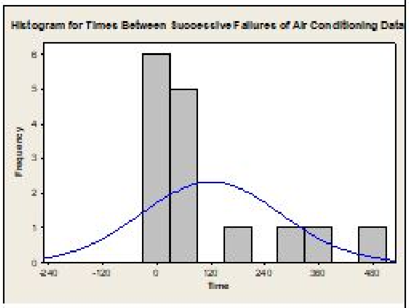
Method		Confidence Interval Limits		
		Lower Limit	Upper Limit	Width
Student-t		12.1380	15.2892	3.1512
Johnson-t		12.1738	15.2525	3.0787
Median-t		7.1738	20.2534	13.0796
Mad-t		7.5124	19.9148	12.4024
AADM-t		5.9415	21.4856	15.5441
MS_T -t	5 %	11.9508	15.4764	3.0256
	10 %	12.2418	15.1854	2.9436
	20 %	12.5797	14.8475	2.2679

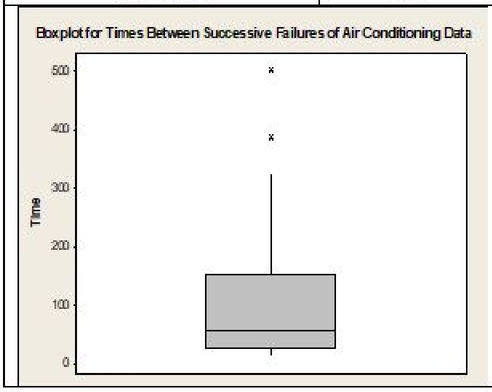
Table 7 shows lower and upper limits for existing CIs and MS_T-t CI. With this information, we obtained widths of CIs. It is seen that, MS_T-t CI AWs have narrower than all others CIs which are consistent with the simulation results.

8.2 Real data II: Failure times of air conditioning data

The second data set was obtained from Shi and Kibria (2007). The data represents the times (*in hours*) between successive failures of air conditioning (AC) equipment in a Boeing 720 airplane for a random sample of 18ACs equipment (Proschan, 1963).

Summary for Times (*in hours*) between Successive Failures of AC Equipment in a Boeing 720	
Descriptive Statistics	
Sample Size (n)	15
Mean	121.267
Median	57
Standard Deviation	154.3
Skewness	1.70
Kurtosis	1.72
Minimum	12
1st Quartile (Q_1)	27
3rd Quartile (Q_3)	153
Maximum	502





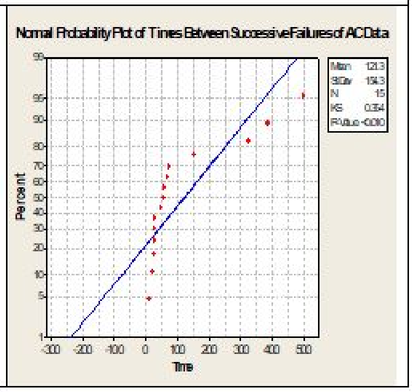


Figure 2: Descriptive statistics for real data II

We are interested to determine the 95% CI for the population mean (μ) of the times between successive failures of air conditioning (AC) equipment in a Boeing 720 airplane. A summary with descriptive statistics is shown in Figure 2. As can be observed, the Kolmogorov-Smirnov (K-S) goodness-of-fit test for normality have a p-value (p-value $<$ 0.010) less than α=0.05. We can say that the data comes from a non-normal distribution. Additionally, the histogram, the box plot, and the normal probability plot of the times between successive failures of air conditioning (AC) equipment in a Boeing 720 airplane data provide good support for the assumption that the population is not normally distributed and the data comes from a positively skewed distribution.

Table 8: The 95% confidence intervals for real data II

Method		Confidence Interval Limits		
		Lower Limit	Upper Limit	Width
Student-t		44.8188	212.3811	167.5623
Johnson-t		50.6552	212.5358	167.8806
Median-t		35.8046	221.3953	185.5907
Mad-t		77.0613	180.1386	103.0773
AADM-t		64.0059	193.1940	129.1881
MS_T -t	5 %	34.4211	222.7788	100.3577
	10 %	59.8470	197.3529	97.5059
	20 %	94.8121	162.3878	67.5757

Table 8 shows that AWs of MS_T-t CI are smaller than all the others considered, which are consistent with the simulation results.

9 Conclusion

Trimmed mean is more convenient for non-normal populations than the sample mean, because it is more robust. Student-t, Johnson-t, Median-t, Mad-t and AADM-t CIs are the most popular methods to estimate mean. However, CPs of MS_T –t CI are closer to nominal than others. In small sample sizes (n = 5, 10, 20), MS_T –t CI performed especially better than existing CIs. Also, as the trimming ratio increased, CPs decreased. It did not perform well for large trimming ratio (20%). MST-t CI has the smaller AW than the others. It is observed that the AWs of CIs are reduced as the sample size increases. When the amount of trimming in the data set increased, CPs and AWs decreased, as expected. Even if the trimming ratio is large enough (20%), AWs of MST-t CI is narrower than the other CIs. CPs and AWs of MST-t CI still have good results when the value of skewness coefficient increases for gamma distribution. Two real data sets are analysed to illustrate the findings of the study and the simulation results are verified. In conclusion, we propose to use MST-t CI for the mean of positively skewed populations.

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