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Interconvertible rules between an aggregative index and a log-change index

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This paper describes interconvertible rules between an aggregative index like the Laspeyres index and a log-change index like the Törnqvist index. Thus we can compare an aggregative index with a log-change index in the same form. Using these rules, we formulate the logarithmic difference between the Laspeyres price index and the Törnqvist price index. One of the rules may be combined with another. By using these combined rules, we can change from given weights to other weights in an aggregative index (or a log-change index) of which the value is invariable.

keywords: price index, quantity index, exact index, substitution bias, logarithmic mean.

1. Introduction

In this paper, we shall show interconvertible rules between an aggregative index (AGI) such as the Laspeyres index and a log-change index (LCI) such as the Törnqvist index. That is: we shall show two rules that will allow conversion of an AGI into an LCI, and two other rules that will allow conversion of an LCI into an AGI. For example, the AGI form of the Laspeyres price index can be converted into the LCI form, and the LCI form of the Törnqvist price index into the AGI form. In this way, we can compare some AGIs with other LCIs in the same form. Without these rules, we have no theoretical account of the difference between an AGI and an LCI.

The Consumer Price Index (CPI) is important in practical matters. This index, for example, is used as a compensation index, a consumption deflator, and a reference

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index for monetary policies (Boskin, 2005; Greenlees and McClelland, 2008; Hill, 2004; McCully et al., 2007; Reinsdorf and Triplett, 2009; Zieschang, 2004). Measuring the CPI or other price indexes, we often utilize the Laspeyres index, the Paasche index, the Fisher index, and the Törnqvist index (see, for example, Diewert, 2004a; Greenlees and Williams, 2009). Using our rules, we can investigate the characteristics of these indexes in detail from various angles. For instance, the Laspeyres price index can be converted into a geometric mean of the price ratios with the weights of the budget shares, which is shown below.

We can assume that the CPI is approximated by the Laspeyres price index and the true cost of living index by the Törnqvist price index. The substitution bias is then the difference between the Laspeyres price index and the Törnqvist price index. While we can measure this bias using actual data, we have not until now had the theoretical or mathematical tools to directly compare their properties. Our rules will help clarify these properties.

Furthermore, our interconvertible rules have relationships with an exact index. The exact index derived by these may be more elegant and may have a simple formula, which is discussed below.

To date, few attempts have been made at determining these rules. Means of combining these rules have never been examined. Our combined rules can derive many AGI forms and LCI forms for some indexes, for example, the Laspeyres price index and the Törnqvist price index. Thus, these additive decompositions and multiplicative decompositions are not unique; see also Balk (2008), Diewert (2002), Diewert (2004c), Reinsdorf et al. (2002).

While we expect that our interconvertible rules will be applied over a vast area, we shall show only three applications to index numbers: 1) the conversion of an AGI into an LCI and an LCI into an AGI, 2) a changed form derived by a combined rule, and 3) the derivation of exact indexes for some utility functions.

This paper is organized as follows. In Section 2, we describe some definitions for an AGI and an LCI, and related topics. In Section 3, we show the interconvertible rules between an AGI and an LCI. In Section 4, we explain how to combine our rules to change from given weights to other weights in an AGI (or an LCI). Applications of our rules as utilized to index numbers are described in Section 5, where the converted forms of the Laspeyres price index and the Törnqvist price index, and the converted and reconverted forms of the Fisher price index are exhibited. In addition, we explain how to derive the exact indexes for the CES utility function and the Klein-Rubin utility function. In Section 6, we illustrate some of the results derived by our rules using actual data. Conclusions are given in Section 7.

2. Some Definitions

In this paper we frequently use the Laspeyres price index (P_L), the Paasche price index (P_P), the Törnqvist price index (P_T), the Montgomery price and quantity indexes (P_M, Q_M), and the Vartia-Sato price and quantity indexes (P_V, Q_V), which are expressed as follows:

$$P_L \equiv \left(\sum p_{1i}q_{0i}\right) / \left(\sum p_{0i}q_{0i}\right),$$

$$P_P \equiv \left(\sum p_{1i}q_{1i}\right) / \left(\sum p_{0i}q_{1i}\right),$$

$$\log P_T \equiv \sum A(w_{11i}, w_{00i}) \log(p_{1i}/p_{0i}),$$

$$\log P_M \equiv \sum \frac{L(e_{11i}, e_{00i})}{L(y_{11}, y_{00})} \log(\frac{p_{1i}}{p_{0i}}),$$

$$\log Q_M \equiv \sum \frac{L(e_{11i}, e_{00i})}{L(y_{11}, y_{00})} \log(\frac{q_{1i}}{q_{0i}}),$$

$$\log P_V \equiv \sum \frac{L(w_{11i}, w_{00i})}{\sum L(w_{11i}, w_{00i})} \log(\frac{p_{1i}}{p_{0i}}),$$

$$\log Q_V \equiv \sum \frac{L(w_{11i}, w_{00i})}{\sum L(w_{11i}, w_{00i})} \log(\frac{q_{1i}}{q_{0i}}),$$

where p_{si} and q_{ti} are, respectively, the price and the quantity of the ith commodity at time s and t, $s(t) = 0$ is the base year, $s(t) = 1$ is the comparison year, $e_{sti} = p_{si}q_{ti}$ is the expenditure, $y_{st} = \sum p_{si}q_{ti}$ is the total expenditure, and $w_{sti} = (p_{si}q_{ti})/(\sum p_{si}q_{ti})$ is the budget share. $A(x_1, x_0) = (x_1 + x_0)/2$ is the arithmetic mean of x_1 and x_0, and $L(x_1, x_0) = (x_1 - x_0)/(\log(x_1/x_0))$ is the logarithmic mean. The cross expenditure, which is expressed as $e_{sti} = p_{si}q_{ti}$ $(s \neq t)$, is a useful concept. Therefore, we also define the total expenditure and the budget share as including this cross expenditure.

In this paper, all variables are assumed to be positive and only natural logarithms are used. In addition, the summation is always made over all the commodities, so the indexes of summation are suppressed.

The above pairs of indexes (P_M, Q_M) and (P_V, Q_V) are the ideal log-change indexes (Balk, 1996; Balk, 2002–3; Balk, 2008; Montgomery, 1937; Sato, 1976; Tsuchida, 1997; Vartia, 1976), which are the only two forms of ideal log-change price and quantity indexes known up to the present. Using our combined rules, we can have many ideal log-change indexes.

These P_L and P_P are the AGIs. We shall define these general forms as follows:

$$\textbf{an AGI is of the form}: \left(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\right)^{\Omega} = \left(\frac{\sum (p_{1i}/p_{0i})z_{0i}}{\sum (p_{0i}/p_{1i})z_{1i}}\right)^{\Omega}.$$

Here, x_i is any positive variable and $z_{si} = p_{si}x_i$. This x_i usually has a dimension denoting quantity but it may have a different dimension. The exponent Ω is positive and dimensionless. We shall refer to both x_i and Ω as the weights of an AGI.

The above P_T, P_M, and P_V are the LCIs. These general forms are as follows:

$$\textbf{an LCI (in the logarithms) is of the form}: \sum \omega_i \log(p_{1i}/p_{0i}).$$

The weight of an LCI is ω_i, which is positive and dimensionless. The sum of the weights, $\sum \omega_i$, may or may not be equal to unity.

Our AGI is easily converted into a (weighted) arithmetic mean index (AMI) of the price ratios as follows:

$$\Big(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\Big)^{\Omega} = \Big(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\Big)^{\Omega-1}\Big(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\Big) = B\sum w_{xi}(\frac{p_{1i}}{p_{0i}}) = \sum w_{bi}(\frac{p_{1i}}{p_{0i}})$$

where $B = ((\sum p_{1i}x_i)/(\sum p_{0i}x_i))^{\Omega-1}$, $w_{xi} = p_{0i}x_i/(\sum p_{0i}x_i)$, and $w_{bi} = Bw_{xi}$. Thus the sum of the weights of the AMI, $\sum w_{bi} = B$, may or may not be equal to unity. This property corresponds to that of our LCI. Any AMI can also be converted into an AGI, which will be shown in Appendix C.

Therefore, our interconvertible rules can be regarded as those between an AMI and an LCI (or a geometric mean index). Our rules can also apply to some indexes and some forms which resemble an AGI or an LCI in appearance but do not belong in this category. These examples will be shown later.

Because the logarithmic mean plays a key role in the following discussion, we shall briefly comment on it; see Balk (2002–3), Balk (2004), Balk (2008), Carlson (1972), Pittenger (1985), Sato (1976), Stolarsky (1975), Tsuchida (1997), Vartia (1976). It has the following properties: the value of $L(x_1, x_0)$ is always positive, $L(x_1, x_0) = L(x_0, x_1)$, $L(x_1, x_0) = x_1$ for $x_1 = x_0$, $L(x_1, x_0) = x_0 L(x_1/x_0, 1)$, and $L(\lambda x_1, \lambda x_0) = \lambda L(x_1, x_0)$ for any positive λ. In addition, it can be approximated by the usual three means: the arithmetic mean, the geometric mean, and the harmonic mean (see Table 1 in Section 6 for a comparison among them). Whenever we use its approximation, we use the arithmetic and geometric means. Besides, the following inequality for x_{1i} and x_{0i} plays an important role below:

$$\sum L(x_{1i}, x_{0i}) \leq L(\sum x_{1i}, \sum x_{0i}). \tag{1}$$

3. Interconvertible Rules

In this section, we shall show interconvertible rules between an AGI and an LCI. Though we shall show the four rules for the price indexes, the same rules also apply to the quantity indexes. First, we describe the two rules that will allow conversion of an AGI into an LCI. Each rule may educe the different weights of an LCI. Next, we describe two other rules that will allow conversion of an LCI into an AGI. The derived weights may also be different from each other.

In the sections below, the same sign may be used with different meanings, and in this event we define it for each case. Each rule is shown in a logarithmic form for convenience in writing. The symbol "⇒"means "be converted into". For example, A ⇒ B means "formula A is converted into formula B".

3.1. Rules that allow conversion of an AGI into an LCI

Rule 1 : $\Omega \log\Big(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\Big) \Rightarrow \Omega \sum \alpha_i \log(\frac{p_{1i}x_i}{p_{0i}x_i}) = \Omega \sum \alpha_i \log(\frac{p_{1i}}{p_{0i}})$.

Here, x_i and Ω are the given weights of the AGI. The weights of the LCI, α_i, are determined in the following way:

$$\alpha_i = L(p_{1i}x_i, p_{0i}x_i)/L(\sum p_{1i}x_i, \sum p_{0i}x_i) \quad \text{and} \quad \sum \alpha_i \leq 1.$$

The following equations, which are used to derive P_M and Q_M in Vartia (1976) and Tsuchida (1997), hold:

$$\log\Big(\frac{\sum e_{11i}}{\sum e_{00i}}\Big) = \sum \frac{L(e_{11i}, e_{00i})}{L(\sum e_{11i}, \sum e_{00i})} \log\left(\frac{e_{11i}}{e_{00i}}\right),$$

$$\log(e_{11i}/e_{00i}) = \log(p_{1i}/p_{0i}) + \log(q_{1i}/q_{0i}).$$

From these, we have the following variant:

$$\log\Big(\frac{\sum e_{10i}}{\sum e_{00i}}\Big) = \sum \frac{L(e_{10i}, e_{00i})}{L(\sum e_{10i}, \sum e_{00i})} \log\left(\frac{e_{10i}}{e_{00i}}\right). \tag{2}$$

Letting $q_{0i} = x_i$ in (2), we obtain the abovementioned rule. Also see (1) above.

Rule 2 : $\Omega \log\Big(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\Big) \Rightarrow \Omega \sum \beta_i \log\left(\frac{p_{1i}x_i}{p_{0i}x_i}\right) = \Omega \sum \beta_i \log(\frac{p_{1i}}{p_{0i}})$.

Here, x_i and Ω are as in Rule 1 above. The weights of the LCI, β_i, are determined in the following way:

$$\beta_i = L(v_{1i}, v_{0i})/\Big(\sum L(v_{1i}, v_{0i})\Big) \quad \text{and} \quad \sum \beta_i = 1,$$

where $v_{si} = p_{si}x_i/(\sum p_{si}x_i)$. Usually β_i is not equal to α_i.

The following equation is used to derive P_V and Q_V in Sato (1976), Vartia (1976), and Tsuchida (1997):

$$\log\Big(\frac{\sum e_{11i}}{\sum e_{00i}}\Big) = \sum\Big(\frac{L(w_{11i}, w_{00i})}{\sum L(w_{11i}, w_{00i})}\Big) \log\left(\frac{e_{11i}}{e_{00i}}\right)$$

from which we have the following variant:

$$\log\Big(\frac{\sum e_{10i}}{\sum e_{00i}}\Big) = \sum \frac{L(w_{10i}, w_{00i})}{\sum L(w_{10i}, w_{00i})} \log\left(\frac{e_{10i}}{e_{00i}}\right). \tag{3}$$

Letting $q_{0i} = x_i$ and $w_{s0i} = p_{si}x_i/(\sum p_{si}x_i) = v_{si}$ in (3), we obtain the above rule.

3.2. Rules that allow conversion of an LCI into an AGI

Rule 3 : $\sum \gamma_i \log\left(\frac{p_{1i}}{p_{0i}}\right) \Rightarrow \Omega \log\left(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\right)$.

Here, γ_i are the given weights of the LCI. The weights of the AGI, x_i and Ω, are determined in the following way:

$$x_i = \frac{\gamma_i}{L(p_{1i}, p_{0i})}, \Omega = L(\sum p_{1i}x_i, \sum p_{0i}x_i), \text{ and } \Omega \geq \sum \gamma_i = \sum L(p_{1i}x_i, p_{0i}x_i).$$

The derived $\sum p_{1i}x_i$ and $\sum p_{0i}x_i$ are dimensionless.

Substituting $e_{10i} = p_{1i}x_i$ and $e_{00i} = p_{0i}x_i$ in (2), and equating $\gamma_i = L(p_{1i}x_i, p_{0i}x_i)$, we have

$$\begin{aligned} \sum \gamma_i \log(\frac{p_{1i}}{p_{0i}}) &= \sum L(p_{1i}x_i, p_{0i}x_i) \log\left(\frac{p_{1i}x_i}{p_{0i}x_i}\right) \\ &= L(\sum p_{1i}x_i, \sum p_{0i}x_i) \log\Big(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\Big). \end{aligned} \tag{4}$$

From $\gamma_i = L(p_{1i}x_i, p_{0i}x_i) = x_i L(p_{1i}, p_{0i})$ and (1), we obtain Rule 3. The second equation in (4) is called the aggregation property of the log-change index by Tsuchida (1997).

Rule 4 : $\sum \gamma_i \log\left(\frac{p_{1i}}{p_{0i}}\right) \Rightarrow (\sum \gamma_i) \log\left(\frac{\sum p_{1i}z_i}{\sum p_{0i}z_i}\right)$.

Here, γ_i are the same as above. The weights of the AGI, z_i, are given as follows:

$$z_i = \delta_i / L(p_{1i}, p_{0i}B) \tag{5}$$

where $\delta_i = \gamma_i/(\sum \gamma_i)$ and $\log B = \sum \delta_i \log(p_{1i}/p_{0i})$. Note that the two values, $\sum p_{1i}z_i$ and $\sum p_{0i}z_i$, are non-dimensional; and $B = (\sum p_{1i}z_i)/(\sum p_{0i}z_i)$.

This rule is obtained the following relationships in which the third equation is the reverse of Rule 2:

$$\begin{aligned} \sum \gamma_i \log(\frac{p_{1i}}{p_{0i}}) &= (\sum \gamma_i) \sum \delta_i \log(\frac{p_{1i}}{p_{0i}}) \\ &= (\sum \gamma_i) \sum \Big(\frac{L(v_{1i}, v_{0i})}{\sum L(v_{1i}, v_{0i})}\Big) \log\Big(\frac{p_{1i}z_i}{p_{0i}z_i}\Big) \\ &= (\sum \gamma_i) \log\Big(\frac{\sum p_{1i}z_i}{\sum p_{0i}z_i}\Big) \end{aligned} \tag{6}$$

where v_{si} and z_i are implicitly given by $\delta_i = L(v_{1i}, v_{0i})/(\sum L(v_{1i}, v_{0i}))$ and $v_{si} = p_{si}z_i/(\sum p_{si}z_i)$.

Solving for z_i in (6) requires a somewhat complicated procedure. Since this procedure has no connection with the main subject, we omit its discussion. Here, we shall only prove that z_i in (5) are the solutions of (6).

From B and $v_{0i} = p_{0i}\delta_i/(L(p_{1i}, p_{0i}B) \sum p_{0i}z_i)$, we get $v_{1i}/v_{0i} = p_{1i}/(p_{0i}B)$ and $v_{0i}L(p_{1i}, p_{0i}B) = p_{0i}\delta_i/(\sum p_{0i}z_i)$. Hence,

$$L(v_{1i}, v_{0i}) = v_{0i}L(v_{1i}/v_{0i}, 1) = v_{0i}L(p_{1i}, p_{0i}B)/(p_{0i}B) = \delta_i/(\sum p_{1i}z_i)$$

and

$$L(v_{1i}, v_{0i})/(\sum L(v_{1i}, v_{0i})) = \delta_i/(\sum \delta_i) = \delta_i,$$

as desired.

It should also be added that the general solutions of (6) are

$$z_i = C\delta_i / L(p_{1i}, p_{0i}B)$$

where C is any pre-determined value such as $\sum \gamma_i$, $\sum p_{si}$, $\sum p_{si}q_{si}$, and so on. C is set to be unity for simplicity.

4. Combined Rules

In this section, we show how to combine the abovementioned rules to change from given weights to other weights in an AGI (or an LCI). Using these combined rules, we can easily derive many sets of weights for an AGI (or an LCI) of which the value is invariable. Although there are eight combined rules in all, four of them are momentous. These four are discussed in this section, and the other four in Appendix A. Since interconvertible rules always have positive weights, the derived weights under combined rules are also positive.

We write a combined rule as "Rule 2 $\bigotimes$ Rule 3". This means the following rules:

$$\begin{aligned} \Omega \log\Big(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\Big) &\Rightarrow \Omega \sum \beta_i \log\Big(\frac{p_{1i}}{p_{0i}}\Big) \\ &= \sum \gamma_i \log\Big(\frac{p_{1i}}{p_{0i}}\Big) \\ &\Rightarrow \Psi \log\Big(\frac{\sum p_{1i}z_i}{\sum p_{0i}z_i}\Big) \end{aligned}$$

where $\gamma_i = \Omega\beta_i$, and the derived weights β_i and (z_i, Ψ) are determined by Rule 2 and Rule 3, respectively. That is: first, we use Rule 2; second, we use Rule 3 with the result of Rule 2.

4.1. Combined rules that change from the given weights to others in an AGI

Given the initial weights (x_i, Ω) of the AGI, we can derive other weights (z_i, Ψ) that satisfy the following equation:

$$\Big(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\Big)^{\Omega} = \Big(\frac{\sum p_{1i}z_i}{\sum p_{0i}z_i}\Big)^{\Psi}. \tag{7}$$

In this changing, there are two cases: one is nondecreasing of the exponent such as (8), and the other is nonincreasing such as (9);

$$0 < \Omega \leq \Psi, \tag{8}$$

and

$$\Omega \geq \Psi > 0. \tag{9}$$

We have (7) and (8) if we use Rule 2 $\bigotimes$ Rule 3, and (7) and (9) if we use Rule 1 $\bigotimes$ Rule 4. These inequalities are due to (1).

These are not so unusual. For example, the Laspeyres price index is written as follows:

$$P_L \equiv \frac{\sum p_{1i}q_{0i}}{\sum p_{0i}q_{0i}} = \Big(\frac{\sum p_{1i}(q_{0i}+x_i)}{\sum p_{0i}(q_{0i}+x_i)}\Big)^{\Omega}$$

where x_i are any positive quantities and Ω is post-determined according to the pre-determined values of x_i, and $(q_{0i}+x_i, \Omega)$ are new weights of P_L. Our combined rules explain how to derive these new weights systematically.

Here we shall define an inverse correspondence of the combined rule. If one combined rule [A] changes from the first set of weights in an AGI (or an LCI) to the second set of weights in an AGI (or an LCI), and if, in addition, another combined rule [B] changes from the second set to the first set; then we say that the combined rule [A] (or [B]) is in inverse correspondence to [B] (or [A]). The above Rule 2 $\bigotimes$ Rule 3 is in inverse correspondence to Rule 1 $\bigotimes$ Rule 4 and vice versa (see Table 4 in Section 6).

We can also use the combined rules iteratively. Thus, we have many sets of weights, (x_i, Ω), (z_i, Ψ), ..., (d_i, Φ), that satisfy the following (10) and (11), or (10) and (12):

$$\Big(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\Big)^{\Omega} = \Big(\frac{\sum p_{1i}z_i}{\sum p_{0i}z_i}\Big)^{\Psi} = \ldots, = \Big(\frac{\sum p_{1i}d_i}{\sum p_{0i}d_i}\Big)^{\Phi}, \tag{10}$$

$$0 < \Omega \leq \Psi \leq \ldots, \leq \Phi, \tag{11}$$

$$\Omega \geq \Psi \geq \ldots, \geq \Phi > 0. \tag{12}$$

4.2. Combined rules that change from the given weights to others in an LCI

Given the weights α_i of the LCI, we can derive other weights β_i that satisfy the following (13) and (14), or (13) and (15):

$$\sum \alpha_i \log(p_{1i}/p_{0i}) = \sum \beta_i \log(p_{1i}/p_{0i}), \tag{13}$$

$$0 < \sum \alpha_i \leq \sum \beta_i, \tag{14}$$

$$\sum \alpha_i \geq \sum \beta_i > 0. \tag{15}$$

We have (13) and (14) if we use Rule 3 $\bigotimes$ Rule 2, while we have (13) and (15) if we use Rule 4 $\bigotimes$ Rule 1. Owing to (1), there are two inequalities (14) and (15). The two combined rules are in inverse correspondence to each other (see Table 4 in Section 6).

We may also use Rule 3 $\bigotimes$ Rule 2 or Rule 4 $\bigotimes$ Rule 1 iteratively. Therefore, we have many weights of the LCI; say the Törnqvist price index, of which the value is invariable.

5. Application to Index Numbers

5.1. Converted forms of the price indexes

5.1.1. Laspeyres price index

Applying Rule 1 to the Laspeyres price index, we have

$$\log P_L \Rightarrow \sum \frac{L(e_{10i}, e_{00i})}{L(y_{10}, y_{00})} \log\Big(\frac{p_{1i}}{p_{0i}}\Big). \tag{16}$$

This converted form in (16) is shown in Balk (2004), Tsuchida (1997), and Vartia (1976).

In addition, applying Rule 2, we have

$$\log P_L \Rightarrow \sum \frac{L(w_{10i}, w_{00i})}{\sum L(w_{10i}, w_{00i})} \log\Big(\frac{p_{1i}}{p_{0i}}\Big) \approx \sum A(w_{10i}, w_{00i}) \log\Big(\frac{p_{1i}}{p_{0i}}\Big) \tag{17}$$

where we used the approximation $L(w_{10i}, w_{00i}) \approx A(w_{10i}, w_{00i})$. This converted form is shown in Balk (2004) and Tsuchida (1997), and is identical with the resultant one in Reinsdorf et al. (2002) that is explained in Appendix B.

The difference between the Laspeyres price index and the Törnqvist price index is called the substitution bias and was measured using actual data (for example, Aizcorbe and Jackman, 1993; Cage et al., 2003; Manser and McDonald, 1988). If we use the approximation in (17), we can account for the substitution bias theoretically. This logarithmic difference is written as follows:

$$\begin{aligned} \log P_L \quad - \quad & \log P_T \approx \frac{1}{2} \sum (w_{10i} - w_{11i}) \log\Big(\frac{p_{1i}}{p_{0i}}\Big) \\ = \quad & \frac{1}{2} \sum (w_{10i} - w_{11i}) \Big(\log \frac{p_{1i}}{p_{0i}} - \log P_P\Big) \\ = \quad & -\frac{1}{2} \sum L(w_{10i}, w_{11i}) \Big(\log \frac{p_{1i}}{p_{0i}} - \log P_P\Big) \Big(\log \frac{q_{1i}}{q_{0i}} - \log Q_P\Big) \end{aligned} \tag{18}$$

where we used the following two equations and Q_P is the Paasche quantity index.

$$\sum (w_{10i} - w_{11i}) \log P_P = 0$$

and

$$\log(w_{10i}/w_{11i}) = -\log(q_{1i}/q_{0i}) + \log Q_P.$$

The Paasche price and quantity indexes can be regarded as weighted means of the price and quantity ratios. Thus, the abovementioned difference approximates the negatively one-half weighted logarithmic covariance of these ratios. If the weighted logarithmic covariance is negative, as is very frequently the case in the consumer goods market, then P_L will be greater than P_T. This approximation error is due to $L(w_{10i}, w_{00i}) \approx A(w_{10i}, w_{00i})$. These errors are shown in Section 6.

Whenever the value of x is close to unity, we can use the approximation that is given by $\log x \approx x - 1$. If this approximation can be used for all logarithmic terms in (18), then

$$P_L - P_T \approx -\frac{1}{2}\sum w_{11i}\Big(\frac{p_{1i}}{p_{0i}} - P_P\Big)\Big(\frac{q_{1i}}{q_{0i}} - Q_P\Big) \tag{19}$$

where we used the following relation:

$$L(w_{10i}, w_{11i}) = \frac{w_{11i}((w_{10i}/w_{11i}) - 1)}{\log(w_{10i}/w_{11i})} \approx w_{11i}.$$

From the result, we may recall Bortkiewicz's notable formula of the difference between P_L and P_P. This formula is that $Q_L(P_P - P_L)$ becomes the weighted covariance of the price and quantity ratios, where Q_L is the Laspeyres quantity index (Balk, 2008; Bortkiewicz, 1923; Diewert, 2004a). See also the difference between P_P and P_T, which will be explained below.

5.1.2. Paasche price index

Applying Rule 1 to the Paasche price index, we have

$$logP_P \Rightarrow \sum \frac{L(e_{11i}, e_{01i})}{L(y_{11}, y_{01})}\log\Big(\frac{p_{1i}}{p_{0i}}\Big).$$

This converted form can be found in Balk (2004).

And applying Rule 2, we have

$$\log P_P \Rightarrow \sum \frac{L(w_{11i}, w_{01i})}{\sum L(w_{11i}, w_{01i})}\log\Big(\frac{p_{1i}}{p_{0i}}\Big) \approx \sum A(w_{11i}, w_{01i})\log\Big(\frac{p_{1i}}{p_{0i}}\Big) \tag{20}$$

where we used the similar approximation above. This converted form is implicitly shown in Balk (2004) and also the same as the resultant one in Reinsdorf et al. (2002) that is explained in Appendix B.

If we use the approximation in (20), we can account for the difference between the Paasche price index and the Törnqvist price index. This logarithmic difference is

$$\begin{aligned}\log P_P \quad - \quad \log P_T &\approx \frac{1}{2}\sum(w_{01i} - w_{00i})\log\Big(\frac{p_{1i}}{p_{0i}}\Big)\\ &= \frac{1}{2}\sum L(w_{01i}, w_{00i})\Big(\log\frac{p_{1i}}{p_{0i}} - \log P_L\Big)\Big(\log\frac{q_{1i}}{q_{0i}} - \log Q_L\Big) \end{aligned}\tag{21}$$

where we used the two equations below.

$$\sum(w_{01i} - w_{00i})\log P_L = 0$$

and

$$\log(w_{01i}/w_{00i}) = \log(q_{1i}/q_{0i}) - \log Q_L.$$

When the logarithmic covariance is negative, P_P will be smaller than P_T.

If the approximations can be used in (21) as in (19), then

$$P_P - P_T \approx \frac{1}{2}\sum w_{00i}\Big(\frac{p_{1i}}{p_{0i}} - P_L\Big)\Big(\frac{q_{1i}}{q_{0i}} - Q_L\Big).$$

5.1.3. Törnqvist price index

By Rule 3, the converted form of the Törnqvist price index is

$$P_T \Rightarrow \Big(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\Big)^{\Omega}$$

where

$$x_i = A(w_{11i}, w_{00i})/L(p_{1i}, p_{0i})$$

and

$$\Omega = L(\sum p_{1i}x_i, \sum p_{0i}x_i) \geq \sum L(p_{1i}x_i, p_{0i}x_i) = \sum A(w_{11i}, w_{00i}) = 1.$$

If we use $L(x_1, x_0) \approx A(x_1, x_0)$ repeatedly, then we derive

$$\Omega \approx \frac{1}{2}\sum \frac{(p_{1i}+p_{0i})A(w_{11i}, w_{00i})}{L(p_{1i}, p_{0i})} \approx \sum A(w_{11i}, w_{00i}) = 1.$$

Thus

$$P_T \approx \frac{\sum(p_{1i}A(w_{11i}, w_{00i})/L(p_{1i}, p_{0i}))}{\sum(p_{0i}A(w_{11i}, w_{00i})/L(p_{1i}, p_{0i}))}. \tag{22}$$

This approximation of the converted index can be found in Reinsdorf (1994).

We may also apply Rule 4. This result is

$$\log P_T \Rightarrow \Big(\sum A(w_{11i}, w_{00i})\Big) \log\Big(\frac{\sum(p_{1i}\delta_i/L(p_{1i}, p_{0i}B))}{\sum(p_{0i}\delta_i/L(p_{1i}, p_{0i}B))}\Big)$$

and

$$P_T \Rightarrow \frac{\sum(p_{1i}A(w_{11i}, w_{00i})/L(p_{1i}, p_{0i}P_T))}{\sum(p_{0i}A(w_{11i}, w_{00i})/L(p_{1i}, p_{0i}P_T))} \tag{23}$$

where

$$\delta_i = A(w_{11i}, w_{00i})/(\sum A(w_{11i}, w_{00i})) \text{ and } B = P_T.$$

Although Reinsdorf et al. (2002) show a convertible formula for an LCI with normalized weights whose sum is equal to unity, their formula yields the same result as Rule 4. Thus, the AGI in (23) is the same as theirs. Note that Rule 4 can also apply to an LCI with non-normalized weights.

5.1.4. CES price index

Our rules can apply to some indexes that do not belong to our AGI or LCI. An example is the CES (or Lloyd-Moulton) price index (Balk, 1999; Diewert, 2004b; Greenlees, 2011; Greenlees and Williams, 2009; Lent and Dorfman, 2009; Lloyd, 1975; Shapiro and Wilcox, 1997). This index is given and rewritten as follows:

$$P_C = \Big(\sum w_{00i}(k_i)^r\Big)^{1/r} = \Big(\frac{\sum w_{00i}(k_i)^r}{\sum w_{00i}}\Big)^{1/r}$$

where $k_i = p_{1i}/p_{0i}$, $r = 1-\sigma$, and σ ($\neq 1$) is the elasticity of substitution.

While we can apply Rule 2, we show only the result by Rule 1. Applying Rule 1 to the last term of the above, we get

$$\log P_C \Rightarrow \frac{1}{r}\sum \alpha_i \log\Big(\frac{w_{00i}(k_i)^r}{w_{00i}}\Big) = \sum \alpha_i \log\Big(\frac{p_{1i}}{p_{0i}}\Big)$$

where

$$\alpha_i = \frac{L(w_{00i}(k_i)^r, w_{00i})}{L(\sum w_{00i}(k_i)^r, \sum w_{00i})} = \frac{w_{00i}L((k_i)^r, 1)}{L(\sum w_{00i}(k_i)^r, 1)}.$$

Using the approximation $L(x_1, x_0) \approx A(x_1, x_0)$ repeatedly, we have

$$\begin{aligned} L(\sum w_{00i}(k_i)^r, \sum w_{00i}) &\approx (1/2)\sum w_{00i}((k_i)^r + 1) \\ &\approx \sum w_{00i}L((k_i)^r, 1). \end{aligned}$$

Therefore

$$\log P_C \approx \sum \frac{w_{00i}L((k_i)^r, 1)}{\sum w_{00i}L((k_i)^r, 1)} \log\Big(\frac{p_{1i}}{p_{0i}}\Big).$$

5.2. Changed forms derived by the combined rule

As an example using the combined rules, we shall consider the Fisher price index P_F. Since $\log P_F = (\log P_L + \log P_P)/2$, and two converted forms of P_L and those of P_P are previously derived, we can quickly obtain $2 \times 2 = 4$ converted forms of P_F. Some converted forms of P_F can be found in Balk (2004) and Reinsdorf et al. (2002). In what follows, we use Rule 2 $\bigotimes$ Rule 3 and the approximation $L(x_1, x_0) \approx A(x_1, x_0)$ to emphasize the similarities between P_F and P_T.

Firstly, we apply Rule 2 to P_L and P_P, and then we obtain

$$\log P_F \approx \sum A(w_{11i}, w_{10i}, w_{01i}, w_{00i}) \log(p_{1i}/p_{0i}) \tag{24}$$

wherein we used the approximations in (17) and (20), and $A(w_{11i}, w_{10i}, w_{01i}, w_{00i})$ is the arithmetic mean of the four budget shares w_{sti}'s.

Secondly, we apply Rule 3 to the above right-hand side. Because there is a similarity between the right-hand side and P_T, this result must also be similar to the approximately converted Törnqvist price index in (22). Thus

$$P_F \approx \frac{\sum(p_{1i}A(w_{11i}, w_{10i}, w_{01i}, w_{00i})/L(p_{1i}, p_{0i}))}{\sum(p_{0i}A(w_{11i}, w_{10i}, w_{01i}, w_{00i})/L(p_{1i}, p_{0i}))}. \tag{25}$$

5.3. Derivations of the exact indexes

Following Diewert (Diewert, 1976; Diewert, 1978), the quantity index $Q(\mathbf{q_1}, \mathbf{q_0})$ is said to be exact if the following two conditions are satisfied: 1) the quantity vector $\mathbf{q_t}$ is a solution of the maximization problem for utility function $U(\mathbf{q_t})$, $t = 0$ and $t = 1$; and 2) $U(\mathbf{q_1})/U(\mathbf{q_0})$ is equal to $Q(\mathbf{q_1}, \mathbf{q_0})$.

The interconvertible rules have relationships with an exact index. If a utility ratio is regarded as our quantity index, then the interconvertible rules can be applied to it[1]. While an exact index sometimes has the same form as a corresponding utility ratio, it may have the converted form from a utility ratio (Diewert, 1976; Diewert, 1978; Diewert, 1981; Diewert, 2009). This converted form of an exact index may be more elegant and simple than the unconverted form. Furthermore, the converted form can be reconverted under our rules. Thus, some utility functions have many exact index formulae[2].

5.3.1. Exact index for the CES utility function

The CES utility function at time t, U_t, is

$$U_t = \Big(\sum a_i (q_{ti})^{-\gamma}\Big)^{-1/\gamma}$$

where $a_i > 0$, $\gamma \neq 0$, and $\gamma > -1$. Applying Rule 2 to the first step in the following equation, we get:

$$-\gamma \log\Big(\frac{U_1}{U_0}\Big) = \log \frac{\sum a_i (q_{1i})^{-\gamma}}{\sum a_i (q_{0i})^{-\gamma}} = \sum \frac{L(v_{1i}, v_{0i})}{\sum L(v_{1i}, v_{0i})} \log\Big(\frac{a_i (q_{1i})^{-\gamma}}{a_i (q_{0i})^{-\gamma}}\Big).$$

Hence

$$\log \frac{U_1}{U_0} = \sum \frac{L(v_{1i}, v_{0i})}{\sum L(v_{1i}, v_{0i})} \log\Big(\frac{q_{1i}}{q_{0i}}\Big). \tag{26}$$

Here $v_{ti} = a_i (q_{ti})^{-\gamma} / \big(\sum a_i (q_{ti})^{-\gamma}\big)$.

Whenever U_t is maximized, the following conditions are satisfied:

$$\partial U_t / \partial q_{ti} = \lambda_t p_{ti} \text{ and } y_{tt} = \sum p_{ti} q_{ti}$$

where λ_t is the marginal utility of the income (=the total expenditure) at time t. From these conditions, we have $v_{ti} = p_{ti} q_{ti} / y_{tt}$ in (26). Hence, we obtain the following result in (27) wherein the Vartia-Sato quantity index Q_V is exact for the CES utility function (for a similar result, see Lau, 1979; Sato, 1976):

$$\log\Big(\frac{U_1}{U_0}\Big) = \sum \frac{L(w_{11i}, w_{00i})}{\sum L(w_{11i}, w_{00i})} \log\Big(\frac{q_{1i}}{q_{0i}}\Big) = \log Q_V. \tag{27}$$

Since the utility function is linearly homogeneous, there is a well-established theorem that states:

$$U_t = \sum (\partial U_t / \partial q_{ti}) q_{ti} = \lambda_t \sum p_{ti} q_{ti} = \lambda_t y_{tt}. \tag{28}$$

From (27), (28), and $P_V Q_V = y_{11}/y_{00}$, we have

$$Q_V = \frac{U_1}{U_0} = \frac{y_{11}/P_V}{y_{00}} = \frac{\lambda_1 y_{11}}{\lambda_0 y_{00}}.$$

Thus, the Vartia-Sato price index P_V has a desirable property for a consumption deflator in this utility function and its reciprocal is equal to the ratio of the marginal utility of the income, λ_1/λ_0.

[1] Our rules may be able to apply to some utility ratios that are not contained in our AGI or LCI

[2] This point was discussed by Diewert (1981).

5.3.2. Exact index for the Klein-Rubin utility function

The Klein-Rubin (or Stone-Geary) utility function at time t, U_t, is

$$\log U_t = \sum a_i \log(q_{ti} - \gamma_i) = \sum a_i \log c_{ti} \tag{29}$$

where $a_i > 0$, $\sum a_i = 1$, $q_{ti} > \gamma_i > 0$, and $c_{ti} = q_{ti} - \gamma_i > 0$.

As is well known, the following Linear Expenditure System is derived under this function:

$$p_{ti}q_{ti} = p_{ti}\gamma_i + a_i\Big(\sum(p_{ti}q_{ti} - p_{ti}\gamma_i)\Big).$$

From this equation, we obtain $a_i = p_{1i}c_{1i}/(\sum p_{1i}c_{1i}) = p_{0i}c_{0i}/(\sum p_{0i}c_{0i})$. Here, we use the geometric mean of the two values for a_i. Thus, $a_i = (G(p_{1i}, p_{0i})G(c_{1i}, c_{0i}))/G(m_{11}, m_{00})$, wherein $G(x_1, x_0) = (x_1x_0)^{0.5}$ is the geometric mean of x_1 and x_0, and $m_{tt} = \sum p_{ti}c_{ti}$.

From (29) and our Rule 4, we have

$$\log(U_1/U_0) = \sum a_i \log(c_{1i}/c_{0i}) = \log\Big((\sum c_{1i}z_i)/(\sum c_{0i}z_i)\Big) \tag{30}$$

where $z_i = a_i/L(c_{1i}, c_{0i}B)$ and $B = U_1/U_0$. Upon substituting a_i and the approximation $L(c_{1i}, c_{0i}B) \approx G(c_{1i}, c_{0i})B^{0.5}$ into (30), we get

$$\frac{U_1}{U_0} = \frac{\sum c_{1i}z_i}{\sum c_{0i}z_i} \approx \frac{\sum c_{1i}G(p_{1i}, p_{0i})}{\sum c_{0i}G(p_{1i}, p_{0i})}. \tag{31}$$

Thus, the approximately exact index for the Klein-Rubin utility function in (31) has a very simple formula. Compare the result in (31) with the Klein-Rubin's. Note that the Klein-Rubin formula is the constant-utility index, not the exact index. Refer to Klein and Rubin (1947–48) and Geary (1950–51). The two terms $\sum c_{1i}G(p_{1i}, p_{0i})$ and $\sum c_{0i}G(p_{1i}, p_{0i})$ are very similar to the supernumerary incomes (Samuelson, 1947–48), $\sum p_{1i}c_{1i}$ and $\sum p_{0i}c_{0i}$.

6. Some Results Using Actual Data

In this section, we explain some of the results derived by our rules using actual data. The data used were yearly expenditures, quantities, and average prices on commodities per household, which consisted of 127 items of the food in Japan for the years 2007–2009 (Statistics Bureau, 2010). Of the food expenditures, only the quantities of these items including the fresh food are reported. The movements of prices and quantities of the fresh food are more pronounced than other commodities in many developed countries such as Japan. Thus, we expect that year-to-year approximated results similar to ours as shown below will be found in many countries and situations.

6.1. Approximation errors

In Section 5, we described the approximate formulae of some price indexes. In there, many approximation errors result from the difference between the logarithmic mean and the arithmetic mean. Thus, we first show this difference.

Table 1 compares the logarithmic mean $L(x_1, x_0)$ of two variables, x_1 and x_0, with the usual three means: the arithmetic mean $A(x_1, x_0)$, the geometric mean $G(x_1, x_0)$, and the harmonic mean $H(x_1, x_0)$. In this comparison, we always set x_0 to be unity. Note that $L(x_1, x_0) = x_0 L(x_1/x_0, 1)$, $A(x_1, x_0) = x_0 A(x_1/x_0, 1)$, $G(x_1, x_0) = x_0 G(x_1/x_0, 1)$, and $H(x_1, x_0) = x_0 H(x_1/x_0, 1)$ hold. Thus, $L(x_1, x_0)/A(x_1, x_0) = L(x_1/x_0, 1)/A(x_1/x_0, 1)$ and so on.

Table 1: Comparison among the different means

Variables					
x_1	x_0	$L(x_1, x_0)$	$A(x_1, x_0)$	$G(x_1, x_0)$	$H(x_1, x_0)$
0.5	1	0.7213	0.7500	0.7071	0.6667
0.6	1	0.7830	0.8000	0.7746	0.7500
0.7	1	0.8411	0.8500	0.8367	0.8235
0.8	1	0.8963	0.9000	0.8944	0.8889
0.9	1	0.9491	0.9500	0.9487	0.9474
1.0	1	1.0000	1.0000	1.0000	1.0000
1.1	1	1.0492	1.0500	1.0488	1.0476
1.2	1	1.0970	1.1000	1.0954	1.0909
1.3	1	1.1434	1.1500	1.1402	1.1304
1.4	1	1.1888	1.2000	1.1832	1.1667
1.5	1	1.2332	1.2500	1.2247	1.2000

Table 1 tells us that the degree of approximation of $A(x_1, x_0)$ to $L(x_1, x_0)$ is very close in the range from $x_1(\text{or} x_1/x_0) = 0.8$ to $x_1(\text{or} x_1/x_0) = 1.2$. In this approximation, the geometric mean is superior to the arithmetic mean and the harmonic mean. Therefore, we can use the geometric means to approximate the logarithmic means in (16), (17), and so forth.

The frequency distribution of the actual ratios of the budget shares and the prices are exhibited in Table 2. Almost all ratios range between 0.8 and 1.2. Thus, these approximations given by $L(x_1/x_0, 1)/A(x_1/x_0, 1) = L(x_1, x_0)/A(x_1, x_0) \approx 1$, wherein x_1/x_0 is w_{10i}/w_{00i}, p_{1i}/p_{0i}, and so forth, will show negligible discrepancies.

Some price indexes for the food expenditures in Japan and their approximations are shown in Table 3. The approximate formulae of P_L, P_P, and P_T are explained in (17), (20), and (22), respectively. We computed the two approximations of P_F: one is the AGI in (25) and the other the LCI in (24). In that section, we formulated two logarithmic differences between the Laspeyres price index and the Törnqvist price index in (18), and between the Paasche price index and the Törnqvist price index in (21). These approximations are also presented in that table. Reflecting the results in Table 1 and Table 2, all the approximations in Table 3 are very good. Thus, we may use these

Table 2: Frequency distribution of the ratios for some ranges

Ratios	Ranges					
	[0.7, 0.8)	[0.8, 0.9)	[0.9, 1)	[1, 1.1)	[1.1, 1.2)	[1.2, 1.3)
2009 / 2008 (Comparison year/ Base year)						
w_{10i}/w_{00i}	0	4	65	52	5	1
w_{11i}/w_{01i}	0	4	62	54	6	1
w_{11i}/w_{00i}	0	4	63	52	8	0
p_{1i}/p_{0i}	0	6	83	35	2	1
2008 / 2007 (Comparison year/ Base year)						
w_{10i}/w_{00i}	0	3	63	50	10	1
w_{11i}/w_{01i}	0	3	63	50	10	1
w_{11i}/w_{00i}	2	6	65	40	12	2
p_{1i}/p_{0i}	0	2	44	63	17	1

approximations in almost all cases. The computed results imply that the two logarithmic covariances in (18) and (21) are negative in Japanese food expenditures.

Table 3: Comparison of true values with their approximations

Price indexes	2009 / 2008		2008 / 2007	
	True values	Approximations	True values	Approximations
P_L	0.982626	0.982631	1.018495	1.018504
P_P	0.981055	0.981059	1.017674	1.017682
P_F(AGI)	0.981840	0.981847	1.018084	1.018089
P_F(LCI)	0.981840	0.981844	1.018084	1.018093
P_T	0.981841	0.981844	1.018084	1.018080
P_L/P_T	1.000800	1.000804	1.000403	1.000412
P_P/P_T	0.999199	0.999203	0.999597	0.999605

6.2. Iterated results by the combined rules

In Section 4, we explained the changing forms of the weights of an AGI and an LCI using the combined rules iteratively. In the changing forms, both the exponents of an AGI and the sums of the weights of an LCI have two cases: one is nondecreasing and the other nonincreasing. Since these topics have never been investigated, we shall illustrate these results taking P_L, P_P, and P_T as examples. The data used were the same for the years 2008–2009.

To P_L and P_P, we first apply Rule 2 $\bigotimes$ Rule 3 three times iteratively, and then Rule 1 $\bigotimes$ Rule 4 five times iteratively. At the start, their exponents Ω's are naturally in unity. These iterated results are exhibited in Table 4. Given that the two combined rules are in inverse correspondence to each other, some coupled values such as the 2nd result and the 4th result are equal. In two price indexes, one of the combined rules is seen to make the exponent of the AGI increase, the other decrease.

To P_T, we first apply Rule 3 $\bigotimes$ Rule 2 three times iteratively, and then Rule 4 $\bigotimes$ Rule 1 five times iteratively. At the start, the sum of the weights, $\sum \omega_i$, is unity from the definition. These iterated results are also shown in Table 4. Since the two combined rules are in inverse correspondence to each other, results like those above are observed. One of the combined rules makes the sum of the weights of the LCI increase, the other decrease.

In computed results, the increasing and decreasing methods for the values of the two exponents of the AGI and the sum of the weights of the LCI are similar to each other. This finding may be interesting and deserve more than a passing notice, though this is beyond the scope of the current discussion.

Table 4: Iterated results of the weights by the combined rules

Indexes' weights	Iterated results					
			Start→	1st→	2nd→	3rd→
	8th←	7th←	6th←	5th←	4th←	
Ω of P_L			1 →	1.000169 →	1.000338 →	1.000508 →
	0.999662 ←	0.999831 ←	1 ←	1.000169 ←	1.000338 ←	
Ω of P_P			1 →	1.000168 →	1.000336 →	1.000504 →
	0.999664 ←	0.999832 ←	1 ←	1.000168 ←	1.000336 ←	
$\sum \omega_i$ of P_T			1 →	1.000169 →	1.000337 →	1.000506 →
	0.999663 ←	0.999832 ←	1 ←	1.000169 ←	1.000337 ←	

7. Conclusion

In this paper, we have shown interconvertible rules between AGIs and LCIs. So far, few attempts have been made at determining these rules. Using these rules, we have converted AGIs such as the Laspeyres price index and the Paasche price index into LCIs, and also converted an LCI such as the Törnqvist price index into AGIs. From these results, we have formulated two logarithmic differences: 1) the difference between the Laspeyres price index and the Törnqvist price index and 2) the difference between the Paasche price index and the Törnqvist price index. In addition, we have derived the exact indexes for some utility functions using our rules.

One of the rules may be combined with another. Using these combined rules, we can change from given weights to other weights in an AGI (or an LCI). Thus, there are many sets of weights of an AGI (or an LCI) of which the value is invariable. As an example using the combined rule, we have derived the approximately converted and reconverted Fisher price indexes.

Furthermore, we compared some price indexes with their derived approximations and illustrated the iterated results by the combined rules using actual data. All our approximations are very close to the true vales.

Appendixes

A. Results of the Other Combined Rules

In this appendix, we briefly discuss the other combined rules that are not covered in Section 4, since these draw insignificant results. The symbol "$\rightarrow$"means "be changed to".

A.1. Rule 1 $\bigotimes$ Rule 3

This combined rule derives the following AGI:

$$\Omega \log\Big((\sum p_{1i}x_i)/(\sum p_{0i}x_i)\Big) \rightarrow \Omega \log\Big((\sum p_{1i}z_i)/(\sum p_{0i}z_i)\Big)$$

where $z_i = x_i/L(\sum p_{1i}x_i, \sum p_{0i}x_i)$. Thus, the resultant formula turns out to be invariant. The other combined rules discussed in this appendix have the same properties.

A.2. Rule 2 $\bigotimes$ Rule 4

This derives

$$\Omega \log\Big((\sum p_{1i}x_i)/(\sum p_{0i}x_i)\Big) \rightarrow \Omega \log\Big((\sum p_{1i}z_i)/(\sum p_{0i}z_i)\Big)$$

where $z_i = x_i/(\sum x_i L(p_{1i}, p_{0i}B))$ and $B = (\sum p_{1i}x_i)/(\sum p_{0i}x_i)$.

A.3. Rule 3 $\bigotimes$ Rule 1

This derives

$$\sum \alpha_i \log(p_{1i}/p_{0i}) \rightarrow \Omega \sum (\alpha_i/\Omega) \log(p_{1i}/p_{0i})$$

where $\Omega = L(\sum p_{1i}x_i, \sum p_{0i}x_i)$ and $x_i = \alpha_i/L(p_{1i}, p_{0i})$.

A.4. Rule 4 $\bigotimes$ Rule 2

This derives the following result:

$$\sum \beta_i \log(p_{1i}/p_{0i}) \to D \sum (\beta_i/D) \log(p_{1i}/p_{0i})$$

where $D = \sum \beta_i$.

B. Correspondence between Our Rules and the Convertible Formulae Derived by Reinsdorf et al.

In this appendix, we describe the correspondence between the results derived by our rules and the convertible formulae shown in Reinsdorf et al. (2002).

B.1. Laspeyres price index

Their formula is the right-hand side of the following:

$$\log P_L \Rightarrow \sum \frac{w_{00i} L(k_i, P_L)}{\sum w_{00i} L(k_i, P_L)} \log\left(\frac{p_{1i}}{p_{0i}}\right)$$

where $k_i = p_{1i}/p_{0i}$. Note that the following two equations hold:

$$\log(w_{10i}/w_{00i}) = \log k_i - \log P_L$$

and

$$P_L(w_{10i} - w_{00i}) = w_{00i}(k_i - P_L).$$

From these, we have $w_{00i}L(k_i, P_L) = P_L L(w_{10i}, w_{00i})$; accordingly the converted form in the above becomes the same as ours in (17). (In this case, we may apply Rule 2 to $P_L = (\sum w_{00i}k_i)/(\sum w_{00i})$ to directly derive their formula. Recall that $v_{1i} = w_{00i}k_i/(\sum w_{00i}k_i) = w_{00i}k_i/P_L$ and $v_{0i} = w_{00i}/(\sum w_{00i}) = w_{00i}$.)

B.2. Paasche price index

Their formula is shown by

$$\log P_P \Rightarrow \sum \frac{w_{11i} L(1/k_i, 1/P_P)}{\sum w_{11i} L(1/k_i, 1/P_P)} \log\left(\frac{p_{1i}}{p_{0i}}\right).$$

The procedures similar to P_L yield $w_{11i}L(1/k_i, 1/P_P) = L(w_{11i}, w_{01i})/P_P$; accordingly this converted form becomes identical with ours in (20). (We may also apply Rule 2 to $P_P = (\sum w_{11i})/(\sum w_{11i}/k_i)$.)

C. Conversion of an AMI into an AGI

Any arithmetic mean index (AMI) of the price ratios is written as follows:

$$\begin{aligned}\sum w_i\Big(\frac{p_{1i}}{p_{0i}}\Big) &= (\sum w_i)\sum\Big(\frac{w_i}{\sum w_i}\Big)\Big(\frac{p_{1i}}{p_{0i}}\Big)\\ &= \Big(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\Big)^{\Omega-1}\Big(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\Big) = \Big(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\Big)^{\Omega}\end{aligned}$$

wherein the following relations are given by

$$\sum w_i = \Big(\frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}\Big)^{\Omega-1},$$

$$\sum\Big(\frac{w_i}{\sum w_i}\Big)\Big(\frac{p_{1i}}{p_{0i}}\Big) = \frac{\sum p_{1i}x_i}{\sum p_{0i}x_i}.$$

Here, w_i, p_{1i}, and p_{0i} are the given variables; and Ω and x_i are the unknown variables. The sum of the weights of our AMI, $\sum w_i$, may or may not be equal to unity. From the second relation, $x_i = Bw_i/p_{0i}$ is found, where B is any constant. The proof is very easy. (Substituting x_i into the right-hand side of that, we immediately obtain the left-hand side of that.) We set $B = 1$ for simplicity. Using these x_i and the first relation, we get Ω. Thus any AMI can be converted into an AGI.

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