Convergence of discontinuous games and essential Nash equilibria

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Abstract: *Let Y be a topological space of non-cooperative games and let F be the map defined on Y such that F(y) is the set of all Nash equilibria of a game y. We are interested in finding conditions on the games which guarantee the upper semicontinuity of the map F. This property of F is a first requirement in order to study the existence of a dense subset Z of Y such that any game y belonging to Z has the following stability property: any Nash equilibria of the game y can be approached by Nash equilibria of a net of games converging to y.*

Keywords: Discontinuous non-cooperative games, better-reply secure games, pseudocontinuous functions, essential Nash equilibria.

1. Introduction.

Let $S_1,\dots,S_n$ be non-empty sets and let $f_1,\dots,f_n$ be real valued functions defined on the Cartesian product of the sets $S_1,\dots,S_n$. The list of data $y=(S_1,\dots,S_n, f_1,\dots,f_n)$ is an *n-player non-cooperative game*: for any $i\in\{1,\dots,n\}$, S_i is the set of strategies of the player i and f_i is the payoff function. If player 1 chooses the strategy x_1, player 2 chooses the strategy x_2 and so on, the corresponding outcomes of the game are: $f_1(x_1,x_2,\dots,x_n)$ for player 1, $f_2(x_1,x_2,\dots,x_n)$ for player 2,…, $f_n(x_1,x_2,\dots,x_n)$ for player n. A list of strategies $x=(x_1,x_2,\dots,x_n)$ is said to be a *profile of strategies*, and a profile of strategies x^* is said to be a *Nash equilibrium* (see Nash (1950)) if, for any player i, $f_i(x^*)\geq f_i(x_i,x^*_{-i})$ for all x_i belonging to S_i, where (x_i,x^*_{-i}) means $(x^*_1,\dots, x^*_{i-1}, x_i, x^*_{i+1},\dots, x^*_n)$. Now, let Y be a set of non-cooperative n-player games endowed with a topology. Following Yu (1999), given a game y, we say that a Nash equilibrium x of the game y is *essential* if for any neighbourhood O of x there exists a neighbourhood N of y such that any game y' belonging to N has at least a Nash equilibrium x' which belongs to O; moreover, we say that the game y is *essential* if any Nash equilibrium of y is essential. As one can see, the idea of essentiality concerns a notion of stability under perturbations on the data. Before to explain this, let us recall some definitions in the setting of set-valued analysis. Let F be a set-valued function (also said a map) defined on Y with values in X. We say that F is *upper semicontinuous* (see, for example, Aliprantis and Border (1999)) at $y\in Y$ if for any open set O containing $F(y)$ there exists a neighbourhood N of y such that $F(y')\subseteq O$ for any $y'\in Y$; we say that F is *lower semicontinuous* at y if for any $x\in F(y)$ and any open neighbourhood O of x there exists a neighbourhood N of y such that $F(y')\cap O$ is non-empty for any y' belonging to N. Now, if F is the map defined on a space of games Y such that $F(y)$ is the set of Nash equilibria of y, then the game y is essential if and only if the map F is lower semicontinuous at y. So, the existence of essential game is nothing but the existence of point of the topological space Y in which the set-valued function F is lower semicontinuous. A crucial aid is the following theorem, due to Fort (1950):

Theorem 1. *Let F be a set-valued function defined on Baire space Y with non-empty and compact values in a metric space X. If F is upper semicontinuous at any point of Y, then there exists a subset Z of Y which is dense and such that F is also lower semicontinuous at any point of Z.*

In light of this theorem, if the topological space of games Y is a Baire space (for the definition of *Baire spaces* see, for example, Aliprantis and Border (1999)) and the space of profiles of strategies is metric, the existence of essential games is guaranteed by conditions which allow the map F to be upper semicontinuous. The focus of this note is just to show under which conditions, remarkable in the framework of strategic choices and weaker than continuity of payoffs, the map F is upper semicontinuous with non-empty and compact values.

2. Preliminaries.

In order to obtain that the set-valued function F satisfies the hypothesis of Theorem 1, by using conditions over the payoffs of games weaker than continuity, let us remind a recent generalization of the continuity of real valued function: see Morgan and Scalzo (2007).

Definition 1. *Let f be a real valued function defined on a topological space X and let $x_0 \in X$. We say that the function f is upper pseudocontinuous at x_0 if*

$$\limsup_{x \to x_0} f(x) < f(x_1)$$

for any $x_1 \in X$ such that $f(x_0) < f(x_1)$. The function f is said to be lower pseudocontinuous at x_0 if

$$f(x_1) < \liminf_{x \to x_0} f(x)$$

for any $x_1 \in X$ such that $f(x_1) < f(x_0)$. Finally, f is said to be pseudocontinuous at x_0 if it is both upper and lower pseudocontinuous.

The class of pseudocontinuous functions play a role in Choice Theory. In fact, as shown in Morgan and Scalzo (2007), if a continuous preference relation defined on a topological space – that is a complete and transitive binary relation such that the upper level sets and the lower level sets are closed – is endowed of numerical representations (also said *utility functions*), then all such representations are pseudocontinuous functions. So, pseudocontinuity is the common topological property among the numerical representations of continuous preference relations.
When a game $y=(S_1,\dots,S_n, f_1,\dots,f_n)$ has the sets S_i compact and convex and the functions f_i pseudocontinuous on $S_1 \times \ldots \times S_n$ and such that $f_i(\cdot, x_{-i})$ is quasi-concave for any x_{-i} and any i, then, in light of Theorem 3.2 in Morgan and Scalzo (2007), y admits Nash equilibria. So, from now on, let Y be the set of all games $y=(S_1,\dots,S_n, f_1,\dots,f_n)$ having the sets of strategies $S_1,\dots,S_n$ non-empty, compact, convex and included, respectively, in the subsets $X_1,\dots,X_n$ of normed spaces, and the payoff functions f_i are pseudocontinuous and bounded on $X = X_1 \times \ldots \times X_n$ and such that $f_i(\cdot, x_{-i})$ is quasi-concave for any x_{-i}. Now, we introduce a suitable topology on Y: following Yu (1999), we first consider the metric on the space of all vector functions $f = (f_1,\ldots,f_n)$, with $f_1,\ldots,f_n$ pseudocontinuous and bounded on X, defined as follows:

$$\rho(f, f') = \sum_{i=1}^{n} \sup_{x \in X} \left| f_i(x) - f'_i(x) \right| ,$$

then , if K_i denote the set of all non-empty, compact and convex subset of X_i, for any i, we take the Vietoris' topology (see Klein and Thopson (1984)) on K_i and so the topology σ on $K = K_1 \times \ldots \times K_n$ which is the product of the Vietoris' topologies on any K_i. Finally, we obtain a topology τ on Y as the product of σ and the topology induced by the metric ρ.

3. The results.

Let $F : Y \to 2^X$ be the set-valued function such that $F(y)$ is the set of all Nash equilibria of the game y, where Y is defined as in the previous section. We have the following result:

Theorem 2. *The set-valued function F has non-empty and compact values and is upper semicontinuous with respect to the topology τ, that is: if an open set O contains the set of Nash equilibria F(y) of a game y and if $(y^{\alpha})_{\alpha}$ is a net of games converging to y in the topology τ, then we have $F(y^{\alpha}) \subseteq O$ for any game y^{α} with $\alpha \succ \alpha_0$ for a suitable index α_0.*

The proof of Theorem 2 can be achieved by the arguments of the proof of Theorem 3.2 in Scalzo (2008). Let us remark that a previous result on the upper semicontinuity of the set-valued function *F* is given in Yu (1999) in the case in which the payoffs of any game are continuous functions. Here, we not only generalize the previous result, but we also obtain a result by using an ordinal topological property, that is the pseudocontinuity.
Theorem 2 can be used in order to state that there exist essential games with pseudocontinuous payoff functions. In fact, if we recognize a non-empty subset Y_1 of *Y* such that Y_1 is a Baire space, from Theorem 2 we know that *F* is upper semicontinuous on Y_1 and from Theorem 1 we know that there exists a subset $Z \subseteq Y_1$ which is dense – that is: the topological closure of *Z* coincides with Y_1 – and such that the map *F* is also lower semicontinuous at *z* for any $z \in Z$, which means that any game $z \in Z$ is essential.
Obviously, the pseudocontinuity is not the only generalization of ordinal character of the continuity in the setting of non-cooperative games. An other remarkable class of discontinuous games is the class of *better-reply secure* games, due to Reny (1999): a game $y=(S_1,\dots,S_n, f_1,\dots,f_n)$ is said to be better-reply secure if for any pair (x^*, u^*) such that x^* is not a Nash equilibrium of *y* and u^* belongs to the closure of the graph of the vector function $(f_1,\dots,f_n)$, then some player *i* has a strategy $x_i^{\wedge}$ such that $f_i(x_i^{\wedge}, x_{-i}) > u_i^* + \varepsilon$ for all x_{-i} belonging in some neighbourhood of x_{-i}^*, where ε is a suitable positive real number. We remark that any game with pseudocontinuous payoff functions is also a better-reply secure: see Proposition 4.1 in Morgan and Scalzo (2007).
Hence a question arises: Does the thesis of Theorem 2 hold for the class of better-reply secure games? The answer of the question is in the following theorem:

Theorem 3. *Let Y_1 be the set of all better-reply secure games with spaces of strategies included, respectively, in $X_1,\dots,X_n$, and let $F_1 : Y_1 \to 2^X$ be the set-valued function such that F(y) is the set of Nash equilibria of y – in light of Theorem 3.1 in Reny (1999), F(y) is non-empty and compact. Then, there exist a game $y \in Y_1$, an open set O containing $F_1(y)$ and a net of games $(y^{\alpha})_{\alpha}$ converging to y such that $F_1(y^{\alpha}) \setminus O$ is non-empty for any α - in other words, the map F_1 is not upper semicontinuous at y.*

For a sketch of proof, it is sufficient to consider the game $G=(S_1,S_2,f_1,f_2)$ and the sequence of games $G^n=(S_1^n,S_2^n,f_1^n,f_2^n)$ such that: $S_1=S_2=[0,1]$, $S_1^n=S_2^n=\left[0,1+\frac{1}{n}\right]$, $f_1(x_1,x_2)=f_1^n(x_1,x_2)=h(x_1)$ and $f_2(x_1,x_2)=f_2^n(x_1,x_2)=h(x_2)$, where the function h is defined as follows:

$$\begin{aligned} h(0) &= 1 \\ h(x) &= 0 \ \forall x \in \left]0,1\right] \\ h(x) &= x \ \forall x \in \left]1,+\infty\right[\end{aligned}.$$

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